Definitions and examples of manifolds

Guillemin-Pollack

In this part we use the Guillemin-Pollack definitions, in particular of a smooth function defined on a subset $X \subset \mathbb{R}^n$: $f : X \to \mathbb{R}^m$ is smooth if for every $x \in X$ there is a neighborhood U of x in $\mathbb{R}^n$ and a smooth function $F_U : U \to \mathbb{R}^m$ that agrees with f on $U \cap X$.

1. For $k < n$ view $\mathbb{R}^k$ as a subset of $\mathbb{R}^n$ via $(x_1, \cdots, x_k) \mapsto (x_1, \cdots, x_k, 0, \cdots, 0)$. Show that a function $f : \mathbb{R}^k \to \mathbb{R}$ is smooth (using the G-P subset definition) if and only if it is smooth in the usual sense.

2. Let $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^k, Z \subset \mathbb{R}^n$. If $f : X \to Y$ and $g : Y \to Z$ are smooth (as maps to the ambient Euclidean spaces) then so is the composition $gf : X \to Z$. If f, g are diffeomorphisms so is gf.

3. Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m, X' \subset \mathbb{R}^k, Y' \subset \mathbb{R}^l$. If $f : X \to X'$ and $g : Y \to Y'$ are smooth, so is $f \times g : X \times Y \to X' \times Y'$. Here we view $X \times Y$ as a subset of $\mathbb{R}^{n+m}$ and $X' \times Y'$ as a subset of $\mathbb{R}^k \times \mathbb{R}^l$.

4. Let $X \subset \mathbb{R}^n$ and let $\Delta = \{(x, x) \in \mathbb{R}^{2n} \mid x \in X\}$ be the diagonal. Show that Δ is diffeomorphic to X. More generally, if $f : X \to Y$ is smooth, then the graph of f

$$graph(f) = \{(x, f(x)) \mid x \in X\}$$

   is diffeomorphic to X. I am suppressing various ambient Euclidean spaces.

5. Show that the map $a : S^n \to S^n$ defined by $a(x) = -x$ is a diffeomorphism.

6. Show that the letters L and I are homeomorphic but not diffeomorphic. Here $I = \{0\} \times \mathbb{R}$ and $L = \{0\} \times [0, \infty) \cup [0, \infty) \times \{0\}$. Hint: View I as a subset of $\mathbb{R}$ and a diffeomorphism $I \to L$ as a local parametrization around the corner point, the derivative should be injective.

Chart definition

7. Show that $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ is not a topological manifold. Hint: How many components does $X \setminus \{(0, 0)\}$ have?

8. Let X be a smooth manifold. Show that the set $C^\infty(X)$ of all smooth functions $X \to \mathbb{R}$ is an algebra, i.e. if $f, g \in C^\infty(X)$ then so are $af + bg$ and fg for any $a, b \in \mathbb{R}$. You can use the fact that this is so when X is an open set in $\mathbb{R}^n$.

9. The group $GL_n(\mathbb{R})$ of $n \times n$ matrices is naturally an open set in the space $M(n)$ of all real $n \times n$ matrices, which is naturally identified with $\mathbb{R}^{n^2}$. Show that the maps $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ defined by $(A, B) \mapsto AB$ and $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ are smooth. This shows that $GL_n(\mathbb{R})$ is a Lie group.

The Regular Value Theorem, Lie groups

10. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = xyz + x^3 + y^3 + z^3$ and let $M_a = f^{-1}(a)$. Show that M_a is a manifold for $a \neq 0$, and that $M_0 \smallsetminus \{(0, 0, 0)\}$ is a manifold.

11. The *unitary group* $U(n)$ is the group of complex $n \times n$ matrices M such that $MM^* = I$, where M^* is transpose followed by conjugation of all entries. Show that $U(n)$ is a compact Lie group.

12. The (real) symplectic group $Sp(2n, \mathbb{R})$ is the group of real $2n \times 2n$ matrices M that satisfy $M\Omega M^t = \Omega$, where Ω is the block matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with blocks of size $n \times n$ and I_n the identity matrix. In other words, if you think of Ω as defining a skew-symmetric bilinear form $(v, w) \mapsto v^t \Omega w$ on $\mathbb{R}^{2n}$, then the condition is that M preserves this form. Show that $Sp(2n, \mathbb{R})$ is a Lie group. Also show that $Sp(2, \mathbb{R}) = SL_2(\mathbb{R})$ (the form $(v, w) \mapsto v^t \Omega w$ is the signed area of the parallelogram spanned by v and w). Hint: Use the argument for $O(n)$ as a template. This time the range of the function F is the space of *skew-symmetric* matrices S, those with $S^t = -S$.