Final

Due at 4 PM on 12/19/14

Do as many problems as you can in 1 1/2 hours.

1. Let
$$M = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \cdot x = y \cdot y = 1, x \cdot y = 0\}$$
where $x \cdot y$ is the usual dot product on vectors in $\mathbb{R}^3$. Show that M is a submanifold of $\mathbb{R}^3 \times \mathbb{R}^3$. What is the dimension of M?

2. Let M, N and X be differentiable manifolds and $Z \subset X$ a differentiable submanifold. Given $x \in N$ let $\iota_x : M \to M \times N$ be the inclusion map. Let $F : M \times N \to X$ be differentiable and let $f_x = F \circ \iota_x$. If M is compact and f_x is transverse to Z show that there is a neighborhood U of x in N such that if $y \in U$ then f_y is transverse to Z.

3. Let $V(x) = \sum f_i(x) \frac{\partial}{\partial x_i}$ be a smooth vector field on $\mathbb{R}^n$ and define $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ by $\omega(x)(v_1, \ldots, v_{n-1}) = \det(V(x)\ v_1\ \cdots\ v_{n-1})$ where the right hand side is the determinant of the matrix of column vectors $V(x), v_1, \ldots, v_{n-1}$. Show that $d\omega = \sum \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$.

4. Let M be a differentiable manifold. Prove that its tangent bundle TM and and its cotangent bundle are isomorphic as smooth vector bundles.

5. Let W be a vector field on a smooth manifold M and assume that V has a flow on that is defined on all of M and for all time. Let V be another vector field on M such that $V - W$ has compact support. Show that V has a flow on all of M defined for all time.

6. Let $M = \mathbb{R}^2 \backslash \{(-1, 0), (1, 0)\}$. Let $\iota_+ : S^1 \to M$ be a diffeomorphism from S^1 to the circle of radius 1 centered at $(1, 0)$ and similarly define $\iota_- : S^1 \to M$ with $(-1, 0)$ the center of the circle. Define a map $\phi : \Omega^1(M) \to \mathbb{R}^2$ by
$$\phi(\omega) = \left(\int_{S^1} (\iota_+)^* \omega, \int_{S^1} (\iota_-)^* \omega \right).$$
Show that ϕ is surjective and conclude that there is a surjective homomorphism from $H^1(M)$ to $\mathbb{R}^2$. (There is, in fact, an isomorphism but you do not need to show this.)