1 Quasiconformal maps

Giving $\mathbb{R}^2$ its usual xy-coordinates we have the identification of $\mathbb{R}^2$ with $\mathbb{C}$ given by $(x, y) \mapsto x + iy$. Let $T\colon \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear map. If we let

$$T_z = \frac{1}{2}(T_x - iT_y) \text{ and } T_{\overline{z}} = \frac{1}{2}(T_x + iT_y)$$

then we can also write $Tz = T_z z + T_{\overline{z}}\overline{z}$. This decomposition is convenient as T is $\mathbb{C}$-linear if and only if $T_{\overline{z}} = 0$ while T is $\mathbb{C}$-anti-linear if and only if $T_z = 0$.

Let $S\colon \mathbb{C} \to \mathbb{C}$ be another $\mathbb{R}$-linear map. For the chain rule it is useful to have formula for $(S \circ T)_z$ and $(S \circ T)_{\overline{z}}$. We compute

$$\begin{aligned} S \circ T(z) &= S(T_z z + T_{\overline{z}}\overline{z}) \\ &= S_z(T_z z + T_{\overline{z}}\overline{z}) + S_{\overline{z}}(\overline{T_z z + T_{\overline{z}}\overline{z}}) \\ &= (S_z T_z + S_{\overline{z}}\bar{T}_{\overline{z}})z + (S_z T_{\overline{z}} + S_{\overline{z}}\bar{T}_z)\overline{z} \end{aligned}$$

and therefore $(S \circ T)_z = S_z T_z + S_{\overline{z}}\bar{T}_{\overline{z}}$ and $(S \circ T)_{\overline{z}} = S_z T_{\overline{z}} + S_{\overline{z}}\bar{T}_z$.

The *Beltrami differential* for T is $\mu_T = T_{\overline{z}}/T_z$. If S is $\mathbb{C}$-linear then we can use the above formula to see that

$$\mu_{S\circ T} = \mu_T \text{ and } \mu_{T\circ S} = \frac{\bar{S}_z}{S_z}\mu_T.$$

In particular, the absolute value $|\mu_T|$ is invariant under both pre- and post-composition by $\mathbb{C}$-linear maps.

The $\mathbb{R}$-linear map takes the unit circle to an ellipse. The Beltrami differential encodes both the ratio of the outradius to the inradius (the *dilatation*) and the directions of maximal and minimal stretch. We leave the following as exercises:

1. $|T_z| + |T_{\overline{z}}|$ is the outradius (amount of maximal stretch); $|T_z| - |T_{\overline{z}}|$ is the inradius (amount of minimal stretch).
2. $\det T = |T_z|^2 - |T_{\overline{z}}|^2$ and therefore T is orientation preserving if $|\mu_T| < 1$, reversing if $|\mu_T| > 1$ and degenerate if $|\mu_T| = 1$.
3. The dilatation of T is $\frac{1+|\mu_T|}{1-|\mu_T|}$.
4. The direction of maximal stretch is $\arg \mu_T$ and the direction of minimal stretch is $\arg \mu_T + \pi/2$.

Let $f\colon \Omega \to \Omega'$ be diffeomorphisms between domains in $\mathbb{C}$. It will be critical later that we recognize the difference between f being differentiable and f having partial derivatives. In particular f is *differentiable* at $z_0 \in \Omega$ if there exists an $\mathbb{R}$-linear map

$$f_*(z_0)\colon \mathbb{C} \to \mathbb{C}$$

such that

$$f(z) - f(z_0) = f_*(z_0)(z - z_0) + O(|z - z_0|).$$

If the partial derivatives exist and are continuous in a neighborhood of z_0 then it is a classical theorem that f is differentiable at z_0 and once we know that f is differentiable it follows that the partial derivatives exist and $f_*(z_0)$ is given by the matrix of partial derivatives. However, it is possible that the partial derivatives exist at z_0 but f is not differentiable.

If f is differentiable at z_0 then the $\mathbb{R}$-linear map can be decomposed into its $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear parts as above. As $f_*(z_0)$ is the matrix of partial derivatives if we let

$$f_z = \frac{1}{2}(f_x - if_y) \text{ and } f_{\overline{z}} = \frac{1}{2}(f_x + if_y)$$

then $f_*(z_0)w = f_z(z_0)w + f_{\overline{z}}(z_0)\bar{w}$. The Beltrami differential, defined above for a single linear map, now becomes a function with

$$\mu_f = f_{\overline{z}}/f_z$$

and the dilatation function

$$K_f = \frac{1 + |\mu_f|}{1 - |\mu_f|}.$$

Then f is a *K-quasiconformal diffeomorphism* if

$$\|K_f\|_\infty = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty} \leq K.$$

The *Jacobian* of f is the function

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2.$$

The following estimate of Grötzsch is fundamental to everything that follows.

Lemma 1.1 *Let*

$$f\colon [0, a] \times [0, 1] \to [0, b] \times [0, 1]$$

be a K-quasiconformal diffeomorphism with $f(0,0) = (0,0)$, $f(a,0) = (b,0)$, $f(0,1) = (0,1)$ and $f(a,1) = (b,1)$. Then $K \geq b/a$ with equality if and only if f is affine.

Proof: The proof has two fundamental ideas. First, the total amount of stretch of the f-image of any horizontal segment will be bounded below as it will start at the vertical line at $x = 0$ and end at $x = b$. Second the pointwise product of the dilatation and the Jacobian is the square of the amount of maximal stretch and therefore bounds above the stretch in any direction.

We apply the fundamental theorem of calculus to f (and view it as a $\mathbb{C}$-valued function) to see that, for each fixed $y \in [0,1]$, we have

$$\int_0^a f_x(x+iy)dx = f(a+iy) - f(0+iy).$$

Observing that $\operatorname{Re} f(a+iy) = b$, $\operatorname{Re} f(0+iy) = 0$ and $f_x = f_z + f_{\overline{z}}$ and taking absolute values we have

$$b \leq \int_0^a (|f_z| + |f_{\overline{z}}|)dx.$$

Next we integrate both sides of this inequality with respect to y and get

$$b \leq \int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|)dx.$$

The integrand can be written as

$$\sqrt{\frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}} \sqrt{|f_z|^2 - |f_{\overline{z}}|^2} = \sqrt{K_f}\sqrt{J_f}$$

and we apply the Cauchy-Schwarz inequality to get

$$b^2 \leq \int_0^1 \int_0^a K_f dxdy \int_0^1 \int_0^A J_f dxdy.$$

The integrand in the first integral is the dilatation and is bounded above by K so the integral is bounded by aK, the area of the domain rectangle times K. The second integrand is the Jacobian so the integral is b, the area of the image rectangle. Therefore we have

$$b^2 \leq (aK)b$$

and rearranging gives the desired lower bound on K.

We now need to check when equality holds. First we observe that

$$b = \int_0^a f_x dx = \int_0^a |f_x| dx$$

if and only if $\operatorname{Im} f_x$ is identically zero and $\operatorname{Re} f_x \geq 0$. Next note that

$$|f_x| = |f_z| + |f_{\overline{z}}|$$

if and only if $\operatorname{Re} f_y$ is identically zero. This implies that derivative is diagonal in the xy-coordinates. For the Cauchy-Schwarz inequality to be an equality then both functions must be constant and therefore both the dilatation and the Jacobian are constant. As the derivative is diagonal, the dilatation is the ratio of the diagonal terms and the Jacobian is the product. If both the ratio and product of two functions is constant the so most be the functions. In particular the derivative is a constant diagonal matrix and f is an affine map as claimed. $\boxed{1.1}$

The assumption that the height of the rectangular is 1 is just to simplify the computation. More generally we have:

Corollary 1.2 *Let R and R' be rectangles of width a and a' and heights b and b'. Let*

$$f\colon R \to R'$$

be a K-quasiconformal diffeomorphism taking horizontal sides to horizontal sides and vertical sides to vertical sides. Then $K \geq \frac{ab'}{a'b}$ with equality if and only if f is affine.

This inequality motivates a more general version of a quasiconformal homeomorphism where we take as an assumption that the above lemma holds. We needs some more definitions to make this precise.

A quadrilateral Q in $\mathbb{C}$ (or more general a Riemann surface) is a Jordan domain with four ordered points z_0, z_1, z_2 and z_3 in ∂Q where the order of the points respects the cyclic order of ∂Q determined by the orientation. We need the following lemma:

Lemma 1.3 *Given any quadrilateral $(Q : z_0, z_1, z_2, z_3)$ there is a biholomorphic map*

$$f\colon Q \to R$$

to a rectangle R taking the z_i to the corners of R and taking the arc of ∂Q between z_0 and z_1 to a horizontal side of R. Furthermore, for any such map the ratio of the height of R to its width is always the same.

Proof: By the Riemann mapping theorem there is a biholomorphic map taking the interior of Q to the upper half plane. By Caratheodory's theorem this map extends to a homeomorphism of Q to the closure of the upper half plane in $\widehat{\mathbb{C}}$. Therefore we can assume that Q is the closure of the upper half plane and the z_i are points in $\mathbb{R} \cup \infty$. Furthermore we can post-compose the Riemann map with a Möbius transformation so that none of the z_i are ∞. In fact we can assume that the z_i are real and $z_0 < z_1 < z_2 < z_3$.

After reducing to this case we construct the desired map via an elliptic integral. Namely let

$$f(z) = \int_i^z \frac{dw}{\sqrt{(w - z_0)(w - z_1)(w - z_2)(w - z_3)}}$$

where the contour integral is over an arc from i to z whose interior is contained in the upper half plane. That this function gives a map to a rectangle is a special case of the Schwarz-Christoffel theorem. For this special case the proof is not difficult.

Let

$$\phi(w) = (w - z_0)(w - z_1)(w - z_2)(w - z_3).$$

This is a holomorphic function with zeros only at the z_i so on any simply connected domain in $\mathbb{C}$ that doesn't contain the z_i we can choose a branch of $\sqrt{\phi}$. On $\widehat{\mathbb{C}}$ the function ϕ is meromorphic with a pole at ∞ of order of 4 and as 4 is even we can take the domain of $\sqrt{\phi}$ to be any simply connected domain in $\widehat{\mathbb{C}}$ that doesn't contain the z_i. The function $\sqrt{\phi}$ will still be meromorphic but the pole at ∞ will be of order 2.

Let Ω be a simply connected domain in $\widehat{\mathbb{C}}$ that contains the upper half plane along with $\mathbb{R} \cup \{\infty\} \backslash \{z_0, z_1, z_2, z_3\}$. Then $\sqrt{\phi}$ is a meromorphic function on Ω with a single pole of order 2 at ∞ and no zeros. Therefore the reciprocal $1/\sqrt{\phi}$ will be a holomorphic function on Ω with a zero of order 2 at ∞ and no other zeros. Then $dw/\sqrt{\phi(w)}$ will be a holomorphic 1-form on Ω and since the zero at ∞ of $1/\sqrt{\phi}$ has order 2 (and there are no other zeros) the 1-form $dw/\sqrt{\phi(w)}$ is non-zero on all of Ω.

The function ϕ is $\mathbb{R}$-valued on $\mathbb{R}\backslash\{z_0, z_1, z_2, z_3\}$ and therefore the argument of $1/\sqrt{\phi}$ is constant on each interval (z_i, z_{i+1}) in $\mathbb{R} \cup \{\infty\}$ (where the i are taken mod 4) and are a multiple of π if ϕ is positive while they are of the form $n\pi + \pi/2$ if ϕ is negative. For any holomorphic function of constant argument θ its contour integral on the $\mathbb{R}$-axis will be a line of slope θ. For the elliptic function f we have that the f-image of (z_0, z_1) and (z_2, z_3) are vertical segments while the f-image of (z_1, z_2) and (z_3, z_0) (by which we mean the segment in $\mathbb{R} \cup \infty$ from z_3 to z_0 that contains ∞) are horizontal.

At z_i, the function $1/\sqrt{\phi}$ is asymptotic to $1/\sqrt{z - z_i}$ so f extends continuously to the z_i and by the previous paragraph $f(\mathbb{R} \cup \{\infty\})$ will be a rectangle. By the maximum principle the image of the upper half plane will be the interior of this rectangle. $\boxed{1.3}$

There is then a unique $a > 0$ and a homeomorphism

$$f \colon Q \to [0, a] \times [0, 1] \subset \mathbb{C}$$

that is holomorphic on the interior of Q with $f(z_0) = 0, f(z_1) = a, f(z_2) = a + i$ and $f(z_3) = i$. We define the *modulus* of Q as $m(Q) = a$. If we let Q^* be the same Jordan domain with the four points permuted by one we see that $m(Q^*) = 1/a$.

We now give the *geometric* definition of a K-quasiconformal homeomorphism: a homeomorphism $f \colon \Omega \to \Omega'$ is K*-quasiconformal* if for all quadrilaterals $Q \subset \Omega$ we have

$$m(f(Q)) \leq Km(Q).$$

Observe that $f(Q)^* = f(Q^*)$ and it follows from our above observation on the modulus of Q^* that this implies that

$$m(f(Q)) \geq K^{-1}m(Q)$$

for all $Q \subset \Omega$. From this it follows f is K-quasiconformal if and only if f^{-1} is K-quasiconformal.

We have the following corollary of Lemma 1.1:

Corollary 1.4 *A K-quasiconformal diffeomorphism is a (geometric) K-quasiconformal homeomorphism.*

The converse of this corollary does not hold. Define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(x+iy) = Kx+iy$. This is an affine map with constant dilatation K. Now choose a continuous map $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^2 = g(z^2)$. There are two possible choices for f; we fix one. Then f is differentiable with constant dilatation K everywhere except for $z = 0$ where f is not differentiable. The proof of Lemma 1.1 works as above for functions with isolated singularities, at least for the inequality. (A little extra thought is required when the minimal dilations is achieved.)

The following important fact follows directly from the definition.

Lemma 1.5 *If f and g are K_1 and K_2-quasiconformal homeomorphisms then $f \circ g$ is a K_1K_2-quasiconformal homeomorphism.*

One might hope that one could weaken the assumption on differentiability to differentiability almost everywhere (a.e.) with a bound on the dilatation. However, this assumption is too weak, as the next example shows. Let $E \subset [0, 1]$ be a set of Lebesgue measure zero and let σ be a positive measure on E without atoms. Such a set and measure can be constructed by taking a homeomorphism from the usual Cantor set to a Cantor set of positive Lebesgue measure and pulling back the measure. We define a function $h\colon [0, 1] \to \mathbb{R}$ by

$$h(x) = \int_0^x d\sigma.$$

Note that h is constant a.e. but not constant. We now define a function

$$f\colon [0, 1] \times [0, 1] \to [0, 1 + \sigma(E)] \times [0, 1]$$

by

$$f(x + iy) = x + h(x) + iy.$$

The map f is a homeomorphism and it takes corners to corners. We also observe that f is differentiable a.e. with $f_{\overline{z}} = 0$ and therefore $\mu_f = 0$ a.e. and the dilatation is zero a.e. However f is not holomorphic and does not satisfy the conclusion of Lemma 1.1. In fact for any K one can find an interval $[a, b] \subset [0, 1]$ such that

$$\frac{f(b) - f(a)}{b - a} > K$$

and therefore f does not satisfy the geometric definition of a K-quasiconformal homeomorphism for any K. We leave this as an exercise. We therefore need to make stronger assumptions on the derivative to get an analytic definition of quasiconformality.

1.1 Extremal length

Let Γ be a family of paths in the $\mathbb{C}$ (or more generally a Riemann surface). Given a conformal metric ρ we let $L_\Gamma(\rho)$ be the infimum of the ρ-length of the paths in Γ and let $\mathbf{A}(\rho)$ be the area of the ρ metric. We define the *extremal length* by

$$\lambda(\Gamma) = \sup_\rho \frac{L_\Gamma(\rho)^2}{\mathbf{A}(\rho)}.$$

Note that the quotient on the right is scale invariant and it will often be convenient to normalize the metrics. For example if we restrict the metrics to have a fixed length or area the supremum will be the same.

In the definition of extremal length it is sufficient to assume that our metrics are smooth. However, it will be more convenient to allow a more general class of metrics. Given any two conformal metrics their ratio is a non-negative function. Here we will allow any conformal metric such that the area is finite and its ratio with a smooth conformal metric is a measurable function.

We make a few elementary observations.

- If Γ is contained in some measurable set E we can assume that ρ is zero outside of E for this will not change $L_\Gamma(\rho)$ and will only decrease $\mathbf{A}(\rho)$.
- As $\lambda(\Gamma)$ is a supremum, for any conformal metric ρ we have

$$\lambda(\Gamma) \geq \frac{L_\Gamma(\rho)^2}{\mathbf{A}(\rho)}.$$

- If $\Gamma' \subset \Gamma$ then for any ρ, $L_{\Gamma'}(\rho) \geq L_\Gamma(\rho)$ so $\lambda(\Gamma') \geq \lambda(\Gamma)$.
- On the other hand, if every path in Γ contains a path in Γ' then $L_{\Gamma'}(\rho) \leq L_\Gamma(\rho)$ so $\lambda(\Gamma') \leq \lambda(\Gamma)$.

Lemma 1.6 *Let R be a rectangle of height a and width b and Γ_R the paths in R connecting the vertical sides. The $\lambda(\Gamma_R) = b/a = m(R)$.*

More generally for a quadrilateral $(Q : z_0, z_1, z_2, z_3)$ let Γ_Q be the paths in Q connecting the z_0z_1-side to the z_2z_3-side. Then $\lambda(\Gamma_Q) = m(Q)$.

Proof: The proof is very similar to the proof of Lemma 1.1. As extremal length is a sup we get an lower bound by choosing any conformal metric on R. If we take the Euclidean metric we get $\lambda(\Gamma) \geq b/a$. For the lower bound take any conformal metric ρ. Then for any fixed y value we have

$$\int_0^b \rho(x+iy)dx \geq L_\Gamma(\rho)$$

and after integrating with respect to y this becomes

$$\int_R \rho dx dx \geq a L_\Gamma(\rho).$$

Viewing the integrand as the product $\rho \cdot 1$ and applying the Cauchy-Schwarz inequality we have

$$ab\,\mathbf{A}(\rho) = \int_R 1 dx dy \int_R \rho^2 dx dy \geq \left(\int_R \rho \cdot 1 dx dy\right)^2.$$

Combining the two inequalities and rearranging this becomes

$$\frac{L_\Gamma(\rho)^2}{\mathbf{A}(\rho)} \leq b/a.$$

As ρ was arbitrary this implies $\lambda(\Gamma) \leq b/a$ and the proof is complete. $\boxed{1.6}$

A metric ρ is *extremal* for Γ if

$$\lambda(\Gamma) = \frac{L_\Gamma(\rho)^2}{\mathbf{A}(\rho)}.$$

Lemma 1.6 can be rephrased as stating that the Euclidean metric on a rectangle R is the extremal metric for Γ_R.

Lemma 1.7 *Let R be a rectangle divided into two quadrilaterals Q_1 and Q_2 by an arc connecting the top to the bottom. If $m(R) = m(Q_1) + m(Q_2)$ then the dividing arc is vertical (and the Q_i are rectangles).*

Proof: We can assume that R has height one and width $m(R)$. Let f_i be the uniformizing conformal map that takes Q_i to the rectangle of height 1 and width $m(Q_i)$. We then define the conformal metric ρ on R to be $|(f_i)_z|$ on Q_i. Any horizontal segment of R has a subsegment connecting the vertical sides of Q_1 and a disjoint subsegment connecting the vertical sides of Q_2. The f_1-image of the first subsegment and the f_2-image of the second subsegment will have length at least $m(Q_1)$ and $m(Q_2)$, respectively, and therefore the ρ-length of the subsegments will be at least $m(Q_1)$ and $m(Q_2)$. It follows that

$$\int_0^{m(R)} \rho(x+iy)dx \geq m(Q_1) + m(Q_2) = m(R)$$

for each fixed $y \in [0,1]$. Subtracting 1 from the integrand and integrating with respect to y this becomes

$$\int_R (\rho - 1) dx dy \geq 0.$$

We also observe that
$$\mathbf{A}(\rho) = m(Q_1) + m(Q_2) = m(R)$$
and therefore
$$\int_R (\rho^2 - 1) dxdy = 0.$$
On the other hand $(\rho - 1)^2$ is non-negative so its integral over R is also non-negative but
$$\begin{aligned}\int_R (\rho - 1)^2 dxdy &= \int_R \left((\rho^2 - 1) - 2(\rho - 1)\right) dxdy \\ &= -2 \int_R (\rho - 1) dxdy \\ &\leq 0\end{aligned}$$
and therefore
$$\int_R (\rho - 1)^2 dxdy = 0$$
and $(\rho - 1) = 0$ a.e. This implies that $|(f_i)_z|$ is constantly equal to 1 and as a holomorphic function of constant modulus is also constant we have that $(f_i)_z$ is constant. Therefore the Q_i are rectangles and the lemma follows. $\boxed{1.7}$

Theorem 1.8 *A 1-quasiconformal homeomorphsim is conformal.*

Proof: Being conformal is local property so it is enough to check that the theorem holds when the domain is a rectangle R. Furthermore a map is conformal if and only if its post-composition with a conformal map is conformal so we can post-compose with a conformal map so that the image is also a rectangle (taking corners to corners). As the map is 1-quasiconformal we can also assume that the image is the same rectangle R.

Let $f\colon R \to R$ be a 1-quasiconformal homeomorphism. Divide R into two rectangles R_1 and R_2 by a vertical segment connecting the top to the bottom. Then $m(R) = m(R_1) + m(R_2)$ and as f is 1-quasiconformal we also have $m(f(R) = R) = m(f(R_1)) + m(f(R_2))$. Therefore, by Lemma 1.7, both $f(R_1)$ and $f(R_2)$ are rectangles and, in fact, as the moduli don't change we must have $f(R_1) = R_1$ and $f(R_2) = R_2$. This implies that every vertical segment is mapped to itself. A similar argument shows that every horizontal segment is also mapped to itself. Any such map must be the identity and hence conformal. $\boxed{1.8}$

For $0 \leq r_1 < r_2 \leq \infty$ let

$$A_{r_1,r_2} = \{z \in \mathbb{C} | r_1 < |z| < r_2\}.$$

Every annulus in $\mathbb{C}$ (or a Riemann surface) is conformally equivalent an A_{r_1,r_2}. This is a consequence of the uniformization theorem.[1] We define the modulus of A_{r_1,r_2} by

$$m(A_{r_1,r_2}) = 2\pi \left(\log \frac{r_2}{r_1}\right)^{-1}.$$

In general, if A is conformally equivalent to A_{r_1,r_2} we define $m(A) = m(A_{r_1,r_2})$. Note that A_{r_1,r_2} is conformally equivalent to A_{s_1,s_2} if and only if $r_2/r_1 = s_2/s_1$ so that while A is not conformally equivalent to a unique annulus of the form A_{r_1,r_2} the modulus is well defined.

As with the quadrilateral we can reinterpret the modulus as an extremal length problem. Given an annulus $A \subset \mathbb{C}$ let Γ_A be the collection of closed curves in A that have winding number 1 around every point in the bounded component of the complement of A.

Lemma 1.9

$$\lambda(\Gamma_A) = m(A)$$

Proof: We can assume that $A = A_{1,s}$. The metric $\rho_0(z) = 1/|z|$, restricted to A, gives the lower bound $\lambda(\Gamma_A) \geq m(A)$. For the upper bound take any metric ρ with support on A. The upper bound then follows almost exactly as in the proof of Lemma 1.6 (which is very similar to Lemma 1.1). In particular, the length of each circle center at 0 of radius $1 < r < s$ must have length $\geq L_{\Gamma_A}(\rho)$ and therefore

$$\int_0^{2\pi} r\rho(re^{i\theta})d\theta \geq L_{\Gamma_A}(\rho)$$

and

$$\int_A \rho d\theta dr = \int_1^s 1/r \left(\int_0^{2\pi} r\rho(re^{i\theta})d\theta\right) dr \geq L_{\Gamma_A}(\rho) \log s.$$

Applying Cauchy-Schwarz to the functions $1/\sqrt{r} \cdot \sqrt{r}\rho$ we have

$$2\pi(\log s)\,\mathbf{A}(\rho) = \int_A 1/r d\theta dr \int_A \rho r d\theta dr \geq \left(\int_A \rho d\theta dr\right)^2 \geq \left(L_{\Gamma_A}(\rho)\log s\right)^2.$$

We rearrange the inequality to get the desired lower bound. $\boxed{1.9}$

[1] If $A \subset \mathbb{C}$ (rather than a general Riemann surface) then it is not hard to see that $\tilde{A} \subset \mathbb{C}$ in which case we just need the Riemann mapping theorem rather than the entire uniformization theorem. In particular we can assume that $0 \in \mathbb{C}$ is contained in the bounded component of the complement of A. The exponential map is a covering map from $\mathbb{C}$ to the punctured plan $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and the pre-image of A in $\mathbb{C}$ is the universal cover $\tilde{A}$.

One advantage of considering annuli rather than quadrilaterals is that for annuli we don't need to worry about boundary behavior. In particular we do not require that the closure of annulus in $\mathbb{C}$ is a closed annulus with boundary. However, it will be convenient at times to restrict to such annuli. In particular, A is a *good* annulus if it is the interior of a compact annulus with boundary. The advantage of good annuli is that they can be subdivided into quadrilaterals. In particular, if we take two disjoint embedded arcs that connect the components of ∂A then the annulus becomes the union of two quadrilaterals. We can use the moduli of the quadrilaterals to bound from below the modulus of the annulus and we will use this to control the modulus of good annuli under a K-quasiconformal homeomorphism.

We first need a lemma about extremal length.

Lemma 1.10 *Let Γ_1 and Γ_2 be two collections of paths and let $\Gamma_1+\Gamma_2$ be paths that are the union of a path in Γ_1 and a path in Γ_2. Then*

$$\lambda(\Gamma_1+\Gamma_2) \geq \lambda(\Gamma_1)+\lambda(\Gamma_2).$$

Proof: Fix conformal metrics ρ_1 and ρ_2 normalized such that

$$L_{\Gamma_i}(\rho_i) = \mathbf{A}(\rho_i)$$

and let $\rho = \max\{\rho_1, \rho_2\}$. Then

$$L_{\Gamma_1+\Gamma_2}(\rho) \geq L_{\Gamma_1}(\rho_1) + L_{\Gamma_2}(\rho_2)$$

and

$$\mathbf{A}(\rho) \leq \mathbf{A}(\rho_1) + \mathbf{A}(\rho_2).$$

Therefore, for any choice of ρ_1 and ρ_2, we have

$$\begin{aligned} \frac{L_{\Gamma_1+\Gamma_2}(\rho)^2}{\mathbf{A}(\rho)} &\geq \frac{(L_{\Gamma_1}(\rho_1)+L_{\Gamma_2}(\rho_2))^2}{\mathbf{A}(\rho_1)+\mathbf{A}(\rho_2)} \\ &= \frac{(L_{\Gamma_1}(\rho_1)+L_{\Gamma_2}(\rho_2))^2}{L_{\Gamma_1}(\rho_1)+L_{\Gamma_2}(\rho_2)} \\ &= L_{\Gamma_1}(\rho_1)+L_{\Gamma_2}(\rho_2) \end{aligned}$$

where in the second line we are using our normalization for the conformal metrics ρ_1 and ρ_2. Given our normalization we also have

$$\lambda(\Gamma_i) = \sup_{L_{\Gamma_i}(\rho_i)=\mathbf{A}(\rho_i)} L_{\Gamma_i}(\rho)$$

so we if let ρ_1 and ρ_2 vary over all conformal metrics (with the given normalization) we get

$$\lambda(\Gamma_1+\Gamma_2) \geq \lambda(\Gamma_1)+\lambda(\Gamma_2).$$

$\boxed{\text{1.10}}$

Lemma 1.11 *Let $f\colon \Omega \to \Omega'$ be a K-quasiconformal homeomorphism. Let $A \subset \Omega$ be "good" annulus. Then*

$$\frac{1}{K} m(A) \leq m(f(A)) \leq K m(A).$$

Proof: A is the conformal image of an annulus $A_{1,r}$. If we take two radial arcs in $A_{1,r}$ then the annulus is divided into two quadrilaterals Q_1 and Q_2 and we check to see that $m(A) = m(Q_1) + m(Q_2)$. We abuse notation and let Q_1 and Q_2 be the image of these quadrilaterals in A. (We are using that A is a good annulus in our claim that the Q_i are quadrilaterals.) By Lemma 1.10

$$\lambda(\Gamma_{f(Q_1)}) + \lambda(\Gamma_{f(Q_2)}) \leq \lambda(\Gamma_{f(Q_1)} + \Gamma_{f(Q_2)}).$$

We also observe that every path in $\Gamma_{f(A)}$ contains a path in $\Gamma_{f(Q_1)} + \Gamma_{f(Q_2)}$ so

$$\lambda(\Gamma_{f(Q_1)} + \Gamma_{f(Q_2)}) \leq \lambda(\Gamma_{f(A)}).$$

As f is K-quasiconformal we have

$$\frac{1}{K} m(Q_i) \leq m(f(Q_i))$$

and using that $m(A) = m(Q_1) + m(Q_2)$ we have

$$\frac{1}{K} m(A) \leq m(f(Q_1)) + m(f(Q_2)).$$

Using Lemmas 1.6 and 1.9 we can replace extremal lengths with moduli and combine all the inequalities to get

$$\frac{1}{K} m(A) \leq m(f(A)).$$

Applying the above inequality to the inverse map f^{-1} we get the claimed upper bound on $m(f(A))$. $\boxed{1.11}$

1.2 Compactness

Quasiconformality is not metric property. However, as is the case with conformal maps, it will often be useful to use the hyperbolic metric to study them. In particular, we'll show that $QC(\mathbb{H}^2; K)$ the family of K-quasiconformal homeomorphism of $\mathbb{H}^2$ are equicontinuous and this, with some additional restrictions, will allows us to establish a compactness theorem. While the final result could be stated without any reference to the hyperbolic metric it will be important in the proof.

The key point is that two points that are close together in the hyperbolic metric on Δ are separated from the boundary by an annulus of small modulus. However, if the two points become far apart after applying a K-quasiconformal map then the annulus will have large modulus by the Grötzsch estimate, a contradiction.

Lemma 1.12 *Given $\epsilon > 0$ there exists a $\delta > 0$ such that if $z_0, z_1 \in \Delta$ with $m(A) < \delta$ and A is an annulus separating z_0 and z_1 from $\partial\Delta$ then $d_{\mathbb{H}^2}(z_0, z_1) < \epsilon$.*

Conversely, given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_{\mathbb{H}^2}(z_0, z_1) < \delta$ then there exists an annulus A separating z_0 and z_1 from $\partial\Delta$ with $m(A) < \epsilon$.

Proof: After performing a conformal automorphism of the disk (a hyperbolic isometry) we can assume that $z_0 = 0$ and $z_1 = r$ for some $0 < r < 1$. We get a lower bound on $\lambda(\Gamma_A)$ by taking the euclidean metric on Δ. In particular for the Euclidean metric every curve in Γ_A has length at least $2r$ and the euclidean area of Δ is π so $\lambda(\Gamma_A) \geq 4r^2/\pi$. As $r \to 0$ we have $d_{\mathbb{H}^2}(z_0, z_1) \to 0$ so there exists an δ' such that if $r < \delta'$ then $d_{\mathbb{H}^2}(z_0, z_1) < \epsilon$. Now choose $\delta = \frac{4(\delta')^2}{\pi}$ and we see that if $m(A) < \delta$ then $r < \delta'$ and therefore $d_{\mathbb{H}^2}(z_0, z_1) < \epsilon$.

For the converse take the annulus $A_{2r,1/2}$. As the distance $d_{\mathbb{H}^2}(z_0, z_1)$ limits to zero so does the modulus of $A_{2r,1/2}$. $\boxed{1.12}$

Proposition 1.13 *The family $QC(\mathbb{H}^2; K)$ is equicontinuous.*

Proof: Fix $\epsilon > 0$. By Lemma 1.12 there exists a $\delta' > 0$ such that for any annulus in Δ of modulus $< \delta$ any two points in the bounded component of the complement will have hyperbolic distance bounded by ϵ. On the other hand, again by Lemma 1.12, we can choose a $\delta > 0$ such that for any two points z_0, z_1 with $d_{\mathbb{H}^2}(z_0, z_1) < \delta$ there is a an annulus A separating z_0 and z_1 from $\partial\Delta$ and $m(A) < \delta/K$. Then for any $f \in QC(\mathbb{H}^2; K)$ we have

$$m(f(A)) \leq Km(A) < \delta$$

so $d_{\mathbb{H}^2}(f(z_0), f(z_1)) < \epsilon$, proving equicontinuity. $\boxed{1.13}$

Clearly $QC(\mathbb{H}^2; K)$ is not compact and does not have compact closure in the space of all homeomorphisms as it contains the group of conformal automorphisms (the isometry group) which does not have compact closure. We will need to make some extra restrictions. We will apply (without proof) some facts from coarse geometry.

Corollary 1.14 *A K-quasiconformal homeomorphism of $\mathbb{H}^2$ is a quasi-isometry (with constants depending only on K).*

Proof: By Proposition 1.13 we can fix an $\epsilon > 0$ and $\delta > 0$ such that if $d_{\mathbb{H}^2}(z_0, z_1) < \delta$ then $d_{\mathbb{H}^2}(f(z_0), f(z_1)) < \epsilon$. For two arbitrary points let n be the smallest integer such that $d_{\mathbb{H}^2}(z_0, z_1) \leq n\delta$. Then

$$d_{\mathbb{H}^2}(f(z_0), f(z_1)) \leq n\epsilon \leq \frac{\epsilon}{\delta} d_{\mathbb{H}^2}(z_0, z_1) + \epsilon.$$

As f^{-1} is also a K-quasi-isometry we have a similar lower bound. $\boxed{1.14}$

We will need to some properties of δ-hyperbolic metrics spaces and quasi-isometries. We'll state what we need in terms of $\mathbb{H}^2$ although they hold in much more generality.

- The hyperbolic plane has natural compactification by S^1. We denote the compact space $\bar{\mathbb{H}}^2$. Every isometry of $\mathbb{H}^2$ extends to a homeomorphism of $\bar{\mathbb{H}}^2$.
- Any two distinct points $x, y \in \bar{\mathbb{H}}^2$ are endpoints of a unique geodesic $\overline{xy}$ in $\mathbb{H}^2$.
- Let $f \colon \mathbb{H}^2 \to \mathbb{H}^2$ be a quasi-isometry. Then f extends continuously to a homeomorphism of $\partial\bar{\mathbb{H}}^2$. In particular, if f is a homeomorphism it extends to a homeomorphism of $\bar{\mathbb{H}}^2$. Furthermore, if f_i are quasi-isometries (with uniform constants) that converge to f in the compact-open topology on $\mathbb{H}^2$ then the extensions of f_i converge (pointwise) to the extension of f.
- There exists a $D > 0$, depending only on the quasi-isometry constants, such that $f(\overline{xy})$ is contained in the D-neighborhood of $\overline{f(x)f(y)}$.
- There exists a $\delta > 0$ such that for any geodesic triangle T (with possibly one or more vertices at ∞) there exists a point z that is within δ of all three sides of T. Furthermore any other such point is a uniformly bounded distance from z. We say that z is a *barycenter* of T.
- Let $v_0, v_1, v_2 \in \bar{\mathbb{H}}^2$ be vertices of a triangle T and $f(v_0), f(v_1)$ and $f(v_2)$ vertices of a triangle T'. Let z be a barycenter of T and z' a barycenter of T'. Then the distance between $f(z)$ and z' is uniformly bounded (with constants only depending on the qc-constants for f).

Theorem 1.15 *The space $QC(\mathbb{H}^2, \{x_0, x_1, x_2\}; K)$ of K-quasiconformal homeomorphisms of the $\mathbb{H}^2$ that pointwise fix the distinct points x_0, x_1 and x_2 on $\partial\mathbb{H}^2$ is compact.*

Proof: We first show that $QC(\mathbb{H}^2, \{x_0, x_1, x_2\}; K)$ has compact closure in the space of all continuous maps of $\mathbb{H}^2$ to itself via Arzela-Ascoli. For this we need to show that for some $z_0 \in \mathbb{H}^2$ the set

$$\{f(z_0) | f \in QC(\mathbb{H}^2, \{x_0, x_1, x_2\}; K)\}$$

is bounded. If this set is bounded for one $z_0 \in \mathbb{H}^2$ then it is bounded for all $z \in \mathbb{H}^2$.

Let z_0 be the barycenter of the ideal triangle with vertices x_0, x_1 and x_2. Then $f(z_0)$ will be uniformly close to the barycenter of the ideal triangle with vertices $f(x_0)$, $f(x_1)$ and $f(x_2)$. However, f fixes these three points so $f(z_0)$ is uniformly close to z_0 and the set is bounded.

Now we show that the closure is contained in the set of homeomorphisms. Assume that $f_i \in QC(\mathbb{H}^2, \{x_0, x_1, x_2\}; K)$ converge in the compact-open topology to f. The property of being a quasi-isometry is preserved in limits so f is proper. It is also a local homeomorphism. For this we use that the maps f_i^{-1} are K-quasiconformal and equicontinuous. In particular given $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_{\mathbb{H}^2}(z_0, z_1) < \delta$ then $d_{\mathbb{H}^2}(f_i^{-1}(z_0), f_i^{-1}(z_1)) < \epsilon$. Conversely if $w_0, w_1 \in \Delta$ with $d_{\mathbb{H}^2}(w_0, w_1) > \epsilon$ then $d_{\mathbb{H}^2}(f_i(w_0), f_i(w_1)) > \delta$ and by passing to limits we have that f is injective.

Now we show that the limit is f is K-quasiconformal. Let $(Q : z_0, z_1, z_2, z_3)$ be a quadrilateral in Δ. We need to show that (after possibly passing to a subsequence) $m(f(Q_i)) \to m(f(Q))$ as $i \to \infty$. Let $\phi_i \colon f(Q_i) \to \mathbb{U}$ be the uniformizing maps. Again by Caratheodory's theorem the ϕ_i extend to homeomorphisms from the closures of the domain and range. We can also normalize the maps such that $\phi_i(0) = f_i(z_0)$, $\phi_i(1) = f_i(z_1)$ and $\phi_i(\infty) = f_i(z_2)$. For each i there will be a negative real number x_i such that $\phi_i(x_i) = f(z_3)$. The modulus $m(f(Q_i))$ will be a continuous function of x_i.

The usual normal families theorems from complex analysis imply that the ϕ_i converge to a holomorphic function ϕ. However, we need to know that this limit is the uniformizing map for the limiting quadrilateral $f(Q)$ and we need to understand the boundary behavior. For this let $\psi \colon \mathbb{U} \to Q$ be the uniformizing map chosen such that after extending to the closure $\psi(0) = 0$, $\psi(1) = z_1$ and $\psi(\infty) = z_2$. Let $x = \psi^{-1}(z_3)$. Let $g_i = \phi_i^{-1} \circ f_i \circ \psi$. Then the g_i are K-quasiconformal homeomorphisms of $\mathbb{U}$ to itself that fix 0, 1 and ∞. By the above paragraphs (after possibly passing to a subsequence) the g_i converge in the compact-open topology to a homeomorphism g that is also a quasi-isometry. In general a family of homeomorphisms of $\bar{\mathbb{H}}^2$ that converge in the compact-open topology when restricted to $\mathbb{H}^2$ do not have to converge on the boundary. However, when the homeomorphisms are quasi-isometries (with uniform constants) we do get convergence on the boundary. Taking the limit of the equation $\phi_i \circ g_i = f_i \circ \psi$ we see that ϕ is the uniformizing map for $f(Q)$ and that $\boxed{1.15}$

Theorem 1.16 *Let X be a finite area complete hyperbolic surface and $QC(X; K)$ the space of K-quasiconformal homeomorphisms of X (with the compact-open topology). Then $QC(X; K)$ is compact.*

Proof: First we see that $QC(X; K)$ is an equicontinuous family by lifting the maps to $\tilde{X} = \mathbb{H}^2$ and applying Theorem 1.13. In fact the maps are (uniform) quasi-isometries and therefore for any $\epsilon > 0$ there is an $\epsilon' > 0$ such that if $z \in X^{\geq\epsilon}$ then $f(z) \in X^{\geq\epsilon'}$. As the ϵ'-thick part of X is compact this implies that for all $z \in X$ the set $\{f(z)|f \in QC(X;K)\}$ is bounded. Therefore, by Arzela-Ascoli, $QC(X;K)$ has compact closure in the space of *continuous* maps of X to itself. If the maps converge on X we can lift them to converge on $\mathbb{H}^2$. By Lemma

To see that the limit is a homeomorphism we recall that is the inverse maps are also K-quasiconformal, equicontinuity implies that any two points some fixed distant apart have image that is a definite distance apart and therefore the limiting map will be injective. the property of being a quasi-isometry also passes to limits

1.3 Quasiconformal maps - analytic definition

Let $I \subset \mathbb{R}$ be an interval and $f: I \to R$ a function. Then f is *absolutely continuous* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $(a_1, b_1), \dots, (a_n, b_n)$ are disjoint intervals in I with

$$\sum |b_i - a_i| < \delta$$

then

$$\sum |f(b_i) - f(a_i)| < \epsilon.$$

The key fact is that absolutely continuous functions satisfy the 2nd fundamental theorem of calculus. That is an absolutely continuous function f is differentiable almost everywhere and

$$f(y) - f(x) = \int_x^y f'(t)dt$$

for $x, y \in I$.

Now let Ω be a domain in $\mathbb{C}$ and $f: \Omega \to \mathbb{C}$ a function. Then f is *absolutely continuous on lines* (ACL) if the restriction of the real and imaginary parts of f to almost every horizontal and vertical segment in Ω is absolutely continuous. We can now give an analytic definition of a K-quasiconformal homeomorphism.

Let $f: \Omega \to \Omega'$ be a homeomorphism between domains in $\mathbb{C}$. Then f is K-quasiconformal (analytic definition) if

1. f is ACL (and therefore f_x and f_y are defined a.e).
2. $|f_{\overline{z}}| \leq K|f_z|$ a.e.

The definition is chosen so that there is enough regularity so that Lemma 1.1 holds.

Lemma 1.17 *Let $f: \Omega \to \Omega'$ be a K-quasiconformal homeomorphism.*

$$\int_E (|f_z|^2 - |f_{\overline{z}}|^2)dxdy \leq \mathbf{A}(f(E))$$

Assuming Lemma 1.17 we have:

Lemma 1.18 *Let*

$$f\colon [0,a]\times[0,1]\to[0,a']\times[0,1]$$

be K-quasiconformal (analytic) with $f(0,0)=(0,0)$, $f(a,0)=(a',0)$, $f(0,1)=(0,1)$ and $f(a,1)=(a',1)$. Then $K\geq a'/a$ with equality if and only if f is affine.

Proof: We have exactly the regularity needed to copy the proof of Lemma 1.1. In particular the ACL assumption implies that

$$\int_0^a f_x(x+iy)dx = f(a+iy)-f(0+iy)$$

for almost every $y\in[0,1]$ and along with the fact partial derivatives are defined a.e. this is enough to see that

$$a'\leq\int_0^1\int_0^a(|f_z|+|f_{\overline{z}}|)dxdy.$$

The integrand is only measurable but we can still apply the Cauchy-Schwarz inequality to get

$$(a')^2\leq\int_0^1\int_0^a\frac{1+|\mu_f|}{1-|\mu_f|}dxdy\int_0^1\int_0^a(|f_z|^2-|f_{\overline{z}}|^2)dxdy.$$

The integrand of this first integral is bounded by K a.e. so the integral is bounded by $Ka\cdot 1$. The second integral is bounded by $a'\cdot 1$ by Lemma 1.17. Therefore $(a')^2\leq(Ka)a'$ and the lemma follows.

If $K=a'/a$ then as before we have the matrix of partial derivatives is diagonal and constant except that hear the condition only holds a.e. The ACL condition is enough to show that f is affine. In particular we have $(\operatorname{Re} f)_x=a'/a$, $(\operatorname{Im} f)_y=1$ and $(\operatorname{Re} f)_y=(\operatorname{Im} f)_x=0$ a.e. and since f is ACL we have that for almost every $y_0\in[0,1]$ there exists a $y_1\in[0,1]$ with $f(x,y_0)=\frac{a'}{a}x+y_1$ and for almost every $x_0\in[0,a]$ there exists an $x_1\in[0,a']$ such that $f(x_0,y)=x_1+y$. By Fubini this implies that $f(x,y)=\frac{a'}{a}x+y$ a.e. and as f is continuous this holds everywhere. $\boxed{1.18}$

Lemma 1.19 *A geometric K-quasiconformal map is ACL.*

Proof: The ACL condition is local so we can assume that the domain is rectangle R. Let R_η be the rectangle with the same base but height η. Then the function $A(\eta)=\mathbf{A}(f(R_\eta))$ is an increasing function of η. Monotonic functions are differentiable a.e. We will show that f is absolutely continuous on every horizontal line of height η were $A'(\eta)$ exists. We can assume that $A'(0)$ exists. The general case follows.

Let $s_1, \dots, s_n$ be a disjoint collection of intervals in the base of R of length $b_1, \dots, b_n$. For now assume that each $f(s_i)$ is rectifiable with length b'_i. We'll show this at the end. Let Q_i be the rectangle in R_η with base s_i and height η.

Fix an $\epsilon_0, \epsilon_1 > 0$. Let $z_0, \dots, z_m$ be an increasing sequence of points in s_i with z_0 and z_m the endpoints and

$$\sum |f(z_{j+1}) - f(z_j)| \geq b'_i - \epsilon_0.$$

Let r_j be the vertical segment in R_η above z_j. Choose η sufficiently small such that the diameter of each $f(r_j)$ is $< \epsilon_1$. Any path connecting the vertical sides of $f(Q_i)$ must intersect each $f(r_j)$ and hence have length

$$\geq \sum \left(|f(z_{j+1}) - f(z_j)| - 2\epsilon_1\right).$$

Combining inequalities we get the modulus estimate

$$m(f(Q_i)) \geq \frac{(b'_i - \epsilon_0 - 2m\epsilon_1)^2}{A_i(\eta)}.$$

In particular, by choosing η sufficiently small we can assume that $\epsilon_0 \leq b'_i/4$ and $\epsilon_1 < \epsilon_0/(2m)$ and therefore

$$m(f(Q_i)) \geq \frac{(b'_i)^2}{4A_i(\eta)}.$$

As there are only finitely many intervals s_i if η is sufficiently small then the above inequality holds for all i.

On the other hand, as f is a geometric K-quasiconformal map, we have

$$m(f(Q_i)) \leq Km(Q_i) = Kb_i/\eta$$

and combining and rearranging we have

$$\frac{(b'_i)^2}{b_i} \leq \frac{4KA_i(\eta)}{\eta}.$$

Summing both sides this becomes

$$\sum \frac{(b'_i)^2}{b_i} \leq 4K\frac{A(\eta)}{\eta}.$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} \left(\sum b'_i\right)^2 = \left(\sum \frac{b'_i}{\sqrt{b_i}} \cdot \sqrt{b_i}\right)^2 &\leq \left(\sum \frac{(b'_i)^2}{b_i}\right) \cdot \left(\sum b_i\right) \\ &\leq 4K\frac{A(\eta)}{\eta} \cdot \left(\sum b_i\right) \end{aligned}$$

and taking the limit as $\eta \to 0$ we have

$$\left(\sum b_i'\right)^2 \leq 4KA'(0) \cdot \left(\sum b_i\right).$$

We are left to show that the $f(s_i)$ is rectifiable. We have essentially done all the necessary work. In particular if $f(s_i)$ is not rectifiable then we can replace the constant b_i' with any $N > 0$ and we would have

$$N^2 \leq 4KA'(0) \cdot \left(\sum b_i\right).$$

As the right hand side doesn't depend on N this is a contradiction. $\boxed{\text{1.19}}$

The ACL condition implies that the partial derivatives of f are defined a.e. By itself this is not enough to prove that f is differentiable almost everywhere.

Lemma 1.20 *Let $f \colon \Omega \to \Omega'$ have continuous partial derivatives in a neighborhood of $z_0 \in \Omega$. Then f is differentiable at z_0.*

Proof: We can assume that $z_0 = 0$. We will show that for all $\epsilon > 0$ there exists a $\delta > 0$ such

$$|f(z) - f(0) - xf_x(0) - yf_y(0)| < \epsilon$$

if $|z| < \delta$. We have the equality

$$\begin{aligned} f(z) - f(0) - xf_x(0) - yf_y(0) &= (f(z) - f(x) - yf_y(x)) \\ &\quad +(f(x) - f(0) - xf_x(0)) \\ &\quad +(y(f_y(x) - f_y(0))). \end{aligned}$$

We'll bound each term by $\epsilon|z|/3$. The key is that if we extend the difference quotients

$$f^h(z) = \frac{f(z+h) - f(z)}{h} \text{ and } f^{ik}(z) = \frac{f(z+ik) - f(z)}{k}$$

to be $f_x(z)$ and $f_y(z)$ when $h = 0$ or $k = 0$ then the extended functions are continuous. In particular we can choose a δ such that if $|z| < \delta$, $|h| < \delta$ and $|k| < \delta$ then $|f^h(z)| < \epsilon/3$ and $|f^{ik}(z)| < \epsilon/3$.

Lemma 1.21 *If $f \colon \Omega \to \Omega'$ is a homeomorphism and has partial derivatives a.e. then f is differentiable a.e.*

Proof: By a (clever) application of Egoroff's theorem we can find a measurable set $E \subset \Omega$ such that the measure of $\Omega - E$ is arbitrarily small and the difference quotients

$$\frac{f(z+h)-f(z)}{h} \text{ and } \frac{f(z+ik)-f(z)}{k}$$

converge uniformly in E to the partial derivatives f_x and f_y. (Egoroff's theorem is for sequences which is why it cannot be applied directly. However, we can define a sequence of functions by

$$\sup_{0<|h|<1/n}\left|\frac{f(z+h)-f(z)}{h} - f_x(z)\right|$$

and apply Egoroff's theorem to this sequence.)

As E is measurable by Fubini's theorem the intersection of E with almost every horizontal and vertical line is measurable. Let E_y be the intersection of E with the horizontal line of height y and similarly define E_y. When E_y (or E^x) is measurable by the Lesbegue density theorem almost every point is a point of density. That is

$$\lim_{\epsilon\to 0}\frac{m((x-\epsilon, x+\epsilon)\cap E_y)}{2\epsilon} \to 1$$

for almost every $x \in E_y$. (Here we are parameterizing points in E_y by their x-coordinate.) Let $E' \subset E$ be the $x+iy \in E$ such that x is a point of density of E_y and y is a point of density of E^x. Fubini's theorem implies that E' has full measure in E. We will show that f is differentiable at every point of E'. We can assume that the point we are checking is $0 \in E'$.

We need to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $z = x+iy$ with $|z| < \delta$ then

$$|f(z) - f(0) - xf_x(0) - yf_y(0)| < \epsilon|z|.$$

For this we rewrite the expression as

$$\begin{aligned} f(z) - f(0) - xf_x(0) - yf_y(0) &= (f(z) - f(x) - yf_y(x)) \\ &\quad +(f(x) - f(0) - xf_x(0)) \\ &\quad +(y(f_y(x) - f_y(0))). \end{aligned}$$

and then bound each of the expressions on the right. We first observe that we get the necessary bound if $x \in E_0$ and y is small. For the first two terms we use that the difference quotients converge uniformly to see that we can choose a δ such that if $|z| < \delta$ (and therefore $|x| < \delta$ and $|y| < \delta$) then

$$\left|\frac{f(z)-f(x)}{y} - f_x(x)\right| < \epsilon/3$$

and

$$\left|\frac{f(x)-f(0)}{x} - f_y(0)\right| < \epsilon/3.$$

For the last term we have that, as the difference quotients are continuous, so are their uniform limits f_x and f_y (on E). In particular, after possibly making δ smaller we have

$$|f_y(x) - f_y(0)| < \epsilon/3.$$

These three bounds give the desired estimate. It is important that while we require that $|z| < \delta$ we only need $x \in E_0$ and there is no further restriction on y. We get a similar estimate if $y \in E^0$. In particular if $x_1, x_2 \in E_0$ and $y_1, y_2 \in E^0$ and all four values are sufficiently small then then the bound will hold for the entire boundary of the rectangle $[x_1, x_2] \times [y_1, y_2]$.

We say that a rectangle is *ϵ-good* if for every z in the boundary we have

$$|f(z) - f(0) - xf_x(0) - yf_y(0)| < \epsilon|z|.$$

Then we can restate the above work to say that there exists a $\delta > 0$ such that the rectangle $[x_1, x_2] \times [y_1, y_2]$ is ϵ-good if $x_1, x_2 \in E_0$, $y_1, y_2 \in E^0$ and $|z| < \delta$ for all points in the rectangle.

To bound the expression for a general $z_0 = x_0 + iy_0$ we first find a small ϵ_0-good rectangle that contains z. Here is where we use that 0 is a point of density of both E_0 and E^0.

We assume that x_0 and y_0 are both positive. The general case will follow after obvious modifications. As 0 is a point of density of E_0 for any ϵ_0 we can find a $\delta_0 > 0$ such that if $0 < x \leq \delta_0$ then

$$m((-x, x) \cap E_0) > \frac{2+\epsilon_0}{1+\epsilon_0}.$$

As

$$m\left(\left(-x, \frac{x}{1+\epsilon_0}\right) \cap E_0\right) \leq m\left(\left(-x, \frac{x}{1+\epsilon_0}\right)\right) = \frac{2+\epsilon_0}{1+\epsilon_0}$$

the interval

$$\left(\frac{x}{1+\epsilon_0}, x\right)$$

contains points in E_0. It is essential that this last statement holds for any $x < \delta_0$. Therefore if $x_0 < \frac{\delta_0}{1+\epsilon_0}$ then there is an $x_1 \in \left(\frac{x_0}{1+\epsilon_0}, x_0\right) \cap E_0$ and an $x_2 \in (x_0, x_0(1+\epsilon_0)) \cap E_0$. Similarly, after possibly making δ_0 smaller, if $y_0 < \frac{\delta_0}{1+\epsilon_0}$ we can find $y_1, y_2 \in E^0$ with $\frac{y_0}{1+\epsilon_0} < y_1 < y_0 < y_2 < y_0(1+\epsilon_0)$. After once again possibly decreasing δ_0 we can assume that $[x_1, x_2] \times [y_1, y_2]$ is an ϵ_0-good rectangle.

For any $z^* \in [x_1, x_2] \times [y_1, y_2]$ we have

$$|z^* - z_0| < \epsilon_0$$

and

$$|z^*| < (1 + \epsilon_0)|z_0|.$$

As f is a homeomorphism, and therefore an open map, the maximum principle applies. In particular, on $[x_1, x_2] \times [y_1, y_2]$ the function $f(z)$, and therefore the function,

$$z \mapsto f(z) - f(0) - x_0 f_x(0) - y_0 f_y(0)$$

has maximum modulus for some $z^* = x^* + iy^*$ on the boundary of the rectangle. Using the above bounds and that the rectangle is ϵ_0-good we have

$$\begin{aligned} |f(z_0) - f(0) - x_0 f_x(0) - y_0 f_y(0)| &\leq |f(z^*) - f(0) - x_0 f_x(0) - y_0 f_y(0)| \\ &\leq |f(z^*) - f(0) - x^* f_x(0) - y^* f_y(0)| \\ &\quad + (x^* - x_0) f_x(0) + (y^* - y_0) f_y(0) \\ &\leq \epsilon_0 (1 + \epsilon_0)|z_0| + \epsilon_0 |f_x(0)||z_0| + \epsilon_0 |f_y(0)||z_0|. \end{aligned}$$

$\boxed{1.21}$

The pull back f^*m of the Euclidean measure is a Borel measure on Ω. By the Lesbegue decomposition theorem it is the sum of a measure that is absolutely continuous with respect to Lesbegue measure and a measure that is singular with respect to Lesbegue measure. Where f is differentiable the Radon-Nikodyn derivative of the absolutely continuous measure is the Jacobian J_f.

The main difficulty of the analytic definition is that it is not clear that is invariant under composition with conformal maps.

Given a function $f\colon \Omega \to \mathbb{C}$ on a domain $\mathbb{C}$, then the measurable functions f_x are and f_y are the *distributional derivatives* of f if for compactly supported smooth functions $\phi\colon \Omega \to \mathbb{C}$ we have

$$\int_\Omega f_x \phi \, dx dy = -\int_\Omega f \phi_x \, dx dy$$

and

$$\int_\Omega f_x \phi \, dx dy = -\int_\Omega f \phi_x \, dx dy$$

where ϕ_x and ϕ_y are the usual partial derivatives of ϕ. If f is smooth then integration by parts gives that the usual derivatives and the distributional derivatives agree. In fact the ACL condition is enough to guarantee this.

Let $f, f_x\colon [0,a] \to \mathbb{R}$ be measurable functions with f continuous such that for all smooth $g\colon [0,a] \to \mathbb{R}$ with compact support in $(0,a)$ we have

$$\int_0^a f_x g dx = -\int_0^a f g' dx.$$

Show that

$$\int_0^t f_x dx = f(t) - f(0)$$

for all $t \in [0,a]$ and conclude that f is absolutely continuous. In fact, we can do better. Show that there exist a countable collection of test functions $g_n\colon [0,a] \to \mathbb{R}$ such that if

$$\int_0^a f_x g dx = -\int_0^a f g' dx$$

then

$$\int_0^t f_x dx = f(t) - f(0).$$

(Hint: We only need to show that the above equation holds for a dense set of $t \in [0,a]$. Choose a countable dense subset $\{t_n\}$ and then for each t_n choose enough functions $g_{n,m}$ such that you can show that

$$\int_0^{t_n} f_x dx = f(t_n) - f(0).$$

)

Lemma 1.22 *If $f\colon \Omega \to \mathbb{C}$ is continuous and has locally integrable distributional derivatives then f is ACL.*

Proof: Being ACL is a local condition so we will assume that the domain is a rectangle R of width a and height b. Let R_η be the rectangle with the same lower side as R but height $\eta \le b$. We choose a test function ϕ to be the product of a function that only depends on x and another that only depends on y. That is let $\phi(x+iy) = g(x)h(y)$. Then we have

$$\int_R f_x g(x)h(y)dxdy = -\int_R f g'(x)h(y)dxdy.$$

We now choose bounded functions h_n such that $h_n \to 1$ (pointwise). As g, g' and f are bounded and f_x is integrable the products $f_x g$ and $f g'$ are integrable and we can apply the dominated convergence theorem to take limits and get

$$\int_R f_x g(x)dxdy = -\int_R f g'(x)dxdy$$

for all g.

Now we observe that that the above equality holds if we replace R with R_η so we have that

$$\int_0^\eta \left(\int_0^a f_x g(x) dx \right) dy = \int_0^\eta \left(- \int_0^a f g'(x) dx \right) dy$$

and therefore we have that

$$\int_0^a f_x g(x) dx = - \int_0^a f g'(x) dx$$

for almost every $y \in [0, b]$. However, the subset of $y \in [0, a]$ where the equality holds will depend on g.

We choose a countable family of g_n as given by the exercise. For each n the set where

$$\int_0^a f_x g_n(x) dx = - \int_0^a f g_n'(x) dx$$

has full measure in $[0, b]$ so the intersection of the sets for all the n will still be full measure and we have that

$$f(t) - f(0) = \int_0^t f(x) dx$$

for all $t \in [0, t]$. This gives the ACL condition for horizontal lines. A similar argument gives the statement for vertical lines. $\boxed{1.22}$

Given a curve γ in the $\mathbb{C}$ let $\bar{\gamma}$ be its reflection across the $\mathbb{R}$-axis and let γ^+ be its image under the folding map $x + iy \mapsto x + i|y|$. Given path family the meaning of $\bar{\Gamma}$ and Γ^+ should be clear.

Proposition 1.23 *If* $\Gamma = \bar{\Gamma}$ *then* $\lambda(\Gamma) = 2\lambda(\Gamma^+)$.

Proof: Let ρ be a metric that is symmetric across the $\mathbb{R}$-axis; that is $\rho(z) = \rho(\overline{z})$. For such a metric $L_\Gamma(\rho) = L_{\Gamma^+}(\rho)$ (with the assumption that $\Gamma = \bar{\Gamma}$) and if we let ρ^+ be the restriction of ρ to the closure of the upper half plane we have $L_{\Gamma^+}(\rho) = L_{\Gamma^+}(\rho^+)$ (for any metric ρ) so $L_\Gamma(\rho) = L_{\Gamma^+}(\rho^+)$ (assuming that ρ is symmetric). We also observe that $\mathbf{A}(\rho) = 2\,\mathbf{A}(\rho^+)$ and therefore

$$\frac{L_\Gamma(\rho)^2}{\mathbf{A}(\rho)} = 2 \frac{L_{\Gamma^+}(\rho^+)^2}{\mathbf{A}(\rho^+)}.$$

Let $\lambda^s(\Gamma)$ be the extremal length where the supremum is restricted to symmetric metrics. Then the above equality implies that $\lambda^s(\Gamma) = 2\lambda(\Gamma^+)$ so we only need to show that $\lambda(\Gamma) = \lambda^s(\Gamma)$. As we are restricting the class of metrics we have $\lambda^s(\Gamma) \leq \lambda(\Gamma)$ so we are left to show the reverse inequality.

Given an arbitrary metric ρ (not necessarily symmetric) let $\rho^s(z) = (\rho(z) + \rho(\overline{z}))/2$ be its symmetrization. For any curve $\gamma \in \Gamma$ its ρ^s length will be the average of the ρ-lengths of γ and $\bar{\gamma}$. Therefore $L_\Gamma(\rho^s) \geq L_\Gamma(\rho)$. On the other hand $\mathbf{A}(\rho^s) = \mathbf{A}(\rho)$ so we have

$$\frac{L_\Gamma(\rho^s)^2}{\mathbf{A}(\rho^s)} \geq \frac{L_\Gamma(\rho)^2}{\mathbf{A}(\rho)}.$$

This implies that $\lambda^s(\Gamma) \geq \lambda(\Gamma)$ and the lemma follows. $\boxed{1.23}$

Let z_0 and z_1 be points in the unit disk Δ and let $\overline{z_0 z_1}$ be the geodesic in the hyperbolic metric on Δ connecting the two points, Let $A_{z_0 z_1} = \Delta \backslash \overline{z_0 z_1}$ be the complementary annulus. The following theorem is due to Grötzsch although we formulate it in a non-standard way.

Theorem 1.24 *Let A be an annulus in Δ that separates $z_0, z_1 \in \Delta$ from $\partial\Delta$. Then $m(A) \geq m(A_{z_0 z_1})$.*

Proof: Let Γ be the collection of closed curves γ in Δ such that both z_0 and z_1 have winding number 1 with respect to γ. Then $\Gamma_A \subset \Gamma$ so $m(A) = \lambda(\Gamma_A) \leq \lambda(\Gamma)$.

We can assume that z_0 and z_1 lie in $\mathbb{R}$. Then $\Gamma = \bar{\Gamma}$ and we can take advantage of Proposition 1.23 to show that $\lambda(\Gamma) = \lambda(\Gamma_{A_{z_0 z_1}})$. The key point is that $\Gamma^+_{A_{z_0 z_1}} = \Gamma^+$ and therefore

$$\lambda(\Gamma_{A_{z_0 z_1}}) = 2\lambda(\Gamma^+_{A_{z_0 z_1}}) = 2\lambda(\Gamma^+) = \lambda(\Gamma)$$

by Proposition 1.23.

We need to justify why $\Gamma^+_{A_{z_0 z_1}} = \Gamma^+$. As $\Gamma_{A_{z_0 z_1}} \subset \Gamma$ we have $\Gamma^+_{A_{z_0 z_1}} \subset \Gamma^+$. Now take $\gamma \in \Gamma$ and we'll show that there is $\gamma' \in \Gamma_{A_{z_0 z_1}}$ with $\gamma^+ = (\gamma')^+$. Note that γ intersects the intervals $(-1, z_0)$ and $(z_1, 1)$ in $\mathbb{R}$ (here we are assuming $z_0 < z_1$) and therefore we can subdivide γ into two arcs a and b into two arcs with endpoints in each of the two intervals. Then γ^+ is subdivided into a^+ and b^+. However, if we let b^- be the reflection of b^+ into the lower half plane and let $\gamma' = a^+ \cup b^-$ then $\gamma' \in \Gamma_{A_{z_0 z_a}}$ as desired.[2] $\boxed{1.24}$

[2]There is one additional subtlety. While γ' will not cross $\overline{z_0 z_1}$ it may intersect it. However, if we expand $\Gamma_{A z_0 z_1}$ to allows such curves the modulus will be the same.