Forms

Miscellaneous

1. Let M be any n-manifold and $TM \to M$ its tangent bundle. Show that the total space is always orientable as a $2n$-manifold (but not always as a vector bundle). Hint: Show that the transition maps between standard charts are orientation preserving.

2. Recall that $\mathbb{R}P^3$ is an orientable manifold. Let $f_n : \mathbb{R}P^3 \to \mathbb{R}P^3$ be given by
$$[X_0 : X_1 : X_2 : X_3] \mapsto [X_0^n : X_1^n : X_2^n : X_3^n]$$
for $n \geqslant 1$. Compute the degree $deg(f_n) \in \mathbb{Z}$. Optional: If you are comfortable with complex manifolds and complex derivatives you could also do it for $\mathbb{C}P^3$ (or $\mathbb{C}P^k$). (All complex manifolds are orientable.)

3. Let V be the vector field in the plane $\mathbb{R}^2 = \mathbb{C}$ defined by $V(z) = z^n$, $n \geqslant 1$. Thus 0 is the only singular point of V. Compute the index $ind_0 V$.

Exterior algebra

4. An alternating 2-tensor $T \in \Lambda^2(V^*)$ is *decomposable* if it can be written as $T = A \wedge B$ for 1-tensors A, B. Show that every alternating 2-tensor in $\Lambda^2((\mathbb{R}^3)^*)$ is decomposable.

5. Show that the 2-tensor $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ in $\Lambda^2((\mathbb{R}^4)^*)$ is not decomposable. Hint: Consider $\omega \wedge \omega$.

6. For $v \in V$ and $\omega \in \Lambda^k(V^*)$ define the *contraction* $v \lrcorner \omega \in \Lambda^{k-1}(V^*)$ (also called *interior product*) by
$$v \lrcorner \omega(v_1, v_2, \cdots, v_{k-1}) = \omega(v, v_1, v_2, \cdots, v_{k-1})$$
   (a) Show that $v \lrcorner (w \lrcorner \omega) = -w \lrcorner (v \lrcorner \omega)$.

   (b) Suppose $v_1, \cdots, v_n$ is a basis of V and $\phi_1, \cdots, \phi_n$ is the dual basis of V^*. Show that
$$v_i \lrcorner (\phi_{i_1} \wedge \phi_{i_2} \wedge \cdots \wedge \phi_{i_k})$$
   equals
$$(-1)^{j-1}\phi_{i_1} \wedge \cdots \wedge \widehat{\phi_{i_j}} \wedge \cdots \wedge \phi_{i_k}$$

if i equals some i_j and otherwise it equals 0. (The hat means omit that term.)

(c) If $\omega_1 \in \Lambda^k(V^*)$ and $\omega_2 \in \Lambda^l(V^*)$ prove the Leibnitz rule:

$$v \lrcorner (\omega_1 \wedge \omega_2) = (v \lrcorner \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (v \lrcorner \omega_2)$$

Hint: Use (b) and linearity of everything.

The significance here is that if ω is a k-form on a manifold X and V is a vector field on X, then the pointwise contraction $V \lrcorner \omega$ is a $(k-1)$-form and this operation will satisfy the Leibnitz rule.

Forms

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = xy - y^2$. Compute the pullback $f^*(\omega)$ of the 1-form $\omega = t\ dt$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (xy^2 + x, y^3 - xy)$. Compute $f^*(\omega)$ and $f^*(\eta)$ where $\omega = x\ dy + y\ dx$ and $\eta = dx\ \wedge dy$.

9. Let X be an oriented n-manifold and $-X$ the same manifold with opposite orientation. Let ω be a compactly supported n-form on X. Show that
$$\int_{-X} \omega = -\int_X \omega$$

10. Let $c : [a, b] \to X$ be a smooth curve and let $f : [c, d] \to [a, b]$ be a smooth map with $f(c) = a$ and $f(d) = b$. If ω is any 1-form on X, show that
$$\int_a^b c^*(\omega) = \int_c^d (cf)^*(\omega)$$
where by $\int_a^b$ we denote integral over $[a, b]$ with the usual orientation from a to b.

11. Let ω be a 1-form on a connected manifold X. Suppose that for every smooth map $f : S^1 \to X$ the integral
$$\int_{S^1} f^*(\omega) = 0$$
Show that there exists a smooth function $g : X \to \mathbb{R}$ such that $\omega = dg$. Hint: Fix a basepoint $x_0 \in X$. Connectivity guarantees that for every $x \in X$ there is a smooth curve $c : [a, b] \to X$ from x_0 to x. Define $g(x) = \int_a^b c^*(\omega)$. First show it doesn't depend on the choice of c.

12. Let S be a surface in $\mathbb{R}^3$ and $\vec{n} = (n_1, n_2, n_3)$ a unit norm vector field along S perpendicular to the tangent space of S at every point. Show that

$$\omega = n_1\ dx_2 \wedge dx_3 + n_2\ dx_3 \wedge dx_1 + n_3\ dx_1 \wedge dx_2$$

is an area form on S, i.e. for any two vectors $v, v' \in T_pS$ the value of $\omega(v, v')$ is (up to sign) the area of the parallelogram in $\mathbb{R}^3$ spanned by v, v'. Hint: you only need to check this on one pair of linearly independent vectors at every point.

13. Using Problem 12 show that the 2-form

$$\omega = x_1\ dx_2 \wedge dx_3 + x_2\ dx_3 \wedge dx_1 + x_3\ dx_1 \wedge dx_2$$

is the area form on S^2.