# Drilling long geodesics in hyperbolic 3-manifolds

K. Bromberg*

September 22, 2006

## 1 Introduction

Given a complete hyperbolic 3-manifold one often wants to compare the original metric to a complete hyperbolic metric on the complement of some simple closed geodesic in the manifold. In some cases this can be done by interpolating between the two metrics using hyperbolic cone-manifolds. We refer to such a deformation as drilling and results which compare the geometry of the original manifold to the geometry of the drilled manifold as *drilling theorems*. The first results of this type are due to Hodgson and Kerckhoff ([HK2]). Their work was extended from finite volume manifolds to geometrically finite manifolds in [Br1]. In [BB] a strong version of the drilling theorem was proved that gave bi-Lipschitz control between the geometry of the two manifolds. In this paper we prove a drilling theorem that allows the geodesic to be arbitrarily long with the tradeoff that it must have a very large tubular neighborhood.

We highlight two applications of this improved drilling theorem to classical conjectures about Kleinian groups. In [BS] the drilling theorem is applied to give a complete proof of the Bers-Sullivan-Thurston density conjecture. In [BBES] we give an alternate proof of the Brock-Canary-Minsky ending lamination classification ([Min], [BCM2], [BCM1]) which takes as its starting point Minsky's a priori bounds theorem ([Min]). Note that there is also an approach to the density conjecture via the ending lamination classification. One can find a more complete history of these two conjectures in the papers cited above.

We now give a more explicit description of the problem. Let $(M, g)$ be a complete hyperbolic 3-manifold and $\gamma$ a simple closed geodesic in $M$. Let $\hat{M} = M \backslash \gamma$. Kerckhoff has observed that one can apply Thurston's hyperbolization theorem to find a complete hyperbolic metric on $\hat{M}$ (see [Ko]). If $M$ is closed or finite volume then by Mostow-Prasad rigidity this metric will be unique. If $(M, g)$ has infinite volume the metric will not be unique. To get a unique metric we need some extra structure. If $(M, g)$ is *geometrically finite* then the higher genus ends of $M$ are naturally compactified by (noded) Riemann

*Supported by a grant from the NSF and a Sloan Foundation Fellowship

surfaces. There will then be a unique complete hyperbolic metric $\hat{g}$ that is geometrically finite and has the same conformal boundary as $(M, g)$.

We can then ask how $(M, g)$ compares to $(\hat{M}, \hat{g})$. For instance there is an inclusion of $(\hat{M}, \hat{g})$ into $(M, g)$. Can this map be made bi-Lipschitz? While this is clearly impossible in a neighborhood of $\gamma$, outside of tubular neighborhood of the geodesic we can control the geometry. Here is a special case of the main theorem of this paper.

**Theorem 1.1** *Given an $L > 0$ and a $K > 1$ there exists an $R > 0$ such that the following holds. Let $(M, g)$ be a geometrically finite hyperbolic 3-manifold and $\gamma$ a closed geodesic of length $\leq L$ with a tubular neighborhood $U$ of radius $R$. Let $(\hat{M}, \hat{g})$ be a geometrically finite hyperbolic structure on $M \backslash \gamma$ with the same conformal boundary as $(M, g)$. Then there exists a $K$-bi-Lipschitz embedding*

$$\phi : M \backslash U \longrightarrow \hat{M}.$$

The following fact may convince the reader why such a theorem should be true. We can produce a metric $\hat{h}$ on $\hat{M}$ of pinched negative curvature such that $\hat{h}$ and $g$ agree on $M \backslash U$. Furthermore for any $\epsilon > 0$ there is an $R > 0$ such that if the radius of the neighborhood $U$ is greater than $R$ then $\hat{h}$ can be chosen to have curvature pinched between $-1 + \epsilon$ and $-1 - \epsilon$. One might hope that there is a small deformation of $\hat{h}$ to a hyperbolic metric.

While this last fact is true and is in fact a consequence of Theorem 1.1 the metric $\hat{h}$ does not play a role in the proof of the theorem. Instead the proof uses the deformation theory of hyperbolic cone-manifolds as developed by Hodgson and Kerckhoff in a series of papers ([HK1, HK2, HK3]). While they only studied finite volume manifolds, there methods were extended to infinite volume, but geometrically finite manifolds, in [Br2, Br1]. A version of Theorem 1.1, where $\gamma$ was assumed to be very short and the bi-Lipschitz constant $K$ depends on the length of $\gamma$, was proved in [BB]. We will use all of this work in this paper.

Here is a brief account of the proof. We view $(M, g)$ is a hyperbolic cone-manifold with singular locus $\gamma$. In the special case of Theorem 1.1 the cone-angle is $2\pi$ but in general we will allow arbitrary cone angles. One then wants to see that there are local deformations decreasing the cone angle. Conditions for such deformations to exist are given in [HK1, HK3] for finite volume manifolds and this work was extended to the infinite volume case in [Br2]. The next step is to show that degenerations of the geometric structure don't occur. For instance we don't want the injectivity radius to decay to zero (see [HK2]) or the infinite volume ends to become degenerate (see [Br1]). Perhaps the most subtle issue is showing that the singular locus doesn't form self intersections. This was done in [HK2] using a packing argument that only works when the singular locus is sufficiently short. This type of argument isn't applicable in the setting of this paper and we need to develop new methods when the singular locus isn't short. Once this is done the rest of the proof follows almost exactly as is [BB].

As the main work in this paper is preventing self-intersections of the singular locus we will say a bit more about how this is done. The details are contained in section 4 and the key estimate is Proposition 4.13. Self-intersections can be prevented by showing that, throughout the deformation, the singular locus has a tubular neighborhood with radius bounded uniformly from below by a positive constant. In such a tubular neighborhood we can explicitly write down the hyperbolic metric and we can decompose an infinitesimal version of the deformation into a model deformation, which can also be explicitly described, and a correction term. As the model is explicit, a direct calculation yields how it changes the tube radius. To analyze the correction term we decompose it into a Fourier series and then derive bounds using brute force estimates. Together, these two things give the estimate in Proposition 4.13.

**Acknowledgements.** Much of this paper was written while the author was visiting IHES. He'd like to thank the Institute for its hospitality.

## 2 Background

Let $N$ be a compact 3-manifold with boundary and $\mathcal{C}$ a collection of simple closed curves in the interior of $N$. Let $M$ be the interior of $N \backslash \mathcal{C}$. A *hyperbolic cone-metric*, $g$, is a complete singular metric on the interior of $N$ that satisfies the following properties. On $M$, $g$ is a Riemannian metric with sectional curvature $-1$; i.e. a hyperbolic metric. At each point $p \in \mathcal{C}$ the metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$$

where $\theta$ is measured modulo the *cone-angle*, $\alpha$. Note that the cone-angle is constant along each connected component of $\mathcal{C}$. If the cone-angle is $2\pi$ for a component $c$ of $\mathcal{C}$ the metric is smooth at this component. For all other cone-angles the metric is singular.

A cone-metric $g$ is a smooth, but incomplete, metric on the non-singular part $M$ of $N$. If the metric is actually complete in a neighborhood of a component $c$ of $\mathcal{C}$ the we say that the cone angle is zero at $c$. As we will see below a deformation that takes the cone angle to zero will limit to a cone-metric with cone angle zero.

In this paper we will study the special class of *geometrically finite hyperbolic cone-manifolds*. To define a geometrically finite cone-manifold we first need the notion of a *complex projective structure* on a surface. Let $S$ be a surface. A complex projective structure is an atlas of charts to the Riemann sphere, $\widehat{\mathbb{C}}$, where the transition maps are restrictions of elements of $PSL_2\mathbb{C}$. Since $PSL_2\mathbb{C}$ is the group of conformal automorphisms of $\widehat{\mathbb{C}}$ a projective structure determines a conformal structure on $S$. Intuitively one can think of a projective structure as a conformal structure with the added notion of round circles.

Let $P(S)$ be the space of projective structures on $S$ and $T(S)$ the Teichmüller space of conformal structures on $S$. There is then a map $P(S) \longrightarrow T(S)$ which assigns to each projective structure its underlying conformal structure. Note that

$$\dim_{\mathbb{C}} P(S) = 2 \dim_{\mathbb{C}} T(S).$$

Both $P(S)$ and $T(S)$ have complex structures and the map $P(S) \longrightarrow T(S)$ is a holomorphic submersion.

The connection between projective structures on surfaces and hyperbolic structures on 3-manifolds is that $\widehat{\mathbb{C}}$ compactifies hyperbolic 3-space, $\mathbb{H}^3$. The group of orientation preserving isometries of $\mathbb{H}^3$, $\mathrm{Isom}^+(\mathbb{H}^3)$, is isomorphic to $PSL_2\mathbb{C}$ and isometries of $\mathbb{H}^3$ extended continuously to conformal automorphisms of $\widehat{\mathbb{C}}$.

We are now ready to define a geometrically finite cone-manifold. Let $\partial_0 N$ be the non-toral components of $\partial N$. The idea is we want the hyperbolic structure to extend to a projective structure on each component of $\partial_0 N$. Formally, we say that a hyperbolic cone-metric $g$ is a geometrically finite if each $p \in \partial_0 N$ has a neighborhood $U$ in $N$ and a map $\psi : U \longrightarrow \mathbb{H}^3 \cup \widehat{\mathbb{C}}$ such that $\psi$ is a diffeomorphism onto its image and $\psi$ restricted to $U \cap \mathrm{int}\, N$ is an isometry. Since isometries of $\mathbb{H}^3$ extend continuously to conformal automorphisms of $\widehat{\mathbb{C}}$ the restriction of $\psi$ to $U \cap \partial_0 N$ defines an atlas of charts for a projective structure on $\partial_0 N$.[1]

The projective structure determined by a geometrically finite cone-metric is the *projective boundary*. The corresponding conformal structure is the *conformal boundary*.

Let $\mathcal{GF}(N, \mathcal{C})$ be the space of geometrically finite hyperbolic cone-metrics on $(N, \mathcal{C})$. We say that two cone-metrics, $g_0$ and $g_1$, are equivalent if there is a diffeomorphism $f : (N, \mathcal{C}) \longrightarrow (N, \mathcal{C})$, isotopic to the identity, such that $g_1 = f^* g_0$. Let $GF(N, \mathcal{C})$ be space of equivalence classes of metrics.

Let

$$\Phi : GF(N, \mathcal{C}) \longrightarrow [0, \infty)^{|\mathcal{C}|} \times T(\partial_0 N)$$

be the map which assigns to each geometrically finite hyperbolic cone-manifold its cone-angles and conformal boundary.

For finite volume cone-manifolds, the following theorem is due to Hodgson and Kerckhoff ([HK1, HK3]). It was extended to geometrically finite manifolds in [Br2].

**Theorem 2.1** *Let $(M, g)$ be a geometrically finite hyperbolic cone-manifold. Assume that all cone angles are $\leq 2\pi$ or that the singular locus has tube radius $\geq \sinh^{-1} \frac{1}{\sqrt{2}}$. Then $\Phi$ is a local diffeomorphism at $(M, g)$.*

Let $(M, g)$ be a geometrically finite hyperbolic cone-manifold satisfying the conditions of Theorem 2.1 and assume that all cone-angles are $\alpha$ and that the conformal boundary of $(M, g)$ is $X$. Set

$$M_t = \Phi^{-1}(t, \ldots, t, X).$$

[1] Are definition of geometrically finite is non-standard as we are not allowing rank one cusps.

By Theorem 2.1 we know that the one-parameter family is defined for some interval $(\alpha', \alpha]$.

There are various geometric quantities and objects that we need to keep track of in the family $M_t$. Let $\mathbb{U}_t(R)$ be the radius $R$ tubular neighborhood of the singular locus in $M_t$. Let $\mathbb{T}_t(\epsilon)$ be the tubular neighborhood of the singular locus whose boundary has injectivity radius $\epsilon$. Let $R_{\max}(t)$ be the supremum of all $R > 0$ such that $\mathbb{U}_t(R)$ is a collection of embedded tubes if $R < R_{\max}(t)$. Let $L_c(t) + \imath\tau_c(t)$ be the complex length of a component $c$ of the singular locus. Here $L_c(t)$ is the length of $c$ and $\tau_c(t)$ is the twisting. Note that the twisting is only defined modulo the cone angle of $c$. We also set $L(t)$ to be the sum of the lengths of all the components of the singular locus.

We will use the following result to prevent degenerations as the cone angle decreases.

**Theorem 2.2 ([Br1])** *Let $M_t$ be a one parameter family of hyperbolic cone-manifolds in $GF(N, \mathcal{C})$ defined for $t \in (\alpha', \alpha]$. Assume that all cone angles are $t$ and the the conformal boundary of $M_t$ is fixed. If $L(t)$ is bounded above and the tube radius of the singular locus is bounded away from zero then $M_t$ limits to a cone-manifold $M_{\alpha'}$ in $GF(N, \mathcal{C})$ as $t \to \alpha'$.*

From this result we see that we need to get lower bounds on the tube radius and upper bounds on the lengths of the singular locus. In [Br1] this we done when the singular locus is sufficiently short.

**Theorem 2.3 ([Br1])** *Given $\alpha_0 > 0$ there exists an $\ell > 0$ such that if $M_\alpha$ is a geometrically finite hyperbolic cone-manifold with cone angles $\alpha \leq \alpha_0$, $L(\alpha) \leq \ell$ and $R_{\max}(\alpha) > \sinh^{-1}\frac{1}{\sqrt{2}}$ then the one-parameter family $M_t$ exists for all $t \in [0, \alpha]$.*

# 3 Families of metrics

Let $(M, g_t)$ be a smooth one-parameter family of hyperbolic metrics. Define $\eta_t \in \mathrm{Hom}(TM, TM)$ by

$$\frac{dg_t(v, w)}{dt} = 2g_t(\eta_t v, w).$$

Note that $\eta_t$ is symmetric since the metric is symmetric.

A family of metrics also determines a smooth family of *developing maps* and *holonomy representations*. The developing map

$$D_t : (\tilde{M}_t, \tilde{g}_t) \longrightarrow \mathbb{H}^3$$

is a local isometry from the universal cover to $\mathbb{H}^3$. The holonomy representation

$$\rho_t : \pi_1(M) \longrightarrow PSL_2\mathbb{C}$$

commutes with the developing map. That is

$$D_t \circ \gamma = \rho_t(\gamma) \circ D_t \tag{3.1}$$

for all $\gamma \in \pi_1(M)$.

By differentiating the developing maps we get a family of vector fields on $v_t$ constructed as follows. Let $p$ be a point in $\tilde{M}$. Then $D_t(p)$ is a smooth path in $\mathbb{H}^3$. Let $v_t(p)$ be the pullback, by $D_t$, of the tangent vector to this path at time $t$. By differentiating (3.1) we see that $v_t - \gamma^* v_t$ will be an infinitesimal isometry in the $g_t$-metric. Another important point is that $\operatorname{sym} \nabla_t v_t = \eta_t$. This follows from the fact that $\frac{dg_t(v,w)}{dt} = \mathcal{L}_{v_t} g_t(v,w)$ where $\mathcal{L}$ is the Lie derivative.

The trace of $\eta_t$ is the divergence of the vector field $v_t$. It measures the volume change of the metrics $g_t$. The traceless part of $\eta_t$ is the *strain*. It measures the change in conformal structure of the family $g_t$.

A vector field $v$ on a hyperbolic manifold is *harmonic* if

$$\nabla^* \nabla v + 2v = 0.$$

Here $\nabla^*$ is the formal adjoint of $\nabla$. The factor of 2 is necessary if we want infinitesimal isometries to be harmonic. It comes from the fact that the Ricci curvature of a hyperbolic manifold is $-2$. We say that $\eta \in \operatorname{Hom}(TM, TM)$ is a harmonic if $\eta = \operatorname{sym} \nabla v$ for a harmonic vector field $v$. If $v$ is also divergence free then $\eta$ is a harmonic strain field. For harmonic strain fields we have the following calculation of the $L^2$-norm.

**Proposition 3.1 ([HK1])** *Let $\eta$ be a harmonic strain field on a compact hyperbolic manifold with boundary $(M, g)$. Then*

$$\int_M \|\eta\|^2 + \|\nabla \eta\|^2 = -\int_{\partial M} *\nabla \eta \wedge \eta$$

*where $\partial M$ is oriented with the outward normal.*

Bounds on the $L^2$-norm of a harmonic vector or strain field can be used to give pointwise norm bounds. These bounds are similar to the mean value theorem for harmonic functions. They take a bound for the $L^2$-norm on a ball a give back a bound on the pointwise norm at the center of the ball.

Here is the version for harmonic vector fields:

**Theorem 3.2** *Let $v$ be a harmonic vector field on a ball $B$ with center $p$. Then*

$$\|v(p)\| \leq \frac{1}{\sqrt{\operatorname{vol}(B)}} \sqrt{\int_B \|v\|^2 dV}.$$

The version for harmonic strain fields is more involved:

**Theorem 3.3** *Let $\eta$ be a harmonic strain field on a ball $B$ of radius $R$ and center $p$. Then*

$$\|\eta(p)\| \leq \frac{3\sqrt{2\,\mathrm{vol}(B)}}{4\pi f(R)}\sqrt{\int_B \|\eta\|^2 dV}$$

*where* $f(R) = \cosh(R)\sin(\sqrt{2}R) - \sqrt{2}\sinh(R)\cos(\sqrt{2}R)$ *for* $R < \frac{\pi}{\sqrt{2}}$.

The mean value theorem for strain fields was proven by Hodgson and Kerckhoff. A proof can be found in [Br1]. The proof of the theorem for vector fields is an easier version of the strain field proof and it can also be found in [Br1].

# 4 In a neighborhood of the singular locus

We will work in cylindrical coordinates for a tubular neighborhood of the singular locus. Let

$$\tilde{U} = \{(r, \theta, z) | r \in \mathbb{R}^+ \text{ and } \theta, z \in \mathbb{R}\}$$

with metric

$$\tilde{g} = dr^2 + \sinh^2 r d\theta^2 + \cosh^2 dz^2.$$

Then $(\tilde{U}, \tilde{g})$ is the universal cover of $\mathbb{H}^3$ with a complete geodesic removed. Let $U$ be the quotient of $\tilde{U}$ by the two isometries

$$(r, \theta, z) \mapsto (r, \theta + \alpha, z)$$

and

$$(r, \theta, z) \mapsto (r, \theta + \tau, z + L).$$

Let $g$ be the induced metric on $U$. We also set $U(R)$ be the set of points in $U$ that are distance $R$ or less from the singular locus. Let $C(R_0, R_1) = U(R_1) - \mathrm{int}\, U(R_0)$ be a collar in $U$.

We fix an orthonormal frame and co-frame by setting

$$e_1 = \frac{\partial}{\partial r}, e_2 = \frac{1}{\sinh r}\frac{\partial}{\partial \theta} \text{ and } e_3 = \frac{1}{\cosh r}\frac{\partial}{\partial z}$$

and letting

$$\{\omega^1, \omega^2, \omega^3\}$$

be the dual co-frame. In all that follows we'll write tensors of type $(1, 1)$ as three-by-three matrices where the $ij$-terms are multiples of $\omega^i \otimes e_j$.

Let $g_t$ be a smooth one-parameter family of hyperbolic cone-metrics on $U$ with $g_0 = g$. Let $\eta$ be the time zero derivative of the $g_t$. Let $\alpha(t)$ be the cone angle, $L(t)$ the length of the singular locus and $\tau(t)$ the twisting, all for the $g_t$-metric. In particular we have $\alpha = \alpha(0)$, $L = L(0)$ and $\tau = \tau(0)$.

In these coordinates we have the following two important harmonic strain fields:

$$\eta_m = \begin{pmatrix} \frac{-1}{\cosh^2 r \sinh^2 r} & 0 & 0 \\ 0 & \frac{1}{\sinh^2 r} & 0 \\ 0 & 0 & \frac{-1}{\cosh^2 r} \end{pmatrix}$$

and

$$\eta_\ell = \begin{pmatrix} \frac{-1}{\cosh^2 r} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{\cosh^2 r+1}{\cosh^2 r} \end{pmatrix}.$$

We will also need the dual of the covariant derivative of $\eta_\ell$. One of the consequences of harmonicity is that this will again be a harmonic strain field. By direct calculation we see that

$$*\nabla\eta_\ell = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tanh r \\ 0 & -\tanh r & 0 \end{pmatrix}.$$

We have the following decomposition for $\eta$.

**Proposition 4.1 ([HK1])** *The derivative $\eta$ decomposes as*

$$\eta = C_m\eta_m + C_\ell\eta_\ell + C_\tau * \nabla\eta_\ell + \eta_c$$

*with $C_m = -\frac{\alpha'(0)}{2\alpha}$, $C_\ell = \frac{\alpha L'(0)-L}{2\alpha L}$, $C_\tau = \frac{\alpha\tau'(0)-\tau}{2\alpha L}$ and $\eta_c = \operatorname{sym}\nabla v_c$ where $v_c$ is a vector field on $U$.*

For a point $p \in U$ let $r_p(t)$ be the distance from $p$ to the singular locus in the $g_t$-metric. To control $r_p(t)$ we will bound the derivative $r'_p(t)$. The bounds on the derivative will be given by bounds on the norm of $\eta$, at least when $\eta$ is a harmonic strain field. Note that although Proposition 4.1 will hold for any $\eta$ the only way we will be able to control the geometry is when $\eta$ is harmonic. As $\eta_m$, $\eta_\ell$ and $*\nabla\eta_\ell$ are all harmonic, the form $\eta$ will be harmonic if and only if $\eta_c$ is harmonic.

Let $(\tilde{U}, \tilde{g}_t)$ be the lift of the $g_t$-metrics. Let

$$D_t : (\tilde{U}, \tilde{g}_t) \longrightarrow \mathbb{H}^3$$

be a smooth family developing maps. The metric completion of $(\tilde{U}, \tilde{g}_t)$ is obtained by adding a single complete geodesic to $\tilde{U}$. The developing maps extends continuously to the metric completion and we can assume that the added geodesic is mapped to a fixed geodesic in $\mathbb{H}^3$.

Let the vector field $v$ on $\tilde{U}$ be the time zero derivative of $D_t$. Let

$$n(p) = g_0\left(v(p), e_1\right)$$

be the component of $v$ normal to the foliation of $\tilde{U}$ consisting of surfaces equidistant to the singular locus.

The following lemma is a simple consequence of our setup.

**Lemma 4.2**

$$n(p) = r_p'(0)$$

Next we decompose the vector field $v$ as

$$v = C_m v_m + C_\ell v_\ell + C_\tau v_\tau + v_c$$

with $\operatorname{sym} \nabla v_m = \eta_m$, etc. Let $n_m$, $n_\ell$, $n_\tau$ and $n_c$ be the corresponding normal components. We bound the four terms separately with the last term being the most difficult to control.

## 4.1 $n_m(p)$

We could explicitly write down $v_m$ and $n_m$. Instead we take a more indirect approach using the radial symmetry of $\eta_m$.

Let $\gamma(s)$ be a path with $s$ defined from $a$ to $b$ and let $|\gamma|(t)$ be its length in the $g_t$-metric. We observe

$$|\gamma|(t) = \int_a^b \sqrt{g_t(\gamma'(s), \gamma'(s))}ds$$

and therefore

$$|\gamma|'(0) = \int_a^b \frac{g(\eta(\gamma'(s)), \gamma'(s))}{\sqrt{g(\gamma'(s), \gamma'(s))}}ds.$$

**Lemma 4.3**

$$n_m(p) = \tanh r_p(0) + \coth r_p(0)$$

**Proof.** Note that $n_m(p)$ depends only on $\eta_m$ and not on the family of metrics $g_t$. In particular, we can assume that $g_t$ is a family of metrics with time zero derivative $\eta = \eta_m$ and that the $g_t$ have the same symmetry as $g$. That is we assume that every isometry $(U, g)$ is also an isometry of $(U, g_t)$.

A *meridian* is a closed geodesic on one of the tori equidistant from the singular locus that bounds a (singular) disk in the solid torus. Parameterize the meridian through $p$ by $\gamma(s) = (r_p(0), s, z)$ where $s$ ranges from 0 to $\alpha$ and $p = (r_p(0), 0, z)$. Note that the symmetry implies that $\gamma$ will still be a meridian in the $g_t$ metric. The length of meridian is completely determined by it radius so the length of $\gamma$ in the $g_t$-metric is

$$|\gamma|(t) = \alpha(t) \sinh r_p(t).$$

Differentiating we see that

$$\begin{aligned} |\gamma|'(0) &= \alpha'(0)\sinh r_p(0) + \alpha\cosh r_p(0) r_p'(0) \\ &= -2\alpha\sinh r_p(0) + \alpha\cosh r_p(0) r_p'(0) \end{aligned}$$

since $\alpha'(0) = -2\alpha$ by Proposition 4.1.

We can also calculate the derivative of the length using $\eta_m$:

$$\begin{aligned} |\gamma|'(0) &= \int_0^\alpha \frac{g(\eta_m(\gamma'(s)), \gamma'(s))}{\sqrt{g(\gamma'(s), \gamma'(s))}} ds \\ &= \int_0^\alpha \frac{1}{\sinh r_p(0)} ds \\ &= \frac{\alpha}{\sinh r_p(0)} \end{aligned}$$

Setting these two expression for $|\gamma|'(0)$ equal to each other and solving for $r_p'(0) = n_m(p)$ gives us the lemma. $\boxed{4.3}$

## 4.2 $n_\ell(p)$

**Lemma 4.4**

$$n_\ell(p) = -\tanh r_p(0).$$

**Proof.** The proof is essentially the same as the proof of Lemma 4.3. As before we assume that the $g_t$ have time zero derivative $\eta = \eta_\ell$ and that the $g_t$ have the same symmetry as $g$. The path $\gamma(s)$ is a meridian in the $g_t$-metrics and we have

$$|\gamma|(t) = \alpha\sinh r_p(t)$$

since the cone angle is $\alpha$ for all $g_0$. Differentiating we have

$$|\gamma|'(0) = \alpha\cosh r_p(0) r_p'(0).$$

We also have

$$\begin{aligned} |\gamma|'(0) &= \int_0^\alpha \frac{g(\eta_\ell(\gamma'(s)), \gamma'(s))}{\sqrt{g(\gamma'(s), \gamma'(s))}} ds \\ &= \int_0^\alpha -\frac{\sinh^2 r_p(0)}{\sinh r_p(0)} ds \\ &= -\alpha\sinh r_p(0). \end{aligned}$$

Again setting the two expressions for $|\gamma|'(0)$ equal to each other and solving for $r_p'(0) = n_\ell(p)$ gives us the lemma. $\boxed{4.4}$

## 4.3 $n_\tau(p)$

**Lemma 4.5**

$$n_\tau(p) = 0.$$

**Proof.** The strain field $*\nabla\eta_\ell$ doesn't change the length of the singular locus or the cone angle so the radii of the equidistant tori is fixed. $\boxed{4.5}$

## 4.4 $n_c(p)$

As we mentioned above we can only control the size of $n_c(p)$ if we assume that $\eta$, and therefore $\eta_c$, are harmonic. As we don't have an explicit description of $\eta_c$ and $v_c$ this will be more difficult than controlling the other terms and it constitutes the main technical work of the paper.

Before stating the main proposition of this section we note that the it is easy to obtain lower bounds on the injectivity radius of a point $p$ in $U$ that is distance $r$ from the singular locus. In particular the injectivity radius at $p$ is bounded below by one half the minimum of $\ell$ and

$$\cosh^{-1}\left(\cosh^2 r - \cos\alpha \sinh^2 r\right).$$

Set the above expression equal to $\ell$ and let $m(\alpha, \ell)$ be the unique positive $r$ that solves this equation.

This subsection will be dedicated to proving the following proposition.

**Proposition 4.6** *Assume that $\eta_c$ is a harmonic strain field. Given $\ell_0$, $L_0$, $\alpha_0$ and $\alpha_1$ there exists a constant $K > 0$ such that the following holds. Let $\mathbf{m} = \max(1, m(\alpha_0, \ell_0))$. If $\ell_0 \le L \le L_0$, $\alpha_0 \le \alpha \le \alpha_1$ and $R \ge \mathbf{m} + 2\ell_0$ then*

$$|n_c(r, \theta, z)| \le Kr\sqrt{\int_{C(\mathbf{m},R)} \|\eta_c\|^2}$$

*for $r \in [\mathbf{m} + \ell_0, R - \ell_0]$.*

Before continuing we also comment on the definition of $\mathbf{m}$. By making the inner radius of the collar greater than $m(\alpha_0, \ell_0)$ we guarantee that the injectivity radius in the collar is greater than $\ell_0/2$. Since the inner radius of the collar is also $\ge 1 > \sinh^{-1}\frac{1}{\sqrt{2}}$ it will be contained in the part of the manifold where the deformation is harmonic. (See Theorem 4.12.) Finally we note the the volume of $C(\mathbf{m}, \mathbf{m} + 2\ell_0)$ is bounded above. All three of these facts will be used below.

Recall in Theorem 3.3 we saw that a bound on the $L^2$-norm of a harmonic strain field in a ball gives a pointwise bound on the norm of the strain at the center of the ball. This bound will depend linearly on the $L^2$-norm and the linearity constant will only depend on the radius of the ball. In particular, there is a constant $K_0 > 0$ so that the pointwise norm of the strain at the center of a ball of radius $\ell_0/2$ is less than $K_0$ times the $L^2$-norm of the strain on the ball. Note that if we write the strain field as a three-by-three matrix in an orthonormal framing then we have the same bound on the absolute value of each term of the matrix.

To prove the proposition we use the Fourier decomposition of $v_c$. That is we can write

$$v_c = \sum (v_{a,b} + w_{a,b})$$

where $v_{a,b}$ has the form

$$f(r)\cos(a\theta + bz) + g(r)\sin(a\theta + bz) + h(r)\sin(a\theta + bz)$$

and $w_{a,b}$ is defined similarly except the $\sin$ and $\cos$ are interchanged. The quantities $a$ and $b$ are of the form $\frac{2\pi n}{\alpha}$ and $\frac{2\pi m + a\tau}{L}$ with $n, m \in \mathbb{Z}$ and we assume that $n$ (and $a$) are non-negative.

We have the following expression for $\operatorname{sym}\nabla v_{a,b}$:

$$\begin{pmatrix} f'\cos & \frac{-af+g'}{2}\sin & \frac{-bf+h'}{2}\sin \\ & (f\coth r + ag)\cos & \frac{bg+ah}{2}\cos \\ & & (f\tanh r + bh)\cos \end{pmatrix}$$

Note that the matrix is symmetric which is why we have only written those terms on the diagonal and above. We have also eliminated the argument for the $\sin$ and $\cos$ which should be $a\theta + bz$. There is a similar expression for $\operatorname{sym}\nabla w_{a,b}$ with the $\sin$ and $\cos$ interchanged. Note that $\operatorname{sym}\nabla v_{a,b}$ and $\operatorname{sym}\nabla w_{a,b}$ are the Fourier decomposition for $\eta_c$ and $\operatorname{tr}(\operatorname{sym}\nabla v_{a,b})$ and $\operatorname{tr}(\operatorname{sym}\nabla w_{a,b})$ are the Fourier decomposition for the divergence of $v_c$. In particular $\operatorname{sym}\nabla v_{a,b}$ and $\operatorname{sym}\nabla w_{a,b}$ are harmonic strain fields and $\operatorname{tr}(\operatorname{sym}\nabla v_{a,b})$ and $\operatorname{tr}(\operatorname{sym}\nabla w_{a,b})$ are zero.

Let

$$\epsilon_{a,b}^2 = \int_{C(\mathbf{m},\mathbf{m}+2\ell_0)} \|\operatorname{sym}\nabla v_{a,b}\|^2$$

and similarly define $\delta_{a,b}$ for $w_{a,b}$. Then by Parseval's equality

$$\int_{C(\mathbf{m},\mathbf{m}+2\ell_0)} \|\eta_c\|^2 = \sum \epsilon_{a,b}^2 + \delta_{a,b}^2.$$

Note that, of course, the $L^2$-norm of $\eta_c$ on $C(\mathbf{m}, \mathbf{m} + 2\ell_0)$ is bounded by the $L^2$-norm on $C(\mathbf{m}, R)$ since $R \geq \mathbf{m} + 2\ell_0$.

**Lemma 4.7** *There exists a constant $K_1$ such that, with the exception of $w_{0,0}$, the vector fields $v_{a,b}$ and $w_{a,b}$ are pointwise bounded by $K_1\epsilon_{a,b}$ and $K_1\delta_{a,b}$, respectively, on $C\left(\mathbf{m} + \frac{1}{2}\ell_0, \mathbf{m} + \frac{3}{2}\ell_0\right)$.*

**Proof.** Every point in $C\left(\mathbf{m}+\frac{1}{2}\ell_0, \mathbf{m}+\frac{3}{2}\ell_0\right)$ is the center of a ball of radius $\ell_0/2$ that is contained in $C(\mathbf{m}, \mathbf{m}+2\ell_0)$ so the $L^2$-norm of $\operatorname{sym}\nabla v_{a,b}$ is bounded by $\epsilon_{a,b}$ on the ball. Therefore each term of $\operatorname{sym}\nabla v_{a,b}$ is bounded in absolute value by $K_0\epsilon_{a,b}$.

We first prove the estimate for $v_{0,0}$. In this case we see that the absolute value of the $\omega^2 \otimes e_2$ term of $\operatorname{sym}\nabla v_{0,0}$ is $\|v_{0,0}\| \coth r$ so $\|v_{0,0}\| \leq K_0\epsilon_{0,0}$.

The estimate for the remaining terms splits into four infinite cases: (1) $a$ and $b$ are not zero but $|b| \leq 1$, (2) $a$ and $b$ are not zero but $|b| > 1$, (3) $a \neq 0$ and $b = 0$, and (4) $a = 0$ and $b \neq 0$. To obtain bounds on $f$, $g$ and $h$ we only need to use the three lower terms of the matrix $\operatorname{sym}\nabla v_{a,b}$ each of which is bounded by $K_0\epsilon_{a,b}$. Note that if $a \neq 0$ then $|a|$ is bounded below by $\frac{2\pi}{\alpha}$. If $a = 0$ and $b \neq 0$ then $|b|$ is bounded below by $\frac{2\pi}{L}$. From here it is easy to derive the required bounds in all four cases.

The terms $w_{a,b}$ are dealt with in exactly the same way except that we do not bound $w_{0,0}$. $\boxed{4.7}$

**Remark.** There is no hope of bounding the norm of $w_{0,0}$ because it could be a (arbitrarily large) rotation about the singular locus in which case its strain will be zero. In fact, one can check to see that this is the only possibility. On the other hand, $w_{0,0}$ has no normal term so it has no effect on the size of $n_c$.

**Lemma 4.8** *There exists a constant $K_2$ such that for $p \in \partial U(\mathbf{m}+\ell_0)$ we have*

$$|n_c(p)| \leq K_2\sqrt{\int_{C(\mathbf{m},R)} \|\eta_c\|^2}.$$

**Proof.** Let $v'_c = v_c - w_{0,0}$. As we remarked above $w_{0,0}$ has no normal term so $n_c(p) = g_0(v'_c(p), e_1)$ and we only need to bound $v'_c$. We already have pointwise bounds on the terms of the Fourier decomposition of $v'_c$ but this does not directly lead to bounds on $v'_c$ itself since we can't bound the series $\sum(\epsilon_{a,b} + \delta_{a,b})$. Instead we will first bound the $L^2$-norm of $v'_c$.

By Lemma 4.7 we see that the $L^2$-norm of $v_{a,b}$ on $C\left(\mathbf{m}+\frac{1}{2}\ell_0, \mathbf{m}+\frac{3}{2}\ell_0\right)$ is bounded by

$$K_1^2\epsilon_{a,b}^2 \operatorname{vol}\left(C\left(\mathbf{m}+\frac{1}{2}\ell_0, \mathbf{m}+\frac{3}{2}\ell_0\right)\right).$$

A similar statement holds for $w_{a,b}$. Once again we use Parseval's equality to get

$$\int_{C\left(\mathbf{m}+\frac{1}{2}\ell_0,\mathbf{m}+\frac{3}{2}\ell_0\right)} \|v'_c\|^2 \leq K_1^2 \operatorname{vol}\left(C\left(\mathbf{m}+\frac{1}{2}\ell_0, \mathbf{m}+\frac{3}{2}\ell_0\right)\right) \int_{C\left(\mathbf{m}+\frac{1}{2}\ell_0,\mathbf{m}+\frac{3}{2}\ell_0\right)} \|\eta_c\|^2.$$

Note that $\operatorname{vol}\left(C\left(\mathbf{m}+\frac{1}{2}\ell_0, \mathbf{m}+\frac{3}{2}\ell_0\right)\right)$ is bounded above by a constant depending only on $L_0$ and $\alpha_1$. Finally we note that if $p \in \partial U(\mathbf{m}+\ell_0)$ then it is the center of a ball of radius

$\ell_0/2$ that is contained in $C\left(\mathbf{m}+\frac{1}{2}\ell_0, \mathbf{m}+\frac{3}{2}\ell_0\right)$. Combining these two facts, with the above inequality and Theorem 3.2 gives us the lemma. $\boxed{4.8}$

We can now prove the main proposition of this subsection.

**Proof of Proposition 4.6.** Note that $\frac{\partial n_c}{\partial r} = g_0(\nabla_{e_1} v_c, e_1)$ which is exactly the $\omega^1 \otimes e_1$ term of $\eta_c$. In particular, this derivative is bounded by

$$K_0 \sqrt{\int_{C(\mathbf{m},R)} \|\eta_c\|^2}$$

at every point in $C(\mathbf{m}+\ell_0, R-\ell_0)$. The estimate is then obtained by integrating this derivative bound and using the bound on $n_c(p)$ for $p \in \partial U(\mathbf{m}+\ell_0)$ given by Lemma 4.8. $\boxed{4.6}$

## 4.5 $L^2$-bounds

For a harmonic strain field $\eta$ we define the function $b_R(\eta)$ by

$$b_R(\eta) = -\int_{\partial U(R)} *\nabla\eta \wedge \eta.$$

By estimating $b_R(\eta)$ we will be able to control the $L^2$-norm of the harmonic strain field on a geometrically finite cone-manifolds.

Recalling the decomposition of $\eta$ in Proposition 4.1 we set $\eta_0 = \eta - \eta_c$. By doing this we have removed the trivial part of the deformation and are left with three terms that we have explicitly described. The following lemma will be key in estimating $b_R(\eta)$:

**Lemma 4.9 ([HK2])**

$$b_R(\eta) = b_R(\eta_0) + b_R(\eta_c)$$

We will always assume that our deformations are parameterized by cone angle. Therefore the constant $C_m$ given in Proposition 4.1 is $-\frac{1}{2\alpha}$. With this assumption Hodgson and Kerckhoff bound $b_R(\eta_0)$ via direct calculations. Here are their estimates.

**Lemma 4.10 ([HK2])** *If $C_m = -\frac{1}{2\alpha}$ then*

$$b_R(\eta_0) \leq \frac{2L}{\alpha^2 \sinh^2 R}.$$

*If we also have $b_R(\eta_0) \geq 0$ then*

$$2\alpha|C_\ell| = \left|\frac{\alpha L'}{L} - 1\right| \leq \frac{1}{\sinh^2 R}.$$

We now have the following corollary of the two previous lemmas and Proposition 3.1

**Corollary 4.11** *If $b_1(\eta_c) \leq 0$ then*

$$\int_{C(1,R)} \|\eta_c\|^2 + \|\nabla \eta_c\|^2 \leq \frac{2L}{\alpha^2 \sinh^2 R}.$$

The point here is that the $L^2$-norm is the difference $b_1(\eta_c) - b_R(\eta_c)$. If the first term is negative then the second term must also be negative and bounded in absolute value by $b_R(\eta_0)$.

## 4.6 Controlling the tube radius

Now we apply our work to a geometrically finite hyperbolic cone-manifold. Let $(M, g_t)$ be a one parameter family geometrically finite cone-manifolds. Let $\eta$ be the derivative of the metrics $g_t$ at time $t = 0$. We say that $\eta$ is a *Hodge form* if the following conditions hold:

1. The form $\eta$ is a harmonic strain field on $M \backslash \mathbb{U}_0(1)$.
2. The $L^2$-norm satisfies the boundary formula

$$\int_{M \backslash \mathbb{U}_0(R)} \|\eta\|^2 + \|\nabla \eta\|^2 = b_R(\eta)$$

   for all $R \in [1, R_{\max}(\alpha)]$.

3. On $\mathbb{U}_0(R_{\max}(\alpha))$ decompose $\eta$ into $\eta_0$ and $\eta_c$ as above. Then $b_1(\eta_c) \leq 0$.

The infinitesimal version of the following theorem was proven by Hodgson and Kerckhoff in [HK3] for finite volume hyperbolic cone-manifolds. (Also see [HK1].) It was extended to geometrically finite manifolds in [Br2]. The form of the theorem given below is derived in [BB] where it can be found as Corollary 6.7.

**Theorem 4.12** *Let $M_t$ be a one-parameter family of geometrically finite cone-manifolds, parameterized by the cone angle $t$. We also assume that the conformal boundary is fixed for the entire family. If $R_{\max}(t) > \sinh^{-1} \frac{1}{\sqrt{2}}$ for all $t$ then we can realize the family by metrics $g_t$ such that the derivative $\eta_t$ are Hodge forms.*

Recall that if $p$ is a point in $M$ then $r_p(t)$ is the distance in $(M, g_t)$ from $p$ to the singular locus. The proposition that follows gathers all the work of this section.

**Proposition 4.13** *Given $\ell_0$, $L_0$, $\alpha_0$ and $\alpha_1$ there exists a constant $R_0 > 0$ such that the following holds. Let $(M, g_t)$ be a one parameter family of geometrically finite cone-manifolds, parameterized by the cone angle $t$ and with fixed conformal boundary. Assume that the derivatives of the metrics $g_t$ are Hodge forms. Let $\mathbf{m} = \max(1, m(\alpha_0, \ell_0))$. If $R_{\max}(t) \geq R_0$ and $r_p(t) \in (\mathbf{m} + \ell_0, R_{\max}(t))$ then $r'_p(t) < 0$.*

**Proof.** The work we have done this section gives us the bound

$$r_p'(t) \leq -\frac{1}{2t}\left(\tanh r_p(t) + \coth r_p(t)\right) + \frac{\tanh r_p(t)}{2t\sinh R_{\max}(t)} + \frac{K\sqrt{2L_0}r_p(t)}{t\sinh R_{\max}(t)}$$

if $R_{\max}(t) \geq \mathbf{m} + 2\ell_0$, $t \in [\alpha_1, \alpha_0]$ and $r_p(t) \in (\mathbf{m} + \ell_0, R_{\max}(t) - \ell_0)$. This first term comes from Lemma 4.3 and the fact that are family is parameterized by cone angle. The second term comes from Lemma 4.4 and the second inequality in Lemma 4.10. The $K$ in the final term is from Proposition 4.6. We also use the inequality in Corollary 4.11 for the last term.

By choosing $R_0$ sufficiently large we can make the final two terms as small as we like when $R_{\max}(t) > R_0$. The only point here is that $\sinh r$ grows much faster than $r$. This will prove the proposition when $r_p(t) \in (\mathbf{m} + \ell_0, R_{\max}(t) - \ell_0)$.

We now need to prove the inequailty for final $\ell_0$-collar of the tube. This is done with three observations. First we note that there is an $\epsilon$ such that the injectivity radius of every point in the collar is $> \epsilon$. Second by choosing $R_0$ sufficiently large we can apply Lemma 4.10 to make the $L^2$-norm of $\eta$ small on $M \backslash \mathbb{U}_t(R_{\max}(t) - \ell_0 - \epsilon)$. Finally we recall that $\frac{\partial n}{\partial r} = g_t(\eta(e_1), e_1)$. Applying Theorem 3.3 to the first two facts we get a pointwise bound on $\eta$ in the collar. Using this bound and the third fact we bound the difference between $r_{p_0}'(t)$ and $r_{p_1}'(t)$ where $r_{p_0}(t) = R_{\max}(t)$ and $r_{p_1}(t) \in (R_{\max}(t) - \ell_0, R_{\max}(t))$. By the previous paragraphs we can assume that $r_{p_0}'$ is uniformly $< 0$. We have just shown that we can make the difference between $r_{p_0}'(t)$ and $r_{p_1}'(t)$ uniformly small so we also have $r_{p_1}'(t) < 0$. $\boxed{4.13}$

## 5 The Drilling Theorem

We are now ready to conclude the proof of the main theorem.

**Proposition 5.1** *Given $\ell_0 > 0$, $L_0 > 0$ and $\alpha_0 > 0$ there exists an $R_0 > 0$ such that the following holds. Let $M_t$ be a one parameter family of geometrically finite cone manifolds, parameterized by cone angle and with fixed conformal boundary. Assume that $L(\alpha_0) < L_0$ and that $R_{\max}(\alpha_0) > R_0$. Then the family $M_t$ can be defined for all $t \in [0, \alpha]$ and either $R_{\max}(t) \geq R_{\max}(\alpha_0)$ or $L(t) \leq \ell_0$.*

**Proof.** We assume that $\ell_0$ is small enough such that if $L(t) \leq \ell_0$ for any $t \leq \alpha_0$ then we can apply Theorem 2.3 to decrease the cone angle to zero.

Let $I$ be the largest connected interval with right endpoint $\alpha_0$ such that for all $t \in I$ we have $\sinh^2 R_{\max}(t) \geq 2$ and $L(t) \geq \ell_0$. By Theorem 4.12, on $I$ we can realize $M_t$ by a family of metrics whose derivatives are Hodge forms. For angles $t$ in $I$ we can integrate the second inequality in Lemma 4.10, to see that $L(t) \leq \sqrt{\frac{t}{\alpha_0}}L(\alpha_0) \leq \sqrt{\frac{t}{\alpha_0}}L_0$. The square root in the inequality comes from our assumption that $\sinh^2 R_{\max}(t) \geq 2$. Let $\alpha'$ be the left endpoint of $I$. By the above bound on $L(t)$ if $\alpha_1 = \frac{\ell_0^2 \alpha_0}{L_0^2}$ then we must have $\alpha' \geq \alpha_1$. If $\alpha' = \alpha_1$ then the length bound gives us that $L(\alpha') = \ell_0$. We need to show that this holds even if $\alpha' > \alpha_1$ for then we can allow Theorem 2.3 take over and and decrease the cone angle all the way to zero.

For all $t$ in $I$ we have $\ell_0 \leq L(t) \leq L_0$ so we can apply Proposition 4.13 to find an $R_0$ such that if $R_{\max}(t) > R_0$ then $r_p'(t) < 0$ for $r_p(t) \in (\mathbf{m} + \ell_0, R_{\max}(t))$. We can also assume that $\sinh^2 R_0 > 2$. This implies that $R_{\max}(t)$ increases as $t$ decreases. In particular, $R_{\max}(\alpha') \geq R_{\max}(\alpha_0) > \sinh^{-1}\sqrt{2}$ so we must have $L(\alpha') = \ell_0$ as desired. $\boxed{5.1}$

Once we know that the cone angle can be decreased to zero we would like to control the geometry throughout the deformation. This was done when the singular locus was short in [BB]. Here is the theorem:

**Theorem 5.2** *Given $\alpha_0 > 0$, $\epsilon_0 > 0$ and $K_0 > 1$ there exists an $\ell_0 > 0$ such that the following holds. Let $M_\alpha$ be a geometrically finite hyperbolic cone-manifold with cone angle $\alpha$. Assume that $\alpha \leq \alpha_0$, the length of the singular locus is $< \ell_0$ and that $R_{\max}(\alpha)$ is $> \sinh^{-1} 1/\sqrt{2}$. Then there a $K_0$-bi-Lipschitz diffeomorphisms*

$$\phi_t : (M_\alpha \backslash \mathbb{T}_{\epsilon_0}(\alpha), \partial \mathbb{T}_{\epsilon_0}(\alpha)) \longrightarrow (M_t \backslash \mathbb{T}_{\epsilon_0}(t), \partial \mathbb{T}_{\epsilon_0}(t))$$

*where $\phi_t$ extends to a homeomorphism between $M_\alpha$ and $M_t$ for $t \in (0, \alpha]$.*

If the length of the singular locus is bounded but not necessarily short then we have the following version of the drilling theorem.

**Theorem 5.3** *Given $\alpha_0 > 0$, $L_0 > 0$ and $K_0 > 1$ there exists an $R_0 > 0$ such that the following holds. Let $M_{\alpha_0}$ be a geometrically finite hyperbolic cone-manifold with cone angle $\alpha_0$. Assume that the length of the singular locus is $< L_0$ and that $R_{\max}(\alpha) > R_0$. Then there are $K_0$-bi-Lipschitz embeddings*

$$\phi_t : M_{\alpha_0} \backslash \mathbb{U}_{\alpha_0}(R_0) \longrightarrow M_t \backslash \mathcal{C}$$

*where $\phi_t$ extends to a homeomorphism from $M_{\alpha_0}$ to $M_t$ for $t \in (0, \alpha_0]$.*

**Proof.** By Theorem 5.2 there is an $\ell_0$ such that such if for some cone angle $\alpha' < \alpha_0$ we have $L(\alpha') \leq \ell_0$ then for all $t \leq \alpha'$ there is a $\sqrt{K_0}$-bi-Lipschitz diffeomorphism $M_{\alpha'} \backslash \mathbb{T}_{\alpha'}(\epsilon)$ to $M_t \backslash \mathbb{T}_t(\epsilon)$. We can assume that $L(\alpha_0) > \ell_0$ for otherwise the result follows from Theorem 5.2.

Now we apply Proposition 5.1 which tells that we can find an $R_1$ such that if $R_{\max}(\alpha_0) > R_1$ then the one-parameter family $M_t$ exists for all $t \in [0, \alpha_0]$ and either $R_{\max}(t) \geq R_{\max}(\alpha_0)$

or $L(t) \leq \ell_0$. As we note in the proof of the proposition $L(t)$ decreases as the cone angle decreases so there is a unique $\alpha'$ such that $L(\alpha') = \ell_0$.

We realize the one-parameter family $M_t$ by metrics $g_t$ as given by Theorem 4.12 so that the derivatives, $\eta_t$, are Hodge forms. Define $\phi'_t$ to be the identity map between $(M, g_\alpha)$ and $(M, g_t)$ for $t \in [\alpha', \alpha]$. By the proof of Proposition 5.1 we see that $\phi'_t(M_\alpha \backslash \mathbb{U}_R(\alpha))$ is contained in $M_t \backslash \mathbb{U}_R(t)$ for all $R \in [R_0, R_{\max}(\alpha_0)]$. By Lemma 4.10 we can make the $L^2$-norm of $\eta_t$ on $M_t \backslash \mathbb{U}_t(R)$ arbitrarily small by choosing $R$ to be large.

Now we need to recall some of the work in [BB]. If $p$ is a point in $M$ that is in the thick part of $(M, g_t)$ for all $t$ then a bound on the $L^2$-norm of $\eta_t$ gives a bound on the pointwise norm of $\eta_t$. Therefore if we can make the $L^2$-norm arbitrarily small then at such points we can make the bi-Lipschitz constant of $\phi_t$ arbitrarily close to one. To control the bi-Lipschitz constant in the thin part we need to modify the maps $\phi_t$. In particular, it was shown that for any $K > 1$ there is a $K' > 1$ such that if $\phi'_t$ is $K'$-bi-Lipschitz on the thick than it can be modified to be $K$-bi-Lipschitz in the thin part. Therefore there is some $R_0 > 0$ that makes the $L^2$-norm of $\eta_t$ on $M_t \backslash \mathbb{U}_t(R_0)$ sufficiently small so that the maps $\phi'_t$ can be modified diffeomorphisms $\phi_t$ that are $\sqrt{K_0}$-bi-Lipschitz on $M_{\alpha_0} \backslash \mathbb{U}_{\alpha_0}(R_0)$.

We now define $\phi_t$ when the cone angle is less than $\alpha'$. Note that we can assume that $\mathbb{U}_{\alpha'}(R_0) \supset \mathbb{T}_{\alpha'}(\epsilon)$ for if not we can simply choose $R_0$ to be larger. To define $\phi_t$ for $t \in [0, \alpha')$ we simply post-compose $\phi_{\alpha'}$ with the $\sqrt{K_0}$-bi-Lipschitz diffeomorphism from $(M_{\alpha'} \backslash \mathbb{T}_{\alpha'}(\epsilon), \partial \mathbb{T}_{\alpha'}(\epsilon)$ to $(M_t \backslash \mathbb{T}_t(\epsilon), \partial \mathbb{T}_t(\epsilon))$ given by Theorem 5.2. $\boxed{5.3}$

# 6 An application

We now describe an an application of the drilling theorem. It is conjectured that for any simple closed geodesic $\gamma$ in a smooth hyperbolic 3-manifold there is a one parameter family of cone-manifold deformations that decreases the cone angle at $\gamma$ from $2\pi$ to zero. We have the following virtual resolution of this conjecture.

**Corollary 6.1** *Let $\gamma$ be a simple closed geodesic in a hyperbolic 3-manifold $(M, g)$. Then there is a cover $(M', g')$ of $(M, g)$ such that $\gamma$ has a homeomorphic lift $\gamma'$ in $(M', g')$ and there is a one parameter family of cone deformations with singular locus $\gamma$ that takes the cone angle from $2\pi$ to zero.*

**Proof.** Using the fact that $\mathbb{Z}$-subgroups of the fundamental group of a hyperbolic 3-manifold are separable, Gabai [Ga] observed that given any $R > 0$ one can find a cover $(M', g')$ of $(M, g)$ such that $\gamma$ lifts homeomorphically to a simple closed geodesic $\gamma'$ such that $\gamma'$ has a tubular neighborhood of radius $R$. Then result then follows from Proposition 5.1. $\boxed{6.1}$

## References

[BB] J. Brock and K. Bromberg. On the density of geometrically finite Kleinian groups. *Acta Math.* **192**(2004), 33–93.

[BBES] J. Brock, K. Bromberg, R. Evans, and J. Souto. *In preparation.*

[BCM1] J. Brock, R. Canary, and Y. Minsky. *In preparation.*

[BCM2] J. Brock, R. Canary, and Y. Minsky. The classification of Kleinian surfaces groups II: the ending lamination conjecture. *2004 Preprint.*

[Br1] K. Bromberg. Hyperbolic cone-manifolds, short geodesics and Schwarzian derivatives. *J. Amer. Math. Soc.* **17**(2004), 783–826.

[Br2] K. Bromberg. Rigidity of geometrically finite hyperbolic cone-manifolds. *Geom. Dedicata* **105**(2004), 143–170.

[BS] K. Bromberg and J. Souto. The density conjecture: A pre-historic approach. *In preparation.*

[Ga] D. Gabai. Homotopy hyperbolic 3-manifolds are virtually hyperbolic. *J. Amer. Math. Soc.* **7**(1994), 193–198.

[HK1] C. Hodgson and S. Kerckhoff. Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery. *J. Diff. Geom.* **48**(1998), 1–59.

[HK2] C. Hodgson and S. Kerckhoff. Universal bounds for hyperbolic Dehn surgery. *Annals of Math.* **162**(2005), 367–421.

[HK3] C. Hodgson and S. Kerckhoff. The shape of hyperbolic Dehn surgery space. *In preparation.*

[Ko] S. Kojima. Deformations of hyperbolic 3-cone manifolds. *J. Diff. Geom.* **49**(1998), 469–516.

[Min] Y. Minsky. The classification of Kleinian surface groups I: models and bounds. *2002 Preprint.*

Department of Mathematics
University of Utah
155 S 1400 E, JWB 233
Salt Lake City, UT 84112