# de Rham cohomology, part 2

The problems this week are harder, so there are fewer of them. Finding Poincaré duals is optional since we didn't cover them in detail.

## More de Rham cohomology

1. Let $Y_{g,n}$ be the orientable surface of genus $g$ with $n$ points removed. Compute $H^*(Y_{g,n})$. Can you find submanifolds whose Poincaré duals generate?

2. Let $X \subset \mathbb{R}^n$ be a closed set. The goal of this exercise is to relate $H^*(\mathbb{R}^n \setminus X)$ and $H^*(\mathbb{R}^{n+1} \setminus X)$.

   Fix a continuous function $f : \mathbb{R}^n \to [0,1]$ such that $f^{-1}(0) = X$. For example $f(y) = \min\{1, d(y,X)\}$ will do. Let
   $$U = \{(y,t) \in \mathbb{R}^n \times \mathbb{R} \mid t < f(y)\}$$
   and
   $$V = \{(y,t) \in \mathbb{R}^n \times \mathbb{R} \mid t > -f(y)\}$$
   Show that $U$ and $V$ are contractible by finding a deformation retraction to hyperplanes $t = -1$ and $t = 1$ respectively. Also show that $U \cap V$ deformation retracts to $\mathbb{R}^n \setminus X$. From the Mayer-Vietoris sequence deduce that for $p \geqslant 2$ we have
   $$H^p(\mathbb{R}^{n+1} \setminus X) \cong H^{p-1}(\mathbb{R}^n \setminus X)$$
   The formula holds even when $p = 1$ or $0$, but one must modify the definition of $H^0$ that effectively removes one dimension (so it is 0 for connected manifolds). That is, one uses *reduced cohomology* $\tilde{H}^0$. If you are curious, $\tilde{H}^0(X)$ is the cokernel of the map $H^0(p) \to H^0(X)$ induced by the constant map $X \to \{p\}$.

3. Using only the Stokes theorem show that an orientable compact manifold doesn't admit a smooth retraction to its boundary.

## Compactly supported de Rham cohomology

4. Let $U, V$ be open subsets of a manifold $M$ with $M = U \cup V$. Show that for any $p$ the sequence
   $$0 \to \Omega^p_c(U \cap V) \xrightarrow{i} \Omega^p_c(U) \oplus \Omega^p_c(V) \xrightarrow{j} \Omega^p_c(M) \to 0$$

is exact, where $i(\omega) = (\omega_U, \omega_V)$ and $j(\eta,\theta) = (\tilde{\eta} - \tilde{\theta})$, with $\omega_U, \omega_V$ denoting the extension of $\omega$ to $U, V$ by 0, and $\tilde{\eta}, \tilde{\theta}$ are the extensions of $\eta, \theta$ by 0 to $M$. This implies, by homological algebra, that there is a long exact sequence

$$\cdots \to H_c^p(U \cap V) \to H_c^p(U) \oplus H_c^p(V) \to H_c^p(M) \to H_c^{p+1}(U \cap V) \to \cdots$$

5. Show that $H_c^p(\mathbb{R}^2 \setminus \{0\}) = \mathbb{R}$ for $p = 1, 2$ and 0 otherwise. Can you find submanifolds whose Poincaré duals generate? Hint: Mayer-Vietoris. There are several ways of doing it, one possibility is to cover $\mathbb{R}^2 \setminus \{0\}$ with two open sets as in Problem 2.. The other is to take $U, V \subset S^2$ to be the complements of the north and south poles respectively and then $U \cap V$ is diffeomorphic to $\mathbb{R}^2 \setminus \{0\}$.

6. Compute $H_c^p(X_n)$ where $X_n$ is $\mathbb{R}^2$ with $n$ points removed. Can you find submanifolds whose Poincaré duals generate compactly supported cohomology? The same question for the regular cohomology.

7. Show that $H_c^p(\mathbb{R}^n \setminus \{0\}) = \mathbb{R}$ for $p = 1, n$ and 0 otherwise. Can you find submanifolds whose Poincaré duals generate?