Homework 5, Math 5510
Problem #45.7 and some notes

#45.7(a) It is clear that $D(A, B) \geq 0$ and $D(A, B) = 0$. We need to show that $D(A, B) = 0$ if and only if $A = B$ and the triangle inequality. Clear if $A = B$ then $D(A, B) = 0$. Now assume that $A \neq B$. Then there must be some point in one set that is not in the other. We can assume there exists an $a \in A \backslash B$. Since B is closed there is an $\epsilon > 0$ such that $B_d(a, \epsilon)$ is disjoint from B. Therefore if $A \subset U(B, \delta)$ then $\delta > \epsilon$ so $D(A, B) \geq \epsilon > 0$.

Now we prove the triangle inequality. Assume $D(A, B) = d_1$ and $D(B, C) = d_2$. Then for all $\epsilon > 0$ we have $A \subset U(B, d_1 + \epsilon/2)$ and $B \subset U(C, d_2 + \epsilon/2)$ and it follows that $A \subset U(U(B, d_1 + \epsilon/2), d_2 + \epsilon/2) \subset U(C, d_1 + d_2 + \epsilon)$. Similarly $C \subset U(A, d_1 + d_2 + \epsilon)$. Therefore $D(A, C) \leq d_1 + d_2$.

#45.7(b) As the hint suggests, for any Cauchy sequence A_n in $\mathcal{H}$ we can pass to a subsequence (which we don't relabel) with $D(A_n, A_{n+1}) < 1/2^n$. We let A be set defined in the hint. We'll show that $A_n \to \bar{A}$ in $\mathcal{H}$.

First we show that $A_n \subset U(\bar{A}, 1/2^{n-2})$. For each point $z \in A_n$ we will show that there exists and $x \in A$ such that $d(z, x) < 1/2^{n-2}$. To do so we claim that there exists a sequence a_k with $a_k \in A_k$, $d(a_k, a_{k+1}) < 1/2^k$ and $a_n = z$. We construct the sequence inductively. First let $a_n = z$ and assume that $\{a_n, a_{n+1}, \ldots, a_k\}$ has been chosen. Since $D(A_k, A_{k+1}) < 1/2^k$ there exists a point in A_{k+1} that is within $1/2^k$ of x_k. Let x_{k+1} be such a point. We similarly define x_k when $k < n$. A standard argument using the sum of a geometric series shows that if $k < j$ then $d(x_k, x_j) < 1/2^{k-1}$ and therefore x_k is Cauchy. As X is complete, x_k will have a limit x which by definition is in A. Since $d(x_k, x_j) < 1/2^{k-1}$ when $k < j$ by taking the limit as $j \to \infty$ we see that $d(x_k, x) \leq 1/2^{k-1}$ and therefore $d(z = x_n, x) \leq 1/2^{n-1} < 1/2^{n-2}$. It follows that $A_n \subset U(A, 1/2^{n-2}) \subset U(\bar{A}, 1/2^{n-2})$.

Now we show that $\bar{A} \subset U(A_n, 1/2^{n-2})$. Let $x \in A$. By definition there exists a sequence x_n with $x_n \in A_n$, $d(x_n, x_{n+1}) < 1/2^n$ and $x_n \to x$. As in the previous paragraph $d(x, x_n) \leq 1/2^{n-1}$ so we have that for all $x \in A$ there exists a point $x_n \in A_n$ with $d(x, x_n) \leq 1/2^{n-1}$. It follows that $A \subset U(A_n, 1/2^{n-1})$ and $\bar{A} \subset \overline{U(A_n, 1/2^{n-1})}$. Since $\overline{U(A_n, 1/2^{n-1})} \subset U(A_n, 1/2^{n-2})$ we have $\bar{A} \subset U(A_n, 1/2^{n-2})$.

It follows that $d(\bar{A}, A_n) < 1/2^{n-2}$ and $A_n \to \bar{A}$ in $\mathcal{H}$.

#45.7(c) Since X is totally bounded for all $\epsilon > 0$ there exists a finite subset $S \subset X$ such that $X = \underset{x \in S}{\cup} B_d(x, \epsilon)$. Let $\mathcal{A}$ be the collection of all subsets of S. We will show that $\mathcal{H} = \underset{A \in \mathcal{A}}{\cup} B_D(A, \epsilon)$. Let $C \in \mathcal{H}$ and let $A = \{x \in S | B_d(x, \epsilon) \cap C \neq \emptyset\}$. Then $A \in \mathcal{A}$ and $C \subset U(A, \epsilon)$ since for all $z \in S$ there exists and $x \in S$ such that $z \in B_d(x, \epsilon)$ (since the ϵ-balls centered at points in S cover X) and therefore $B_d(x, \epsilon) \cap S \neq \emptyset$ so $x \in A$.

We also have $A \subset U(C, \epsilon)$ since for all $x \in A$, $B_d(x, \epsilon) \cap C \neq \emptyset$ so there exists a point $z \in C$ with $d(x, z) < \epsilon$. Since $C \subset U(A, \epsilon)$ and $A \subset U(C, \epsilon)$ we have $D(A, C) < \epsilon$ and $C \in B_D(A, \epsilon)$. Therefore $\mathcal{H}$ is totally bounded.

#45.7(d) A totally bounded, complete metric space is compact so by (b) and (c), $\mathcal{H}$ is compact.

Iteration on the Hausdorff metric. There is a very interesting construction of fractals using the Hausdorff metric on closed subsets of $\mathbb{R}^n$.

Let $\mathcal{H}$ be the collection of closed, bounded subsets of $\mathbb{R}^n$ with the Hausdorff metric. In homework you have shown that $\mathcal{H}$ is a complete metric space. We will define a contraction from $\mathcal{H}$ to itself. For $i = 1, \ldots, k$ let $f_i \colon \mathbb{R}^n \to \mathbb{R}^n$ be a contraction. That is assume that there exists a $c < 1$ such that $\|f_i(x) - f_i(y)\| \leq c\|x - y\|$ where $\|\|$ is the usually Euclidean metric on $\mathbb{R}^n$. We then define a map $F \colon \mathcal{H} \to \mathcal{H}$ by $F(A) = f_1(A) \cup \cdots \cup f_k(A)$. In the Hausdorff metric we then have $D(F(A), F(B)) \leq cD(A, B)$. (Why?) It then follows from the contraction mapping principle (also a homework problem) that there exists a unique $A \in \mathcal{H}$ such that $F(A) = A$.

For particular choices of the f_i we can get some interesting sets A that are fixed. For example if we take $n = 1$ we can take $f_1(x) = x/3$ and $f_2(x) = (x - 1)/3$. These are both contractions with $c = 1/3$. You should then check that the standard middle thirds Cantor set if fixed by F.

For a more general version of the above example let $f_1(x) = x/k$ and $f_2(x) = (x-1)/k$ where $k \in (0, 1)$. If $k \geq 1/2$ then the fixed set will be the interval $[0, 1]$. If $k < 1/2$ then the fixed set will still be homeomorphic to the standard Cantor set. However the "size" of the Cantor set will vary. To define this precisely we need to discuss Hausdorff measure. You can also construct many well known fractals in the plane and higher dimensions using this construction. There are lots of references for this. The definition of Hausdorff dimension is a little bit complicated but you certainly have the tools to understand it. If anyone is interested I am happy to discuss this outside of class.