Math 6510 - Homework 6
Due in class on 11/5/13

1. Let M_0, M_1 and N be differentiable manifolds and $f : M_0 \times M_1 \to N$ a smooth map. Then the map $F : M_0 \times TM_1 \to TN$ defined by $F(x_0, v) = (f_*(x_0, x_1))(0, v)$ is smooth where $v \in T_{x_1}M_1$. (You don't need to prove this but you should make sure that you know why its true!).

   Let G be a Lie group and define $f : G \times G \to G$ by $f(a, b) = ab$. Then use the above fact to show that a left-invariant vector field is smooth by showing that F restricted to $G \times \{v\}$ is an embedding where $v \in T_{id}G$.

2. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a one-dimensional sub-algebra. Show that for any left-invariant vector field $X \in \mathfrak{h}$ there is a flow ϕ_t defined for all $t \in \mathbb{R}$ with $\phi_t \in H \subset G$ where H is the Lie subgroup of G with Lie algebra $\mathfrak{h}$. Show that the map $t \mapsto \phi_t$ is an onto homomorphism from the additive group $\mathbb{R}$ to H.

3. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $X \in \mathfrak{g}$ be a left-invariant vector field and $\phi_t \in G$ the associated flow. Show that $ad_g X = X$ if and only if g commutes with ϕ_t. Conclude that ad_g is the identity on $\mathfrak{g}$ if and only if g is in the center of G.

4. Define a map from $\mathfrak{g}$ to G as follows. For $X \in \mathfrak{g}$ let ϕ_t^X be the associated flow. Define $\exp(X) = \phi_1^X$. Note that $\mathfrak{g}$ is a vector space so $T_0\mathfrak{g} = \mathfrak{g}$ and $\exp_*(0) : \mathfrak{g} \to T_{id}G = \mathfrak{g}$. Show that $\exp_*(0) = id$.

5. Let $G = GL(n)$ and recall that $\mathfrak{g} = M(n)$, the space of $n \times n$ matrices. Show that

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$