Math 6210 - Homework 2
Due at 4 PM on 9/15/05

From Rudin: Chapter 1, # 3,5,6

1. Show that every Riemann integrable function is measurable (with respect to the σ-algebra $\mathcal{M}$ on which Lesbesgue measure is defined). Show that the Riemann integral and Lesbesgue integral have the same value. You can just do the case for a 1 variable functions on $[0,1]$. Here is an outline of a proof.

   (a) Show that there exists a measurable function g with $f = g$ a.e.

   (b) Recall that we defined a "measure" for any subset of $\mathbb{R}^n$ but this "measure" was only countably additive on a certain σ-algebra $\mathcal{M}$. Show that any set with measure zero is in $\mathcal{M}$.

   (c) Let f_0 and f_1 be functions such that $f_0 = f_1$ a.e. Show that if f_0 is measurable then f_1 is measurable.

   (d) Conclude that f is measurable.

2. For this problem we will define and study a notion of convergence of positive measures on $(\mathbb{R}^n, \mathcal{B})$ (recall that $\mathcal{B}$ is the σ-algebra of Borel sets). Let μ_i and μ be positive measures. Then $\mu_i \to \mu$ if for every continuous, compactly supported function

$$f : \mathbb{R}^n \longrightarrow [0, \infty]$$

   we have

$$\int_{\mathbb{R}^n} f d\mu_i \to \int_{\mathbb{R}^n} f d\mu.$$

   (a) Let

$$g_i : \mathbb{R}^n \longrightarrow [0, \infty]$$

   be measurable functions. As in class, for each g_i define a measure ϕ_i by the formula

$$\phi_i(E) = \int_E g_i dm$$

   where m is Lesbesgue measure. Let g also be a non-negative measurable function and assume that

$$\lim_{i \to \infty} \int_{\mathbb{R}^n} |g - g_i| dm = 0.$$

   Let ϕ be the measuring corresponding to g. Show that $\phi_i \to \phi$.

   (b) Let μ_k be the counting measures defined in class. Show that $\mu_k \to m$. (Here is the definition of μ_k. For each non-negative integer k define

$$\frac{1}{2^k}\mathbb{Z}^n = \left\{ \vec{x} \in \mathbb{R}^n | 2^k \vec{x} \in \mathbb{Z}^n \right\}.$$

   Then the measure μ_k is defined by the formula

$$\mu_k(E) = \sum_{x \in \frac{1}{2^k}\mathbb{Z}^n} \frac{1}{2^k} \mu_x(E)$$

where

$$\mu_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is the atomic measure with support at x.)