Math 6510 - Homework 1
Due in class on 9/9/14

1. Find a differentiable atlas for $S^1 \times S^1$,

2. Show that if M and N are differentiable manifolds then $M \times N$ is a differentiable manifold.

3. Find a differentiable atlas of $\mathbb{R}$ such that the identity map is not smooth.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. Show that there exists $c_1, c_2 \in \mathbb{R}^+$ such that $c_1|v| \leq |Tv| \leq c_2|v|$.

5. Let $U, V \subset \mathbb{R}^n$ be open and $f : U \to V$ a smooth homeomorphism such that $f_*(x)$ is an isomorphism for some $x \in U$. Show that f^{-1} is differentiable at $y = f(x)$ and that $(f^{-1})_*(y) = (f_*(x))^{-1}$.

   Here is an outline of how to prove this: To simplify notation let $g = f^{-1}$ and $T = (f_*(x))^{-1}$. We want show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|w| < \delta$ then

$$\frac{|g(y+w) - (g(y) + Tw)|}{|w|} < \epsilon.$$

   If w is small then $y + w \in V$ then there exists $v \in \mathbb{R}^n$ such that $x + v = g(y + w)$.

   (a) Show that

$$-(g(y+w) - (g(y) + Tw)) = T(f(x+v) - (f(x) + f_*(x)v))$$

   and conclude that

$$\frac{|g(y+w) - (g(y) + Tw)|}{|w|} < \|T\| \frac{|f(x+v) - (f(x) + f_*(x)v|)}{|w|}$$

   where $\|T\| = \sup_{|v| \leq 1} |Tv|$.

   (b) To finish the proof we need to show that there exists a neighborhood V_0 of y and a $c > 0$ such that $|w| \geq c|v|$ for all w with $y + w \in V_0$. Assume this is false and show that there is a sequence $\lambda_i w_i$ and $\sigma_i v_i$ with $|w_i| = |v_i| = 1$, $g(y + \lambda_i w_i) = x + \sigma_i v_i$ and $\lambda_i/\sigma_i \to 0$. On the other hand use the fact that f is differentiable at x to show that $\liminf_{i\to\infty} \lambda_i/\sigma_i > 0$ to obtain a contradiction and show that c can be chosen with $2c = \liminf_{i\to\infty} \lambda_i/\sigma_i > 0$.

   (c) Use (a) and (b) to show that if $w \in V_0$ then

$$\frac{|g(y+w) - (g(y) + Tw)|}{|w|} < \frac{\|T\|}{c} \frac{|f(x+v) - (f(v) + f_*(x)v)|}{|v|}$$

   and conclude the proof that $g = f^{-1}$ is differentiable at y.

6. For $i = 0, 1$ let $U_i \subset \mathbb{R}^k$ be open and $\phi_i : U_i \to \mathbb{R}^n$ smooth, injective maps such that $\phi_0(U_0) = \phi_1(U_1)$ and $(\phi_i)_*(x)$ is injective for all $i \in U_i$. Show that there exists a diffeomorphism $f : U_0 \to U_1$ such that $\phi_0 = \phi_1 \circ f$.