Math 6510 - Homework 4

Due in class on 11/11/14

1. The space of linear isomorphisms of $\mathbb{R}^n$ to itself, $Isom(\mathbb{R}^n, \mathbb{R}^n)$ can be identified with an open subspace of $\mathbb{R}^{n^2}$ and is a differentiable manifold. Show that $Isom(\mathbb{R}^n, \mathbb{R}^n)$ has exactly two components and T and S are in the same component then either they are both orientation preserving or they are both orientation reversing. Conclude that if T is orientation preserving it homotopic to the identity and if it is orientation reversing than it is homotopic to the isomorphism that changes the sign of the first coordinate and is the identity on all other coordinates.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that 1 is not an eigenvalue. Define a map $f : S^{n-1} \to S^{n-1}$ by $f(x) = (T - id)x/|(T - id)x|$. Show that $\deg f = 1$ if $T - id$ is orientation preserving and $\deg f = -1$ if $T - id$ is orientation reversing. Note that 0 is a Lefschetz fixed point of T. Show that $L_0(T) = \deg f$.

3. Let $U \subset \mathbb{R}^n$ be a neighborhood of 0 and $f : U \to \mathbb{R}^n$ be a differentiable map with 0 a Lefschetz fixed point. For small t define a map $f_t : S^{n-1} \to S^{n-1}$ by $f_t(x) = (f(tx) - tx)/|f(tx) - tx|$. Show that $\deg f_t = L_0(f)$ for sufficiently small t.

4. Let $B \subset \mathbb{R}^n$ be a closed ball and $B_1, \dots, B_k \subset B$ a collection of open balls whose closure is disjoint and contained in the interior of B. Let $M = B \backslash \{B_1, \dots, B_k\}$ and let $N \cup N_1 \cup \dots \cup N_k = \partial M$ be the $k + 1$ components of ∂M labeled in the obvious way. Give $\mathbb{R}^n$ its standard orientation and orient each N_i such that if $x \in N_i$, $\{v_1, \dots, v_{n-1}\}$ is an oriented basis of $T_x N_i$, and $n \in T_x\mathbb{R}^n$ is a outward pointing normal vector than $\{n, e_1, \dots, e_{n-1}\}$ is an oriented basis of $T_x\mathbb{R}^n$. Let $f : M \to N$ a differentiable map and show that $\deg f|_N = \sum_{i=1}^k \deg f|_{N_i}$.

5. Let V be a finite dimensional vector space and V^* its dual. Define a map $f : V \times V^* \to \mathbb{R}$ by $f(v, \sigma) = \sigma(v)$. Also define maps $f^v : V^* \to \mathbb{R}$ and $f_\sigma : V \to \mathbb{R}$ by $f^v(\sigma) = f_\sigma(v) = \sigma(v) = f(v, \sigma)$. Calculate $(f^v)_*(\sigma) : T_\sigma V \to T_{\sigma(v)}\mathbb{R}$ and $(f_\sigma)_*(v) : T_v V^* \to T_{\sigma(v)}\mathbb{R}$. Use this to calculate $f_*(v, \sigma) : T_{(v,\sigma)} V \times V^* \to T_{f(v,\sigma)}\mathbb{R}$.