Introductory topics in Kleinian grops and hyperbolic 3-manifolds

Convex hull problems

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1. Show that a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$ acts properly discontinuously on $\mathbb{H}^3$.

   For any two points p_1 and p_2 in $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ there is a unique geodesic with endpoints p_1 and p_2. A set K in $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ is *convex* if whenever p_1 and p_2 are contained in K then this geodesic is also in K.

2. If K is a closed convex set in $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ show that for every $p \in \mathbb{H}^3$ there is a unique ball centered at p that intersects K in exactly one point. If $p \in \widehat{\mathbb{C}}$ show that there is a unique horoball centered at p that intersects K in exactly one point. Note that we allow the ball and horoball to be single point.

   Define a map $\pi_K : \mathbb{H}^3 \cup \widehat{\mathbb{C}} \longrightarrow K$ by setting $\pi_K(p)$ to be the point of intersection given in the previous problem. The map π_K is the *nearest point retraction* onto K.

3. Show that π_K is continuous and $\pi_K(p) = p$ if and only if $p \in K$.

4. If K is Γ-invariant show that π_K commutes with the action of Γ.

   The *convex hull*, $CH(\Lambda)$, of a set Λ is the smallest closed convex set that contains Λ.

5. Show that the convex hull is well defined.

   The *limit set* $\Lambda = \Lambda(\Gamma)$ of Kleinian group Γ is the smallest, non-empty, closed Γ-invariant subset of $\widehat{\mathbb{C}}$.

6. Show that $CH(\Lambda)$ is Γ-invariant.

   The *domain of discontinuity*, $\Omega = \Omega(\Gamma)$, for Γ is the complement of the limit set. That is $\Omega = \widehat{\mathbb{C}} \backslash \Lambda$.

7. Use the nearest point retraction to show that Γ acts properly discontinuously on Ω.