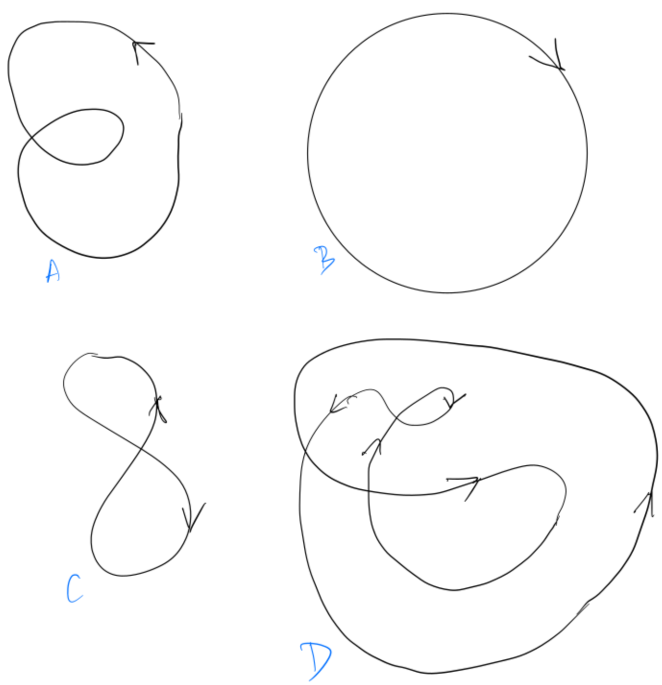
Homework 7: Argument Principle, Rouché, Maximum Modulus Principle

Argument Principle

1. The curve A below is the image of $|z| = 1$ under the map $z \mapsto (z + \frac{1}{2})^2$. All pictures are drawn up to isotopy, i.e. deformations that don't change crossings. Can you determine the complementary component that contains 0? Prove that there are no holomorphic functions defined in $\{|z| < 1.1\}$ that map $|z| = 1$ to any of the curves B,C,D. Orientations are induced by the counterclockwise orientation of $|z| = 1$. Hint: For the curve D consider a double cover. Also note that B is the image of $|z| = 1$ under $z \mapsto \frac{1}{z}$, but that's not defined at $z = 0$.



2. How many roots does $z^4 + z + 1 = 0$ have in the first quadrant? Hint: Consider the image under the map of the boundary of a big quarter disk.

Rouché's theorem.

3. Show that all roots of $z^4 + z^3 + 1 = 0$ have norm $< \frac{3}{2}$.

4. Prove the following special case of Brouwer's fixed point theorem (which states that any continuous map of a closed disk to itself has a fixed point). Let f be holomorphic in $\{|z| < 1.1\}$ and assume it maps $\{|z| \leq 1\}$ to $\{|z| < 1\}$. Then f has a fixed point.

5. How many roots does $z^4 - 3z + 1 = 0$ have in $\{|z| < 1\}$?

6. How many roots does $z^4 + z^3 - 4z + 1 = 0$ have in $\{1 < |z| < 3\}$?

7. How many roots does $e^z - 4z^n + 1 = 0$ have in $\{|z| < 1\}$?

Maximum Modulus Principle.

8. Suppose f is a holomorphic function in a neighborhood of $\{1 \leq |z| \leq R\}$. Suppose that $|f(z)| \leq 1$ when $|z| \leq 1$ and $|f(z)| \leq R$ when $|z| = R$. Prove that $|f(z)| \leq |z|$ when $1 \leq |z| \leq R$.

9. Let f be a nonconstant entire function and let $M_r = \max_{|z|=r} |f(z)|$. Show that $r \mapsto M_r$ is strictly increasing.

10. Prove the Minimum Modulus Principle: If $\Omega \subseteq \mathbb{C}$ is a domain, $f : \Omega \to \mathbb{C}$ holomorphic and nonconstant, and if $f(z) \neq 0$ for $z \in \Omega$, then $|f|$ does not attain its infimum on Ω.

11. Give an alternative proof (called the "nobody is above average" proof) of the Maximum Modulus Principle, as follows. Suppose $z_0 \in \Omega$, $f : \Omega \to \mathbb{C}$ is holomorphic, and $|f(z_0)| \geq |f(z)|$ for $z \in \Omega$. Let $r > 0$ such that $\{|z - z_0| \leq r\} \subset \Omega$. Recall the Mean Value Property of holomorphic functions

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

Show that this implies that $|f(z)|$ is constant in a neighborhood of z_0 and then use Problem #2 from Homework #1.