Haar Functions

Define functions $e_n^k : [0,1] \to \mathbb{C}$ by

$$e_n^k(x) = \begin{cases} 2^{n/2} & \frac{k-1}{2^n} \leq x < \frac{k-1/2}{2^n} \\ -2^{n/2} & \frac{k-1/2}{2^n} \leq x \leq \frac{k}{2^n} \\ 0 & \text{otherwise} \end{cases}$$

for integers $n = 0, 1, \ldots$ and $1 \leq k \leq 2^n$. Define $e_0^0(x) = 1$.

These functions are called *Haar functions* and form an orthonormal basis of $L^2[0,1]$. We saw in class that an orthonormal set is a basis if and only the following holds:

If $f \in L^2[0,1]$ and $(f, e_i^j) = 0$ for all e_i^j then $f = 0$.

Below are some problems that work through the proof of this fact. You should try to do all of them but I've tried to write it so each problem is essentially self-contained.

1. Show that the e_n^k form an orthonormal set. That is show that $(e_n^k, e_n^k) = 1$ and $(e_n^k, e_i^j) = 0$ if $n \neq i$ or $k \neq j$.

2. Show that

$$\begin{aligned} (\chi_{[0,3/4]}, e_0^0) &= 3/4 \\ (\chi_{[0,3/4]}, e_0^1) &= 1/4 \\ (\chi_{[0,3/4]}, e_1^2) &= \sqrt{2}/4. \end{aligned}$$

Show for all other e_n^k that $(\chi_{[0,3/4]}, e_n^k) = 0$. Show that

$$\chi_{[0,3/4]} = \frac{3}{4} e_0^0 + \frac{1}{4} e_0^1 + \frac{\sqrt{2}}{4} e_1^2.$$

Make similar calculations for $\chi_{[0,1/2]}$, $\chi_{[1/2,1]}$ and $\chi_{[0,1/4]}$. You should get

$$\chi_{[0,1/2]} = \frac{1}{2} e_0^0 + \frac{1}{2} e_0^1,$$

$$\chi_{[0,1/2]} = \frac{1}{2} e_0^0 - \frac{1}{2} e_0^1,$$

and

$$\chi_{[0,1/4]} = \frac{1}{4} e_0^0 + \frac{1}{4} e_0^1 + \frac{\sqrt{2}}{4} e_1^1.$$

3. If $k \le 2^n$ show that

$$e_{n+1}^k(x) = \begin{cases} \sqrt{2}e_n^k(2x) & x \le 1/2 \\ 0 & x > 1/2 \end{cases}$$

and if $k > 2^n$ show that

$$e_{n+1}^k(x) = \begin{cases} \sqrt{2}e_n^{k-2^n}(2x-1) & x \ge 1/2 \\ 0 & x < 1/2. \end{cases}$$

4. We want to prove that every characteristic function of the form $\chi_{[0,k/2^n]}$ is a finite linear combination of the e_i^j. This is a bit tricky and we will need to use induction.

   We will induct on n. Let P_n be the statement that for all integers k with $1 \le k \le 2^n$ the characteristic function $\chi_{[0,k/2^n]}$ is finite linear combination of the e_i^j.

   To use induction we need to show that two things are true. First we need to see that P_0 is true. Then we need to show that if P_n is true then so is P_{n+1}.

   (a) Show that P_0 is true. This should be very easy! Also note that by problem 2 we know that P_1 and P_2 are also true.

   (b) Show that $\chi_{[0,2c]}(2x) = \chi_{[0,c]}(x)$.

   (c) Now we begin the heart of the problem. Assume that P_n is true. We use this to show that $\chi_{[0,k/2^{n+1}]}$ is a finite linear combination the e_i^j. We first do the case when $k \le 2^n$.

   Since P_n is true there are finitely many constants c_i^j and functions e_i^j such that

   $$\chi_{[0,k/2^n]}(x) = \sum c_i^j e_i^j(x).$$

   From (b) we know that $\chi_{[0,k/2^{n+1}]}(x) = \chi_{[0,k/2^n]}(2x)$. Use this and problem 3 to show that

   $$\chi_{[0,k/2^{n+1}]}(x) = \begin{cases} \sum c_i^j e_i^j(2x) & x \le 1/2 \\ 0 & x > 1/2 \end{cases}$$

   and therefore

   $$\begin{aligned} \chi_{[0,k/2^{n+1}]}(x) &= \frac{1}{\sqrt{2}} \sum_{(i,j)\neq(0,0)} c_i^j e_{i+1}^j(x) + c_0^0 \chi_{[0,1/2]}(x) \\ &= \frac{1}{\sqrt{2}} \sum_{(i,j)\neq(0,0)} c_i^j e_{i+1}^j(x) + \frac{c_0^0}{2}(e_0^0(x) + e_0^1(x)). \end{aligned}$$

   Note that problem 3 doesn't apply to e_0^0 and this part of the sum needs to be dealt with separately.

(d) Now we need to take care of the case when $k > 2^n$. We first observe that if $k > 2^n$ then

$$\chi_{[0,k/2^{n+1}]} = \chi_{[0,1/2]} + \chi_{[1/2,k/2^{n+1}]}$$

as functions in $L^2[0,1]$. Note that in the usual sense these functions aren't equal since

$$\chi_{[0,k/2^{n+1}]}(1/2) = 1 \neq 2 = \chi_{[0,1/2]}(1/2) + \chi_{[1/2,k/2^{n+1}]}(1/2).$$

We already know that $\chi_{[0,1/2]}$ is a finite linear combination of e_0^0 and e_0^1 so we only need to show that $\chi_{[1/2,k/2^{n+1}]}$ is a finite linear combination of the e_i^j. Observe that

$$\chi_{[1/2,k/2^{n+1}]}(x) = \chi_{[0,\frac{k-2^n}{2^n}]}(2x-1).$$

Applying the assumption that P_n is true we have

$$\chi_{[0,\frac{k-2^n}{2^n}]}(x) = \sum c_i^j e_i^j(x)$$

where the right-hand side is a finite sum.

We now use the second half of problem 3 to see that

$$\chi_{[1/2,k/2^{n+1}]}(x) = \begin{cases} \sum c_i^j e_i^j(2x-1) & x \geq 1/2 \\ 0 & x < 1/2 \end{cases}$$

and therefore

$$\begin{aligned} \chi_{[1/2,k/2^{n+1}]}(x) &= \frac{1}{\sqrt{2}} \sum_{(i,j)\neq(0,0)} c_i^j e_{i+1}^{j+2^n}(x) + c_0^0 \chi_{[1/2,1]}(x) \\ &= \frac{1}{\sqrt{2}} \sum_{(i,j)\neq(0,0)} c_i^j e_{i+1}^{j+2^n}(x) + \frac{c_0^0}{2}(e_0^0(x) - e_0^1(x)). \end{aligned}$$

5. Let $f \in L^2[0,1]$ such that $(f, e_i^j) = 0$ for all e_i^j. Show that

$$\int_0^{k/2^n} f(x)dx = 0$$

for all non-negative integers n with $1 \leq k \leq 2^n$.

6. A *diadic rational* is a number of the form $k/2^n$ where n is a positive integer and k is an integer. Show that the diadic rationals are dense in $\mathbb{R}$. Here is one way to do this:

(a) Let x and y be real numbers with $x > y$. Show that there exists a positive integer n such that $2^n x - 2^n y > 1$.

(b) Let x and y be real numbers with $x - y > 1$. Show that there exists an integer k such that $x > k > y$. (Hint: Let k be the smallest integer such that $k > y$. Why does such a k exist?)

(c) Use (a) and (b) to show that if x and y are real numbers with $x > y$ then there exists a diadic rational $k/2^n$ with $x > k/2^n > y$.

(d) Let x be a real number. By (c) for a positive integer i we can find a diadic rational $k_i/2^{n_i}$ such that $x - 1/i < k_i/2^{n_i} < x$. Show that $k_i/2^{n_i} \to x$ and conclude that the diadic rationals are dense in $\mathbb{R}$.

7. Let A be a dense subset of $\mathbb{R}$. Show that for every x in $\mathbb{R}$ there is an increasing sequence $a_i \in A$ such that $a_i \to x$. (Hint: Fix some $a_0 \in A$ with $a_0 < x$. Now inductively define the sequence by choosing a_i such that $\max\{a_{i-1}, x - 1/i\} < a_i < x$.)

8. Let x be a positive real number and x_i an increasing sequence of positive real numbers such that $x_i \to x$. Let $f : [0, x] \to \mathbb{R}$ be a non-negative measurable function. Show that

$$\int_0^{x_i} f(y)dy \to \int_0^x f(y)dy.$$

(Hint: Apply the Lesbegue Monotone Convergence Theorem to the functions $f\chi_{[0,x_i]}$ on the interval $[0, x]$.)

Now let $f : [0, x] \to \mathbb{R}$ be any measurable function such that

$$\int_0^x |f(y)|dy < \infty$$

and show that

$$\int_0^{x_i} f(y)dy \to \int_0^x f(y)dy.$$

9. Let $f \in L^2[0, 1]$ such that $(f, e_i^j) = 0$ for all e_i^j. Use (5-8) to show that

$$\int_0^x f(y)dy = 0$$

for all $x \in [0, 1]$ and therefore

$$\int_{x_0}^{x_1} f(y)dy = 0$$

for all $x_0, x_1 \in [0,1]$ or more generally if A is a finite union of intervals in $[0,1]$ then

$$\int_A f(y)dy = 0.$$

10. We need to recall a few things from measure theory. The *symmetric difference* of two sets is

$$S(A,B) = (A - B) \cup (B - A).$$

We then define a distance between two sets by

$$d(A,B) = d(S(A,B)).$$

An *elementary subset* of $\mathbb{R}$ is a finite union of bounded intervals. Recall that for any measurable subset E of $\mathbb{R}$ with $m(E) < \infty$ then there exist elementary sets A_i such that $d(A_i, E) \to 0$.

(a) Find a subsequence A_{i_k} such that $d(A_{i_k}, E) < 1/2^k$ and then note that

$$\sum_k d(A_{i_k}, E) = \sum_k m(S(A_{i_k}, E)) < \infty.$$

(b) Show that $\chi_{A_{i_k}} \to \chi_E$ pointwise almost everywhere and therefore $f\chi_{A_{i_k}} \to f\chi_E$ pointwise a.e. for any function f. (Hint: Apply the Borel-Cantelli Lemma to the sets $S(A_{i_k}, E)$.)

11. We now want to show that for any measurable subset E of $[0,1]$ that $\int_E f(x)dx = 0$.

(a) Note that

$$\int_0^1 |f(x)|dx \le \left| \int_0^1 f(x)dx \right| \le |(f, e_0^0)| \le \|f\|_2 < \infty.$$

We want to apply Lebesgue's Dominated Convergence with $|f|$ the bounding function.

(b) Let A_i be elementary sets with $d(A_i, E) < 1/2^i$. Apply (10) and the dominated convergence theorem to the functions $f\chi_{A_i}$ and $f\chi_E$ to see that

$$\int_{A_i} f(x)dx \to \int_E f(x)dx$$

and conclude that $\int_E f(x)dx = 0$.

12. We are now ready to finish. Again let $f \in L^2[0,1]$ such that $(f, e_i^j) = 0$ for all e_i^j. Conclude that for every measurable set $E \subset [0,1]$ we have $\int_E f(x)dx = 0$. (This should just be putting together the previous problems.)

    Now define functions $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{-f, 0\}$ and sets $E^+ = \{x \in [0,1] | f(x) \geq 0\}$ and $E^- = \{x \in [0.1] | f(x) < 0\}$. Note that

    (a) $f = f^+ - f^-$;

    (b) $\int_0^1 f^+(x)dx = \int_{E^+} f^+(x)dx = \int_{E^+} f(x)dx = 0$;

    (c) $\int_0^1 f^-(x)dx = \int_{E^-} f^-(x)dx = \int_{E^-} f(x)dx = 0$;

    We have seen in class that if the integral of a non-negative function is 0 then the function is 0 a.e. Conclude that f^+ and f^- are 0 a.e. and therefore f is 0 a.e.