Cone-manifolds and the density conjecture

Jeffrey F. Brock and Kenneth W. Bromberg

Abstract

We give an expository account of our proof that each cusp-free hyperbolic 3-manifold M with finitely generated fundamental group and incompressible ends is an algebraic limit of geometrically finite hyperbolic 3-manifolds.

1. Introduction

The aim of this paper is to outline and describe new constructions and techniques we hope will provide a useful tool to study deformations of hyperbolic 3-manifolds. An initial application addresses the following conjecture.

Conjecture 1.1 (Bers–Sullivan–Thurston. The Density Conjecture). Each complete hyperbolic 3-manifold M with finitely generated fundamental group is an algebraic limit of geometrically finite hyperbolic 3-manifolds.

Algebraic convergence of M_n to M refers to convergence in the *algebraic deformation space* or in the topology of convergence on generators of the holonomy representations

$$\rho_n \colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3).$$

The approximating manifolds $M_n = \mathbb{H}^3/\rho_n(\pi_1(S))$ are *geometrically finite* if the *convex core* of M_n, the minimal convex subset homotopy equivalent to M_n, has finite volume. We give an expository account of our progress toward Conjecture 1.1 [BB02].

Theorem 1.2. *Let M be a complete hyperbolic 3-manifold with no cusps, finitely generated fundamental group, and incompressible ends. Then M is an algebraic limit of geometrically finite hyperbolic 3-manifolds.*

Our result represents an initial step in what we hope will be a general geometrically finite approximation theorem for *topologically tame* complete hyperbolic 3-manifolds, namely, for each such manifold M that is homeomorphic to the interior of a compact 3-manifold.

Indeed, the clearly essential assumption in our argument is that M is tame; we make direct use of the following theorem due to Bonahon and Thurston (see [Bon86, Thu79]).

Theorem 1.3 (Bonahon–Thurston). *Each cusp-free complete hyperbolic 3-manifold M with finitely generated fundamental group and incompressible ends is geometrically and topologically tame.*

The tameness of a complete hyperbolic 3-manifold with finitely generated fundamental group reduces to a consideration of its *ends* since every such 3-manifold M contains a *compact core*, namely, a compact submanifold $\mathscr{M}$ whose inclusion is a homotopy equivalence. Each end e of M is associated to a component E of $M \setminus \mathrm{int}(\mathscr{M})$, which we typically refer to as an "end" of M, assuming an implicit choice of compact core. An end E is *incompressible* if the inclusion of E induces an injection $\pi_1(E) \hookrightarrow \pi_1(M)$. The end E is *geometrically finite* if it has compact intersection with the convex core. Otherwise, it is *degenerate*.

For a degenerate end E, *geometric* tameness refers to the existence of a family of simple closed curves on the closed surface $S = \partial \mathscr{M} \cap E$ whose geodesic representatives leave every compact subset of E. Using interpolations of pleated surfaces, Thurston showed that a geometrically tame end is homeomorphic to $S \times \mathbb{R}^+$, so M is topologically tame if all its ends are geometrically finite or geometrically tame (R. Canary later proved the equivalence of these notions [Can93]).

1.1. Approximating the ends

Our approach to Theorem 1.2 will be to approximate the manifold M *end by end*. Such an approach is justified by an *asymptotic isolation* theorem (Theorem 1.7) that isolates the geometry of the ends of M from one another when M is obtained as a limit of geometrically finite manifolds. Each degenerate end E of M has one of two types: E has either

   I. *bounded geometry:* there is a uniform lower bound to the length of the shortest geodesic in E, or

   II. *arbitrarily short geodesics:* there is some sequence γ_n of geodesics in E whose length is tending to zero.

Historically, it is the latter category of ends that have been persistently inscrutable (they are known to be generic [McM91, CCHS01]). Our investigation of such ends begins with another key consequence of tameness, due to J. P. Otal (see [Ota95], or his article [Ota02] in this volume). Before discussing this result, we introduce some terminology.

If E is an incompressible end of M, the cover $\widetilde{M}$ corresponding to $\pi_1(E)$ is homotopy equivalent to the surface $S = \partial \mathscr{M} \cap E$. Thus, $\widetilde{M}$ sits in the *algebraic deformation*

space AH(S), namely, hyperbolic 3-manifolds *M* equipped with homotopy equivalences, or *markings*, $f : S \to M$ up to isometries that preserve marking and orientation (see [Thu86b], [McM96]). The space *AH(S)* is equipped with the algebraic topology, or the topology of convergence of holonomy representations, as described above. Theorem 1.3 guarantees each $M \in AH(S)$ is homeomorphic to $S \times \mathbb{R}$; Otal's theorem provides deeper information about how short geodesics in *M* sit in this product structure.

Theorem 1.4 (Otal [Ota95]). *Let M lie in AH(S). There is an* $\varepsilon_{\mathrm{knot}} > 0$ *so that if* $\mathscr{A}$ *is any collection of closed geodesics so that for each* $\gamma \in \mathscr{A}$ *we have*

$$\ell_M(\gamma) < \varepsilon_{\mathrm{knot}}$$

then there exists a collection of distinct real numbers $\{t_\gamma \mid \gamma \in \mathscr{A}\}$ *and an ambient isotopy of* $M \cong S \times \mathbb{R}$ *taking each* γ *to a simple curve in* $S \times \{t_\gamma\}$.

Said another way, sufficiently short curves in *M* are simple, unknotted and pairwise unlinked with respect to the product structure $S \times \mathbb{R}$ on *M*.

Otal's theorem directly facilitates the *grafting* of tame ends that carry sufficiently short geodesics. This procedure, introduced in [Brm02b], uses embedded end-homotopic annuli in a degenerate end to perform 3-dimensional version of grafting from the theory of projective structures (see e.g. [McM98, GKM00]). In section 3 we will describe how successive graftings about short curves in an end *E* of *M* can be used to produce a sequence of projective structures with holonomy $\pi_1(M)$ whose underlying conformal structures X_n reproduce the asymptotic geometry of the end *E* in a limit.

Our discussion of ends *E* with bounded geometry relies directly on a large body of work of Y. Minsky [Min93, Min94, Min00, Min01] which has recently resulted in the following *bounded geometry theorem.*

Theorem 1.5 (Minsky. Bounded Geometry Theorem). *Let M lie in AH(S), and assume M has a global lower bound to its injectivity radius* inj: $M \to \mathbb{R}^+$. *If* $N \in AH(S)$ *has the same end-invariant as that of M then* $M = N$ *in AH(S).*

In other words, there is an orientation preserving isometry $\varphi : M \to N$ that respects the homotopy classes of the markings on each. The "end invariant" $\nu(M)$ refers to a union of invariants, each associated to an end *E* of *M*. Each invariant is either a Riemann surface in the *conformal boundary* ∂M that compactifies the end, or an *ending lamination*, namely, the support $|\mu|$ of a limit $[\mu]$ of simple closed curves γ_n whose geodesic representatives in *M* that exit the end *E* (here $[\mu]$ is the limit of $[\gamma_n]$ in Thurston's projective measured lamination space $\mathscr{PL}(S)$ [Thu79, Thu86b]).

Minsky's theorem proves Theorem 1.2 for each M with a lower bound to its injectivity radius, since given any end invariant $\nu(M)$ there is some limit M_∞ of geometrically finite manifolds with end invariant $\nu(M_\infty) = \nu(M)$ (see [Ohs90, Bro00]).

1.2. Realizing ends on a Bers boundary

Grafting ends with short geodesics and applying Minsky's results to ends with bounded geometry, we arrive at a *realization* theorem for ends of manifolds $M \in AH(S)$ in some *Bers compactification*.

Theorem 1.6 (Ends are Realizable). *Let $M \in AH(S)$ have no cusps. Then each end of M is realized in a Bers compactification.*

We briefly explain the idea and import of the theorem. The subset of $AH(S)$ consisting of geometrically finite cusp-free manifolds is the *quasi-Fuchsian locus $QF(S)$*. In [Ber60] Bers exhibited the parameterization

$$Q\colon \text{Teich}(S) \times \text{Teich}(S) \to QF(S)$$

so that $Q(X,Y)$ contains X and Y in its conformal boundary; $Q(X,Y)$ *simultaneously uniformizes* the pair (X,Y). Fixing one factor, we obtain the *Bers slice $B_Y = \{Q(X,Y) \mid Y \in \text{Teich}(S)\}$*, which Bers proved to be precompact. The resulting compactification $\overline{B_Y} \subset AH(S)$ for Teichmüller space has frontier ∂B_Y, a *Bers boundary* (see [Ber70a]).

We say an end E of $M \in AH(S)$ is *realized by Q in the Bers compactification* $\overline{B_Y}$ if there is a manifold $Q \in \overline{B_Y}$ and a marking preserving bi-Lipschitz embedding $\phi\colon E \to Q$ (see Definition 4.2).

The cusp-free manifold $M \in AH(S)$ is *singly-degenerate* if exactly one end of M is compactified by a conformal boundary component Y. In this case, the main theorem of [Brm02b] establishes that M itself lies in the Bers boundary ∂B_Y, which was originally conjectured by Bers [Ber70a]. Theorem 1.6 generalizes this result to the relative setting of a given incompressible end of M, allowing us to pick candidate approximates for a given M working end-by-end.

1.3. Candidate approximates

To see explicitly how candidate approximates are chosen, let M have finitely generated fundamental group and incompressible ends. For each end E of M, Theorem 1.6 allows us to choose $X_n(E)$ so that the limit of $Q(X_n(E),Y)$ in $\overline{B_Y}$ realizes the end E. Then we

simply let M_n be the geometrically finite manifold homeomorphic to M determined by specifying the data

$$(X_n(E_1), \ldots, X_n(E_m)) \in \text{Teich}(\partial \mathcal{M})$$

where $\mathcal{M}$ is a compact core for M; $\text{Teich}(\partial \mathcal{M})$ naturally parameterizes such manifolds (see section 5). The union $X_n(E_1) \cup \ldots \cup X_n(E_m)$ constitutes the conformal boundary ∂M_n.

To conclude that the limit of M_n is the original manifold M, we must show that limiting geometry of each end of M_n does not depend on limiting phenomena in the other ends. We show ends of M_n are *asymptotically isolated*.

Theorem 1.7 (Asymptotic Isolation of Ends). *Let N be a complete cusp-free hyperbolic 3-manifold with finitely generated fundamental group and incompressible ends. Let M_n converge algebraically to N. Then up to bi-Lipschitz diffeomorphism, the end E of M depends only on the corresponding sequence $X_n(E) \subset \partial M_n$.*

(See Theorems 4.1 and 5.1 for a more precise formulation).

When $N \in AH(S)$ is singly-degenerate, the theorem is well known (for example see [McM96, Prop. 3.1]). For N not homotopy equivalent to a surface, the cover corresponding to each end of N is singly-degenerate, so the theorem follows in this case as well.

The ideas in the proof of Theorem 1.7 when N is doubly-degenerate represent a central focus of this paper. In this case, the cover of N associated to each end is again the manifold N and thus not singly-degenerate, so the asymptotic isolation is no longer immediate. The situation is remedied by a new technique in the cone-deformation theory called the *drilling theorem* (Theorem 2.3).

This drilling theorem allows us to "drill out" a sufficiently short curves in a geometrically finite cusp-free manifold with bounded change to the metric outside of a tubular neighborhood of the drilling curve. When quasi-Fuchsian manifolds $Q(X_n, Y_n)$ converge to the cusp-free limit N, any short geodesic γ in N may be drilled out of each $Q(X_n, Y_n)$.

The resulting drilled manifolds $Q_n(\gamma)$ converge to a limit $N(\gamma)$ whose higher genus ends are bi-Lipschitz diffeomorphic to those of N. In the manifold $N(\gamma)$, the rank-2 cusp along γ serves to insulate the geometry of the ends from one another, giving the necessary control. (When there are no short curves, Minsky's theorem again applies).

The drilling theorem manifests the idea that the thick part of a hyperbolic 3-manifold with a short geodesic looks very similar to the thick part of the hyperbolic 3-manifold obtained by removing that curve. We employ the cone-deformation theory of C. Hodgson and S. Kerckhoff to give analytic control to this qualitative picture.

1.4. Plan of the paper

In what follows we will give descriptions of each facet of the argument. Our descriptions are expository in nature, in the interest of conveying the main ideas rather than detailed specific arguments (which appear in [BB02]). We will focus on the case when *M* is homotopy equivalent to a surface, which presents the primary difficulties, treating the general case briefly at the conclusion.

In section 2 we provide an overview of techniques in the deformation theory of hyperbolic cone-manifolds we will apply, providing bounds on the metric change outside a tubular neighborhood of the cone-singularity under a change in the cone-angle. In section 3 we describe the grafting construction and how it produces candidate approximates for the ends of *M* with arbitrarily short geodesics. Section 4 describes the asymptotic isolation theorem (Theorem 1.7), the realization theorem for ends (Theorem 1.6), and finally how these results combine to give a proof of Theorem 1.2 when *M* lies in $AH(S)$. The general case is discussed in section 5.

1.5. Acknowledgments

2. Cone-deformations

Over the last decade, Hodgson and Kerckhoff have developed a powerful rigidity and deformation theory for 3-dimensional hyperbolic cone-manifolds [HK98]. While their theory was developed initially for application to closed hyperbolic cone-manifolds, work of the second author (see [Brm00]) has generalized this rigidity and deformation theory to *infinite volume* geometrically finite manifolds.

The cone-deformation theory represents a key technical tool in Theorem 1.2. Let *N* be a compact, hyperbolizable 3-manifold with boundary; assume that ∂N does not contain tori for simplicity. Let *c* be a simple closed curve in the interior of *N*. A *hyperbolic cone-metric* is a hyperbolic metric on the interior of $N \setminus c$ that completes to a singular metric on all of the interior of *N*. Near *c* the metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$$

where θ is measured modulo the *cone-angle*, α.

Just as $\mathbb{H}^3$ is compactified by the Riemann sphere, complete infinite volume hyperbolic 3-manifolds are often compactified by projective structures. If a hyperbolic cone-metric is so compactified it is *geometrically finite without rank-one cusps*. As we have excised the presence of rank-one cusps in our hypotheses, we simply refer to such metrics as geometrically finite.

A projective structure on ∂N has an underlying conformal structure; we often refer to ∂N together with its conformal structure as the *conformal boundary of N*.

Theorem 2.1. *Let M_α denote N with a 3-dimensional geometrically finite hyperbolic cone-metric with cone-angle α at c. If the cone-singularity has tube-radius at least $\sinh^{-1}\left(\sqrt{2}\right)$, then nearby cone-metrics are locally parameterized by the cone-angle and the conformal boundary.*

Here, the *tube-radius* about c is the radius of the maximally embedded metric tube about c in M_α.

This local parameterization theorem was first proven by Hodgson and Kerckhoff for closed manifolds with cone-angle less than 2π and no assumption on the size of the tube radius [HK98]. In the thesis of the second author [Brm00], Hodgson and Kerckhoff's result was generalized to the setting of general geometrically finite cone-manifolds, where the conformal boundary may be non-empty. The replacement of the cone-angle condition with the tube-radius condition is recent work of Hodgson and Kerckhoff (see [HK02a] in this volume).

Theorem 2.1 allows us to decrease the cone-angle while keeping the conformal boundary fixed at least for cone-angle near α. We need more information if we wish to decrease the cone-angle all the way to zero.

Theorem 2.2 ([Brm02a]). *Let M_α be a 3-dimensional geometrically finite hyperbolic cone-metric with cone-angle α. Suppose that the cone-singularity c has tube-radius at least $\sinh^{-1}\left(\sqrt{2}\right)$. Then there exists an $\varepsilon > 0$ depending only on α such that if the length of c is less than ε there exists a one-parameter family M_t of geometrically finite cone-metrics with cone-angle t and conformal boundary fixed for all $t \in [0, \alpha]$.*

2.1. The drilling theorem

When the cone-angle α is 2π the hyperbolic cone-metric M_α is actually a smooth hyperbolic metric. When the cone-angle is zero the hyperbolic cone-metric is also a smooth complete metric; the curve c, however, has receded to infinity leaving a rank-two cusp, and the complete hyperbolic metric lives on the interior of $N \setminus c$. We call $N \setminus c$ with its complete hyperbolic metric M_0 the *drilling along c* of M_α.

Applying the analytic tools and estimates developed by Hodgson and Kerckhoff [HK02b], we obtain infinitesimal control on the metric change outside a tubular neighborhood of the cone-singularity under a change in the cone-angle. Letting $U_t \subset M_t$ denote a standard tubular neighborhood of the cone-singularity we obtain the following *drilling theorem*, which summarizes the key geometric information emerging from these estimates.

Theorem 2.3 (The Drilling Theorem). *Suppose M_α is a geometrically finite hyperbolic cone-metric satisfying the conditions of Theorem 2.2, and let M_t be the resulting family of cone-metrics. Then for each $K > 1$ there exists an $\varepsilon' > 0$ depending only on α and K such that if the length of c is less than ε', there are diffeomorphisms of pairs*

$$\phi_t : (M_\alpha \setminus U_\alpha, \partial U_\alpha) \longrightarrow (M_t \setminus U_t, \partial U_t)$$

so that ϕ_t is K-bi-Lipschitz for each $t \in [0, \alpha]$, and ϕ_t extends over U_α to a homeomorphism for each $t \in (0, \alpha]$.

3. Grafting short geodesics

A simple closed curve γ in $M \in AH(S)$ is *unknotted* if it is isotopic in M to a simple curve γ_0 in the "level surface" $S \times \{0\}$ in the product structure $S \times \mathbb{R}$ on M. For such a γ, there is a bi-infinite annulus A containing γ representing its free homotopy class so that A is isotopic to $\gamma_0 \times \mathbb{R}$. Let A^+ denote the sub-annulus of A exiting the positive end of M, let A^- denote the sub-annulus of A exiting the negative end.

The *positive grafting* $\mathrm{Gr}^+(\gamma, M)$ *of M along γ* is the following surgery of M along the *positive grafting annulus A^+*.

1. Let $M_{\mathbb{Z}}$ denote the cyclic cover of M associated to the curve γ. Let

    $$F \colon S^1 \times [0, \infty) \to A^+$$

    be a parameterization of the grafting annulus and let $F_{\mathbb{Z}}$ be its lift to $M_{\mathbb{Z}}$.

2. Cutting M along A^+ and $M_{\mathbb{Z}}$ along $A_{\mathbb{Z}}^+ = F_{\mathbb{Z}}(S^1 \times [0, 1))$, the complements $M \setminus A^+$ and $M_{\mathbb{Z}} \setminus A_{\mathbb{Z}}^+$ each have two isometric copies of the annulus in their metric completions $\overline{M \setminus A^+}$ and $\overline{M_{\mathbb{Z}} \setminus A_{\mathbb{Z}}^+}$: the *inward annulus* inherits an orientation from F that agrees with the orientation induced by the positive orientation on $M \setminus A^+$ and the *outward annulus* inherits the opposite orientations from F and $M \setminus A^+$. The complement $M_{\mathbb{Z}} \setminus A_{\mathbb{Z}}^+$ also contains an *inward* and *outward* copy of $A_{\mathbb{Z}}^+$ in its metric completion.
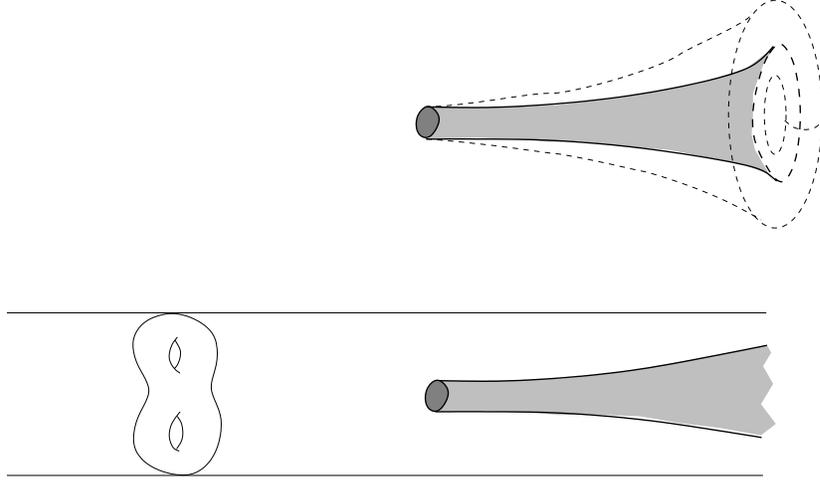
Figure 1: The grafting annulus and its lift.

3. Let F^{in} and F^{out} denote the natural parameterizations of the inward and outward annulus for the metric completion of $M \setminus A^+$ induced by F and let $F_{\mathbb{Z}}^{\mathrm{in}}$ and $F_{\mathbb{Z}}^{\mathrm{out}}$ be similarly induced by $F_{\mathbb{Z}}$.

4. Let ϕ be the isometric gluing of the inward annulus for $\overline{M_{\mathbb{Z}} \setminus A_{\mathbb{Z}}^{\pm}}$ to the outward annulus for $\overline{M \setminus A^+}$ and the outward annulus of $\overline{M_{\mathbb{Z}} \setminus A_{\mathbb{Z}}^+}$ to the inward annulus of $\overline{M \setminus A^+}$ so that

$$\phi(F^{\mathrm{in}}(x,t)) = F_{\mathbb{Z}}^{\mathrm{out}}(x,t) \quad \text{and} \quad \phi(F^{\mathrm{out}}(x,t)) = F_{\mathbb{Z}}^{\mathrm{in}}(x,t)$$

(the map ϕ on the geodesic $\widetilde{\gamma} \subset M_{\mathbb{Z}}$ should just be the restriction covering map $M_{\mathbb{Z}} \to M$).

The result $\mathrm{Gr}^+(M,\gamma)$ of positive grafting along γ is no longer a smooth manifold since its metric is not smooth at γ, but $\mathrm{Gr}^+(M,\gamma)$ inherits a smooth hyperbolic metric from M and $M_{\mathbb{Z}}$ away from γ.

3.1. Graftings as cone-manifolds.

Otal's theorem (Theorem 1.4) guarantees that a sufficiently short closed geodesic γ^* is unknotted. In this case, the positive grafting $\mathrm{Gr}^+(M,\gamma^*)$ along the closed geodesic γ^* is well defined, and the singularity has a particularly nice structure: since the singularity is a geodesic, the smooth hyperbolic structure on $\mathrm{Gr}^+(M,\gamma^*) \setminus \gamma^*$ extends to
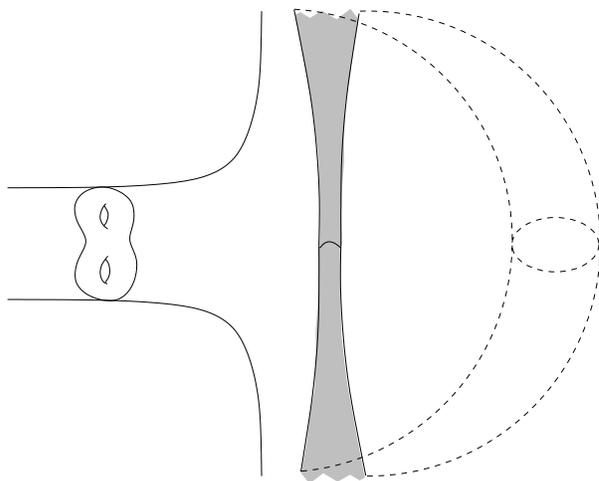
Figure 2: Grafting: glue the wedge $M_{\mathbb{Z}} \setminus A_{\mathbb{Z}}^+$ along the completion of $M \setminus A^+$.

a hyperbolic cone-metric on $\mathrm{Gr}^+(M, \gamma^*)$ with cone-singularity γ^* and cone-angle 4π at γ^* (cf. [Brm02b]).

3.2. Simultaneous grafting

We would like to apply the cone-deformation theory of section 2 to the grafting $\mathrm{Gr}^+(M, \gamma^*)$. The deformation theory applies, however, only to geometrically finite hyperbolic cone-manifolds. The grafting $\mathrm{Gr}^+(M, \gamma^*)$ alone may not be geometrically finite if the manifold M is doubly-degenerate. Indeed, in the doubly-degenerate case positive grafting produces a geometrically finite *positive* end, but to force geometric finiteness of *both* ends, we must perform negative grafting as well.

Let γ and β be two simple unknotted curves in M that are also *unlinked*: γ is isotopic to a level surface in the complement of β. Then γ is homotopic either to $+\infty$ or to $-\infty$ in the complement of β. Assume the former. Then we may choose a positive grafting annulus A_γ^+ for γ and a negative grafting annulus A_β^- for β and perform *simultaneous grafting* on M: we simply perform the grafting surgery on A_γ^+ and A_β^- at the same time.

By Otal's theorem, when γ^* and β^* are sufficiently short geodesics in the hyperbolic 3-manifold M, they are simple, unknotted and unlinked. If γ^* is homotopic to $+\infty$ in $M \setminus \beta^*$, the simultaneous grafting

$$\mathrm{Gr}^{\pm}(\beta^*, \gamma^*, M)$$

produces a hyperbolic cone-manifold with two cone-singularities, one at γ^* and one at β^*, each with cone-angle 4π.

We then prove the following theorem.

Theorem 3.1 (Simultaneous Graftings). *Let γ^* and β^* be two simple closed geodesics in M as above. Then the simultaneous grafting $\mathrm{Gr}^{\pm}(\beta^*, \gamma^*, M)$ is a geometrically finite hyperbolic cone-manifold.*

The proof applies the theory of geometric finiteness for variable negative curvature developed by Brian Bowditch [Bow94] [Bow95], to a variable negative curvature smoothing $\mathcal{M}$ of $\mathrm{Gr}^{\pm}(\beta^*, \gamma^*, M)$ at its cone-singularities. Using these results, we obtain the following version of Canary's geometric tameness theorem [Can93] for Riemannian 3-manifolds with curvature pinched between two negative constants, or *pinched negative curvature* (we omit the cusped case as usual).

Theorem 3.2 (Geometric Tameness for Negative Curvature). *Each end E of the topologically tame 3-manifold $\mathcal{M}$ with pinched negative curvature and no cusps satisfies the following dichotomy: either*

1. *E is geometrically finite: E has finite volume intersection with the convex core of $\mathcal{M}$, or*

2. *E is simply degenerate: there are essential, non-peripheral simple closed curves γ_n on the surface S cutting off E whose geodesic representatives exit every compact subset of E.*

In our setting, any simple closed curve η on S whose geodesic representative η^* avoids the cone-singularities of $\mathrm{Gr}^{\pm}(\beta^*, \gamma^*, M)$ projects to a closed geodesic $\pi(\eta^*)$ in M under the natural local isometric covering

$$\pi \colon \mathrm{Gr}^{\pm}(\beta^*, \gamma^*, M) \setminus \beta^* \sqcup \gamma^* \to M.$$

The projection π extends to a homotopy equivalence across $\beta^* \sqcup \gamma^*$, so the image $\pi(\eta^*)$ is the geodesic representative of η in M. Though π is not proper, we show that any sequence η_n of simple closed curves on S whose geodesic representatives in $\mathrm{Gr}^{\pm}(\beta^*, \gamma^*, M)$ leave every compact subset must have the property that $\pi(\eta_n^*)$ leaves every compact subset of M. This contradicts bounded diameter results from Thurston's theory of pleated surfaces [Thu79], which guarantee that realizations of $\pi(\eta_n^*)$ by pleated surfaces remain in a compact subset of M. The contradiction implies that grafted ends are geometrically finite, proving Theorem 3.1.

The simultaneous grafting $\mathrm{Gr}^+(\beta^*, \gamma^*, M)$ has two components in its *projective boundary at infinity* to which the hyperbolic cone-metric extends. Already, we can give an outline of the proof of Theorem 1.2 in the case that each end of the doubly-degenerate manifold $M \in AH(S)$ has arbitrarily short geodesics.

Here are the steps:

I. Let $\{\gamma_n^*\}$ be arbitrarily short geodesics exiting the positive end of M and let $\{\beta_n^*\}$ be arbitrarily short geodesics exiting the negative end of M. Assume γ_n^* is homotopic to $+\infty$ in $M \setminus \beta_n^*$.

II. The simultaneous graftings

$$\mathrm{Gr}^{\pm}(\beta_n^*, \gamma_n^*, M) = M_n^c$$

have projective boundary with underlying conformal structures X_n on the negative end of M_n^c and Y_n on the positive end of M_n^c.

III. By Theorem 3.1 the manifolds M_n^c are geometrically finite hyperbolic cone-manifolds (with no cusps, since M has no cusps).

IV. Applying Theorem 2.3, we may deform the cone-singularities at γ_n^* and β_n^* back to 2π fixing the conformal boundary of M_n^c to obtain quasi-Fuchsian hyperbolic 3-manifolds $Q(X_n, Y_n)$.

V. Since the lengths of γ_n^* and β_n^* are tending to zero, the metric distortion of the cone-deformation outside of tubular neighborhoods of the cone-singularities is tending to zero. Since the geodesics γ_n^* and β_n^* are exiting the ends of M, larger and larger compact subsets of M are more and more nearly isometric to large compact subsets of $Q(X_n, Y_n)$ for n sufficiently large. Convergence of $Q(X_n, Y_n)$ to M follows.

Next, we detail our approach to the general doubly-degenerate case, which handles ends with bounded geometry and ends with arbitrarily short geodesics transparently.

4. Drilling and asymptotic isolation of ends

It is peculiar that manifolds $M \in AH(S)$ of *mixed type*, namely, doubly-degenerate manifolds with one bounded geometry end and one end with arbitrarily short geodesics, present some recalcitrant difficulties that require new techniques. Here is an example of the type of phenomenon that is worrisome:

4.1. Example

Consider a sequence $Q(X_n, Y)$ tending to a limit Q_∞ in the Bers slice B_Y for which Q_∞ is partially degenerate, and for which Q_∞ has arbitrarily short geodesics. Allowing Y to vary in Teichmüller space, we obtain a *limit Bers slice B_∞ associated to the sequence* $\{X_n\}$ (this terminology was introduced by McMullen [McM98]). The limit Bers slice B_∞ is an embedded copy of Teich(S) in $AH(S)$ consisting of manifolds

$$M(Y') = \lim_{n \to \infty} Q(X_n, Y') \quad \text{where } Y' \text{ lies in } \text{Teich}(S).$$

Each $M(Y')$ has a degenerate end that is bi-Lipschitz diffeomorphic to Q_∞ (see, e.g., [McM96, Prop. 3.1]), but the bi-Lipschitz constant depends on Y'.

If, for example, δ is a simple closed curve on S and $\tau^n(Y) = Y_n$ is a divergent sequence in Teich(S) obtained via an iterated Dehn twist τ about δ, a subsequence of $\{M(Y_n)\}_{n=1}^\infty$ converges to a limit M_∞, but there is no *a priori* reason for the degenerate end of M_∞ to be bi-Lipschitz diffeomorphic to that of $M(Y)$. The limiting geometry of the ends compactified by Y_n could, in principle, bleed over into the degenerate end, causing its asymptotic structure to change in the limit. (We note that such phenomena would violate Thurston's *ending lamination conjecture* since M_∞ has the same ending lamination associated to its degenerate end as does $M(Y)$).

4.2. Isolation of ends

For a convergent sequence of quasi-Fuchsian manifolds $Q(X_n, Y_n) \to N$, we seek some way to isolate the limiting geometry of the ends of $Q(X_n, Y_n)$ as n tends to infinity. Our strategy is to employ the drilling theorem in a suitably chosen family of convergent approximates $Q(X_n, Y_n) \to N$ for which a curve γ is short in $Q_n = Q(X_n, Y_n)$ for all n. We prove that drilling γ out of each Q_n to obtain a drilled manifold $Q_n(\gamma)$ produces a sequence converging to a drilled limit $N(\gamma)$ whose higher genus ends are bi-Lipschitz diffeomorphic to those of N.

An application of the covering theorem of Thurston and Canary [Thu79, Can96] then demonstrates that the limiting geometry of the negative end of N depends only on the sequence $\{X_n\}$ and the limiting geometry of the positive end of N depends only on the sequence $\{Y_n\}$.

When N has no such short geodesic γ, the ends depend only on the end invariant $\nu(N)$, since in this case N has bounded geometry and Theorem 1.5 applies. These arguments are summarized in the following isolation theorem for the asymptotic geometry of N (cf. Theorem 1.7).

Theorem 4.1 (Asymptotic Isolation of Ends). *Let $Q(X_n, Y_n) \in AH(S)$ be a sequence of quasi-Fuchsian manifolds converging algebraically to the cusp-free limit manifold N. Then, up to marking and orientation preserving bi-Lipschitz diffeomorphism, the positive end of N depends only on the sequence $\{Y_n\}$ and the negative end of N depends only on the sequence $\{X_n\}$.*

We now argue that as a consequence of Theorem 4.1 we need only show that each end of a doubly-degenerate manifold M arises as the end of a singly-degenerate manifold lying in a Bers boundary.

Definition 4.2. Let E be an end of a complete hyperbolic 3-manifold M. If E admits a marking and orientation preserving bi-Lipschitz diffeomorphism to an end E' of a manifold Q lying in a Bers compactification, we say E is *realized in a Bers compactification by Q*.

If, for example, the positive end E^+ of M is realized by Q_∞^+ on the Bers boundary ∂B_X then there are by definition surfaces $\{Y_n\}$ so that $Q(X, Y_n)$ converges to Q_∞^+, so E^+ depends only on $\{Y_n\}$ up to bi-Lipschitz diffeomorphism. Arguing similarly, if E^- is realized by Q_∞^- on the Bers boundary ∂B_Y, the approximating surfaces $\{X_n\}$ for which $Q(X_n, Y) \to Q_\infty^-$ determine E^- up to bi-Lipschitz diffeomorphism.

By an application of Theorem 4.1, if the manifolds $Q(X_n, Y_n)$ converge to a cusp-free limit N, then the negative end E_N^- is bi-Lipschitz diffeomorphic to E^- and the positive end E_N^+ is bi-Lipschitz diffeomorphic to E^+. We may glue bi-Lipschitz diffeomorphisms

$$\psi^- : E_N^- \to E^- \quad \text{and} \quad \psi^+ : E_N^+ \to E^+$$

along the remaining compact part to obtain a global bi-Lipschitz diffeomorphism

$$\psi : N \to M$$

that is marking and orientation preserving. By applying Sullivan's rigidity theorem [Sul81a], ψ is homotopic to an isometry, so $Q(X_n, Y_n)$ converges to M.

4.3. Realizing ends in Bers compactifications

To complete the proof of Theorem 1.2, then, we seek to realize each end of the doubly-degenerate manifold M on a Bers boundary; we restate Theorem 1.6 here.

Theorem 4.3 (Ends are Realizable). *Let $M \in AH(S)$ have no cusps. Then each end of M is realized in a Bers compactification.*

In the case that M has a conformal boundary component Y, the theorem asserts that M lies within the Bers compactification $\overline{B_Y}$. This is the main result of [Brm02b], which demonstrates all such manifolds are limits of quasi-Fuchsian manifolds.

We are left to attend to the case when M is doubly-degenerate. As one might expect, the discussion breaks into cases depending on whether an end E has bounded geometry or arbitrarily short geodesics. We discuss the positive end of M; one argues symmetrically for the negative end.

1. If a bounded geometry end E has ending lamination ν, choose a measured lamination μ with support ν and a sequence of weighted simple closed curves $t_n\gamma_n \to \mu$. Choose Y_n so that $\ell_{Y_n}(\gamma_n) < 1$.

2. If γ_n^* are arbitrarily short geodesics exiting the end E, we apply the drilling theorem to $\mathrm{Gr}^{\pm}(\gamma_0, \gamma_n, M)$ to send the cone-angles at γ_0^* and γ_n^* to 2π. The result is a sequence $Q(X, Y_n)$ of quasi-Fuchsian manifolds.

We wish to show that after passing to a subsequence $Q(X, Y_n)$ converges to a limit Q_∞ that realizes E on the Bers boundary ∂B_X.

Bounded geometry. When E has bounded geometry, we employ [Min00] to argue that its end invariant ν has *bounded type*. This condition ensures that any end with ν as its end invariant has bounded geometry. The condition $\ell_{Y_n}(\gamma_n) < 1$ guarantees that $\ell_{Y_n}(t_n\gamma_n) \to 0$ so that any limit Q_∞ of $Q(X, Y_n)$ has ν as its end-invariant (by [Bro00], applying [Ber70a, Thm. 3]). We may therefore apply a relative version of Minsky's ending lamination theorem for bounded geometry (see [Min94], and an extension due to Mosher [Mos01] that treats the case when the manifold may not possess a *global* lower bound to its injectivity radius) to conclude that Q_∞ realizes E.

Arbitrarily short geodesics. If E has an exiting sequence $\{\gamma_n\}$ of arbitrarily short geodesics, we argue using Theorem 2.3 that $Q(X, Y_n)$ converges in the Bers boundary ∂B_X to a limit Q_∞ that realizes E.

4.4. Binding realizations

As a final detail we mention that to apply Theorem 4.1, we require a *convergent* sequence $Q(X_n, Y_n) \to N$ so that the limit $Q^- = \lim Q(X_n, Y_0)$ realizes the negative end E^- of M and the limit $Q^+ = \lim Q(X_0, Y_n)$ realizes the positive end E^+.

By an application of [Bro01], the realizations described in our discussion of Theorem 1.6 produce surfaces $\{X_n\}$ and $\{Y_n\}$ that converge up to subsequence to laminations in Thurston's compactification of Teichmüller space that *bind the surface S*.

Thus, an application of Thurston's *double limit theorem* (see [Thu86b, Thm. 4.1], [Ota96]) implies that $Q(X_n, Y_n)$ converges to a cusp-free limit N after passing to a subsequence.

5. Incompressible ends

We conclude the paper with a brief discussion of the proof of Theorem 1.2 when M is not homotopy equivalent to a closed surface.

Since M has incompressible ends, Theorem 1.3 implies that M is homeomorphic to the interior of a compact 3-manifold N. Equipped with a homotopy equivalence or *marking* $f: N \to M$, the manifold M determines an element of the *algebraic deformation space* $AH(N)$ consisting of all such marked hyperbolic 3-manifolds up to isometries preserving orientation and marking, equipped with the topology of algebraic convergence.

By analogy with the quasi-Fuchsian locus, the subset $AH(N)$ consisting of M' that are geometrically finite, cusp-free and homeomorphic to M is parameterized by the product of Teichmüller spaces

$$\mathrm{Teich}(\partial N) = \prod_{X \subset \partial N} \mathrm{Teich}(X).$$

In this situation, the cover $\widetilde{M}$ corresponding to an end E of M lies in $AH(S)$. Theorem 1.6 guarantees that if E is degenerate it is realized on a Bers boundary; indeed, since M is cusp-free and M is not homotopy equivalent to a surface, it follows that $\widetilde{M}$ is itself singly-degenerate, so Theorem 1.6 guarantees that $\widetilde{M}$ lies in a Bers compactification.

The remaining part of Theorem 1.2, then, follows from the following version of Theorem 1.7.

Theorem 5.1 (Asymptotic Isolation of Ends II). *Let M be a cusp-free complete hyperbolic 3-manifold with incompressible ends homeomorphic to* $\mathrm{int}(N)$. *Let $M_n \to M$ in $AH(N)$ be a sequence of cusp-free geometrically finite hyperbolic manifolds so that each M_n is homeomorphic to M. Let $(E^1, \ldots, E^m)$ denote the ends of M, and let $\partial M_n = X_n^1 \sqcup \ldots \sqcup X_n^m$ be the corresponding points in $\mathrm{Teich}(\partial N)$. Then, up to marking preserving bi-Lipschitz diffeomorphism, E^j depends only on the sequence $\{X_n^j\}$.*

In the case not already covered by Theorem 4.1, the covers of M_n corresponding to a fixed boundary component are quasi-Fuchsian manifolds $Q(Y_n, X_n^j)$. Their limit is

the singly-degenerate cover of M corresponding to E^j, so the surfaces Y_n range in a compact subset of Teichmüller space.

Again, it follows that the marked bi-Lipschitz diffeomorphism type of the end E does not depend on the surfaces Y_n. Theorem 1.2 then follows in this case from an application of Theorem 1.6 to each end degenerate end E of M, after an application of Sullivan's rigidity theorem [Sul81a].

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Jeffrey F. Brock

Department of Mathematics
University of Chicago
5734 S. University Ave
Chicago, IL 60637-1546
USA

brock@math.uchicago.edu

Kenneth W. Bromberg

Mathematics 253-37
Caltech
Pasadena CA 91125
USA

bromberg@its.caltech.edu