Tangent Spaces and Morse functions

Tangent spaces

1. Show that for k odd the k-sphere S^k admits a nowhere vanishing vector field. Note: It is a deep result that the only spheres with a *trivial* tangent bundle are S^1, S^3 and S^7.

2. Let G be a Lie group and $v \in T_1G$ a vector in the tangent space of the identity. For $g \in G$ define the vector $V(g) \in T_g(G)$ as

$$V(g) = dL_g(v)$$

where $L_g : G \to G$ is the diffeomorphism defined by $x \mapsto gx$ (left translation by g). Show that V is a smooth vector field on G. Deduce that Lie groups have a trivial tangent bundle. Hint: If $m : G \times G \to G$ is multiplication, show that the following composition is V:

$$G \to T(G) \times T(G) \to T(G)$$

where the first map is inclusion $g \mapsto ((g, 0), (1, v))$ (i.e. zero section in the first coordinate and constant $v \in T_1(G)$ in the second), and the second map is derivative dm of m.

Morse functions

3. Analyze the behavior of the given function $f : \mathbb{R}^2 \to \mathbb{R}$ at the origin. Is the critical point nondegenerate? If it is, determine the index.

   (a) $f(x, y) = x^2 + 4y^3$

   (b) $f(x, y) = x^2 - 2xy + y^2$

   (c) $f(x, y) = x^2 + y^4$

   (d) $f(x, y) = x^2 + 11xy + y^2/2 + x^6$

   (e) $f(x, y) = 10xy + y^2 + 75y^3$

4. Let $S^n = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$ and define $f : S^n \to \mathbb{R}$ by $f(x_1, x_2, \cdots, x_{n+1}) = x_{n+1}$. Show that f is a Morse function, and compute all critical points and their indices.

5. View $\mathbb{R}P^2$ as the quotient of the antipodal map $a : S^2 \to S^2$, $a(x, y, z) = (-x, -y, -z)$, where $S^2 \subset \mathbb{R}^3$ is the unit sphere as usual. Show that the map $f : \mathbb{R}P^2 \to \mathbb{R}$ defined by $f([x, y, z]) = x^2 + 2y^2 + 3z^2$ is a Morse function and compute all critical points and their indices.

6. Let $U \subseteq \mathbb{R}^n$ be open. Show that a smooth function $f : U \to \mathbb{R}$ is Morse iff
$$\det(H)^2 + \sum_i \left(\frac{\partial f}{\partial x_i}\right)^2 > 0$$
where $H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is the Hessian.

7. Prove the stability of Morse functions: Let X be a compact manifold and $F : X \times P \to \mathbb{R}$ smooth for a manifold P. If F_{p_0} is Morse for $p_0 \in P$, show that F_p is Morse for all p in a neighborhood of p_0. Hint: Use Problem 6.

8. Let X be a compact manifold. Show that there is a Morse function $f : X \to \mathbb{R}$ that has distinct values at distinct critical points of X. Hint: Start with a Morse function $g : X \to \mathbb{R}$ with critical points $x_1, \cdots, x_k$, fix mesa functions $\rho_i : X \to [0, 1]$ that are 1 in a small neighborhood of x_i and 0 outside a slightly larger neighborhood. Then let $f = g + \sum a_i \rho_i$ for small and generic a_i.