Midterm, Math 5510
October 23rd, 2015
Justify all of your work.

1. Let $\mathcal{B} = \{[a,b)|a,b \in \mathbb{R}, a < b\}$ and $\mathcal{B}' = \{[a,b)|a,b \in \mathbb{Q}, a < b\}$ be collections of subsets of $\mathbb{R}$.

   (a) Show that both collections form a basis for a topology.
   Solution: Given $x \in \mathbb{R}$ there exists $a, b \in \mathbb{Q}$ such that $a < x < b$. Then $[a,b)$ contains x and is in both $\mathcal{B}$ and $\mathcal{B}'$.
   Also note that for any B_1 and B_2 in $\mathcal{B}$ that have non-empty intersection, we have $B_1 \cap B_2$. The same statement holds for elements in $\mathcal{B}'$.

   (b) Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Show that a sequence x_n in $\mathbb{R}$ converges to $x \in \mathbb{R}$ in $\mathcal{T}$ if and only x_n converges in the usual topology on $\mathbb{R}$ and $x_n \geq x$ for all but finitely many n.
   Solution: First assume that $x_n \to x$ in $\mathcal{T}$. Then if $x \in (a,b)$ (a basis element for the standard topology) there exists a $c \in \mathbb{R}$ with $a < c < x$ and we have that $x \in [c,b)$ and there exists an N such that if $n > N$ then $x_n \in [c,b) \subset (a,b)$. Therefore $x_n \to x$ in the standard topology.
   To see that $x_n \geq x$ for all but finitely many x we note that for all but finitely many n we have $x_n \in [x,b)$ for any $b > x$.
   Now assume that $x_n \to x$ in the usual topology and that $x_n \geq x$ for all but finitely many n. The if $x \in [a,b)$ we have that $x \in (a-1,b)$ so there exists an N_1 such that if $n > N_1$ then $x_n \in (a-1,b)$. But for all but finitely many n we have $x_n \geq x$ so there exists an N_2 such that if $n > N_2$ then $x_n \geq x$. Let $N = \max\{N_1, N_2\}$. Then if $n > N$, $x_n \in (a-1,b)$ and $x_n \geq x$ so $x_n \in [x,b) \subset [a,b)$ and $x_n \to x$ in $\mathcal{T}$.

   (c) Let $\mathcal{T}'$ be the topology generated by $\mathcal{B}'$. Show that a sequence x_n converges to $x \in \mathbb{R}$ in $\mathcal{T}'$ if and only if x_n converges in the usual topology and either x is irrational or $x_n \geq x$ for all but finitely many n.
   Solution: If $x \in \mathbb{Q}$ then the proof of both implications is exactly the same as in (b). If x is irrational then we need to show that convergence in $\mathcal{T}'$ is equivalent to convergence in the standard topology. If $x_n \to x$ in $\mathcal{T}'$ then the proof that $x_n \to x$ in the standard topology is also the same as in (b). All we are left to show is that convergence in the standard topology implies convergence in $\mathcal{T}'$. Let $x \in [a,b) \in \mathcal{B}'$. Since $a \in \mathbb{Q}$ and x is irrational we have that $x > a$ and $x \in (a,b)$. Since $x_n \in x$ in the standard topology this implies that there exists an $N > 0$ such that if $n > N$ then $x_n \in (a,b) \subset [a,b)$. Therefore $x_n \to x$ in the standard topology.

2. Give an example of two topologies on the same set where one topology is connected and the other is not. You cannot use either the discrete or the indiscrete topology.

Solution: Let $X = \{a, b, c\}$ be a set with two elements. Then $\mathcal{T} = \{\emptyset, \{a\}, X\}$ is a connected topology and $\mathcal{T}' = \{\emptyset, \{a\}, \{b, c\}, X\}$ is disconnected since $\{a\}$, $\{b, c\}$ are a separation.

Another example is $\mathcal{T}$ on $\mathbb{R}$ from Problem 1 which is disconnected ($\bigcup_{x<0} [x, 0)$, $\bigcup_{x>0} [0, x)$ are a separation.) and the standard topology on $\mathbb{R}$ which is connected.

3. (a) Define the closure $\bar{A}$ of a set A.

   Solution: The closure $\bar{A}$ of A is the intersection of all closed sets that contain A.

   (b) Let A and B be subset of topological space X. Prove or disprove (by finding a counterexample) that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

   Solution: The set $\bar{A} \cup \bar{B}$ is closed and contains $A \cup B$ (since the closure of a set contains the set itself and the union of two closed sets is closed). Therefore $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

   Similarly $\overline{A \cup B}$ is a closed set that contains A so $\bar{A} \subset \overline{A \cup B}$ and $\overline{A \cup B}$ contains B and $\bar{B} \subset \overline{A \cup B}$. This implies that $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

   These two inclusions imply that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

4. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$. Show that if Y is compact, then X is compact. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.]

Solution: Let $\mathcal{U}$ be an open cover of X. We claim that for each $y \in Y$ there exists an open set W_y such that $p^{-1}(W_y)$ is covered by finitely many open sets in $\mathcal{U}$. For each $y \in Y$, $p^{-1}(\{y\})$ is compact and non-empty (since p is surjective). Since it is covered by $\mathcal{U}$ there is a finite subcover $U_1, \ldots, U_n$. Let $U = \cup U_i$. Then $X \backslash U$ is closed and $p(X \backslash U)$ is closed in Y, since p is a closed map. Let $W_y = Y \backslash p(X \backslash U)$. Then $p^{-1}(W_y)$ is contained in U so it is covered by finitely many sets in $\mathcal{U}$ so we have proved the claim.

The sets W_y form an open cover of the compact set Y so there exists a finite subcover $W_{y_1}, \ldots, W_{y_k}$. Since each of the $p^{-1}(W_{y_i})$ is covered by finitely many sets in $\mathcal{U}$ and the union of the finitely many $p^{-1}(W_{y_i})$ cover X the cover $\mathcal{U}$ has a finite subcover.