de Rham cohomology

Computations

The main tools are Mayer-Vietoris and homotopy invariance (Poincaré lemma).

1. Compute the de Rham cohomology of the Möbius strip.

2. Compute by induction on n the de Rham cohomology of $\mathbb{R}^2$ with n points removed. You can assume that the points are on the x-axis, although that doesn't affect the answer.

3. Let M be a manifold. A *good cover* of M is a finite open cover $\{U_1, U_2, \cdots, U_m\}$ such that for any subcollection the intersection

$$U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}$$

   is either empty or contractible. Show that de Rham cohomology $H^p(M)$ of M is finite dimensional for all p. Hint: Induct on m.

   Remark: Compact smooth manifolds admit a good cover. E.g. fix a Riemannian metric, and then it turns out that all sufficiently small balls are convex, in the sense that any two points are connected by a unique geodesic (shortest path), which implies that any finite cover by such balls is good.

Real projective space

4. Show that $H^p(\mathbb{R}P^n) = 0$ when $0 < p < n$. Hint: Given a closed p-form ω on $\mathbb{R}P^n$, pull it back to S^n by the projection $\pi : S^n \to \mathbb{R}P^n$ and use your knowledge of $H^p(S^n)$ to find a $(p-1)$-form η on S^n with $d\eta = \pi^*\omega$. Then arrange by averaging that $a^*\eta = \eta$ for the antipodal map $a : S^n \to S^n$ and show that η induces a form on $\mathbb{R}P^n$.

5. Finish the computation of $H^p(\mathbb{R}P^n)$ by showing that $H^n(\mathbb{R}P^n)$ is 0 when n is even and it is $\mathbb{R}$ when n is odd. (Recall that $\mathbb{R}P^n$ is orientable iff n is odd.) Hint: Use Mayer-Vietoris for open sets U, V analogous to the sets we used for $\mathbb{C}P^n$ in class, and show that it gives an exact sequence

$$0 \to H^{n-1}(\mathbb{R}P^{n-1}) \to \mathbb{R} \to H^n(\mathbb{R}P^n) \to 0$$

   and then induct on n starting with $\mathbb{R}P^1 = S^1$.

The n-torus

6. Think of the 2-torus T^2 as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. the forms dx, dy on $\mathbb{R}^2$ are translation invariant and descend to T^2, and they are closed. Show that their cohomology classes are independent in $H^1(T^2)$ by finding a closed submanifold along which the integral of one is 0 and of the other it is nonzero. (Perhaps a good warmup is to argue that the class of dx generates $H^1(S^1)$.)

7. Generalize Problem 6 to the n-torus and show that $\dim H^k(T^n) \geqslant \binom{n}{k}$.

8. Show that in fact $\dim H^k(T^n) = \binom{n}{k}$ using the Mayer-Vietoris sequence and induction on n as follows. Write $S^1 = U \cup V$ where U, V are open intervals and $U \cap V$ consists of two intervals. Thus $T^n = T^{n-1} \times U \cup T^{n-1} \times V$. In the M-V sequence we have a map

$$H^k(T^{n-1} \times U) \oplus H^k(T^{n-1} \times V) \to H^k(T^{n-1} \times (U \cap V))$$

which can be identified with

$$H^k(T^{n-1}) \oplus H^k(T^{n-1}) \to H^k(T^{n-1}) \oplus H^k(T^{n-1})$$

The image of this map is the diagonal (this follows from the Poincaré lemma). Show that this implies that we have an exact sequence

$$0 \to H^{k-1}(T^{n-1}) \to H^k(T^n) \to H^k(T^n) \to 0$$

and then induct.

Miscellaneous

9. Let M be a closed (compact, no boundary) orientable manifold of dimension n and let ω be an $(n-1)$-form on M. Show that $d\omega$ vanishes at some point of M.

10. Let $M \subset \mathbb{R}^n$ be a closed $(n-1)$-manifold. Show that M separates $\mathbb{R}^n$ into at least 2 components. Hint: If $\mathbb{R}^n \setminus M$ is connected, show that there is a map $S^1 \to \mathbb{R}^n$ with $I_2(f, M) = 1 \mod 2$. Why is that a contradiction? You can use the fact that connected manifolds are smooth-path connected (can you prove it?)

11. If M is also connected show that it separates $\mathbb{R}^n$ into exactly 2 components and it is the boundary of the closure of each, so in particular M is orientable. So for example $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$. Hint: Tubular neighborhood theorem.