Math 6510 - Homework 6
Due in class on 12/9/14

1. Let $T \in \mathcal{T}^k(V)$ and $S \in \mathcal{T}^l(V)$ be tensors on a vector space V with $Alt(S) = 0$. Show that $Alt(T \bigotimes S) = Alt(S \bigotimes T) = 0$. This is in Spivak and I sketched how to prove this in class but it is in good exercise in the linear algebra we are using to try to write down a complete proof on your own.

2. $\Lambda(M)$ is the vector space of smooth vector fields on M. Let $\bar{T} : \Lambda(M) \times \cdots \times \Lambda(M) \to C^\infty(M)$ be a function such that $\bar{T}(V_1, \dots, fV_i + gV_i', \dots, V_k) = f\bar{T}(V_1, \dots, V_i, \dots, V_k) + g\bar{T}(V_1, \dots, V_i', \dots, V_k)$. Show that there exists a $T \in \mathcal{T}^k(M)$ with $T = \bar{T}$. In particular, you should show that if $V_1, \dots, V_k$ and $W_1, \dots, W_k$ are vector fields with $V_i(x) = W_i(x)$ then $\bar{T}(V_1, \dots, V_k)(x) = \bar{T}(W_1, \dots, W_k)(x)$.

   (a) First assume that there is a neighborhood U of x such that $V_i|_U = W_i|_U$. You can use that fact that there is a smooth function $\psi : M \to \mathbb{R}$ with $supp(\psi) \subset U$ and $\psi(x) = 1$.

   (b) Let $E_1, \dots, E_n$ be smooth vector fields that are a basis for each tangent space in U. Assume that $V_i(x) = 0$ and show that $T(V_1, \dots, V_k)(x) = 0$ by using the fact there are smooth functions $f_j : M \to \mathbb{R}$ with $V_i = \sum f_j E_j$ on U.

   (c) Use (a) and (b) to finish the proof.

3. Let β be a k-form on the product manifold $M \times N$. We say that β is tangent to M if $\beta(\dots, V, \dots) = 0$ when V is tangent to N.

   For the product manifold $M \times I$, where I is an interval, let $\pi_I : M \times I \to I$ be the projections to I. If ω is a k-form on $M \times I$ show that there exists a k-form α and a $k-1$-form η on $M \times I$ such that
   $$\omega = \alpha + \pi_I^* dt \wedge \eta$$
   with α and η tangent to M.

4. Let $G = Isom^+(\mathbb{R}^2)$ be the group of orientation preserving isometries of $\mathbb{R}^2$. We will view points in $\mathbb{R}^2$ as column vectors. If $T \in G$ then T is of the form $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ where $A \in SO(2)$. Define vector fields on $\mathbb{R}^2$ by $V_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$, $V_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $V_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $\mathfrak{g}$ be the vector space spanned by V_1, V_2 and V_3. That is $\mathfrak{g}$ is the vector space of vector fields on $\mathbb{R}^2$ that are linear combinations of the V_i.

   (a) Given $T \in G$ and $V \in \mathfrak{g}$ show that $T_* V \in \mathfrak{g}$ and therefore $T_* : \mathfrak{g} \to \mathfrak{g}$ is a linear map.

   (b) Using the V_i as a basis for $\mathfrak{g}$ write T_* as a 3×3-matrix and show that G is a subgroup of $GL(n)$.