Math 2270 Homework, due Sept 2, 2015.

(1) In class we saw that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field. Generalizing this, assume that $c \in \mathbb{Q}$, that $\gamma \in \mathbb{R} - \mathbb{Q}$ and $\gamma^2 = c$. Prove that

$$\mathbb{Q}(\gamma) = \{a + b\gamma \mid a, b \in \mathbb{Q}\}$$

is a field. Thus e.g. $\mathbb{Q}(\sqrt{5})$ is a field.

(2) From the axioms for a vector space show that $c \cdot \vec{O} = \vec{O}$ for all $c \in K$.

(3) From the axioms for a vector space show that if $c \cdot u = \vec{O}$ then $c = 0$ or $u = \vec{O}$.

(4) Check that

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x = y \text{ and } y = 5z\}$$

is a subspace of $\mathbb{R}^3$. Find a generating set for W.

(5) Express $(0, 0, 1) \in \mathbb{R}^3$ as a linear combination of $v_1 = (1, 1, 1)$, $v_2 = (0, 1, -1)$ and $v_3 = (0, 1, 0)$.

(6) Show that the functions $f(t) = \cos t$ and $g(t) = \sin t$ are linearly independent in the vector space of all functions $\mathbb{R} \to \mathbb{R}$.

(7) Suppose that two vectors $v, w \in V$ are linearly dependent and that $w \neq \vec{O}$. Show that $v = cw$ for some $c \in K$.

(8) Show that the vectors $v_1 = (a, b)$ and $v_2 = (c, d)$ in $\mathbb{R}^2$ are linearly independent if and only if $ad - bc \neq 0$. Hint: Use the previous problem.