Notes and problems on compactness

Let $\mathcal{O}$ be a collection of open sets in $\mathbb{R}^n$. Then $\mathcal{O}$ is an *open cover* of a set $A \subset \mathbb{R}^n$ if $A \subset \bigcup_{U \in \mathcal{O}} U$.

A set K is *compact* if every open cover has a finite subcover. That is K is compact if for every open cover $\mathcal{O}$ there are sets $U_1, \dots, U_k \in \mathcal{O}$ such that

$$K \subset \bigcup_{i=1}^{k} U_i.$$

Theorem 1 *A compact set is closed.*

Proof. We will prove the contrapositive. Assume that A is not closed. We will construct an open cover that has no finite subcover. Since A is not closed there exists a sequence $\{x_i\}$ in A that converges to some $x \notin A$. Note that

$$\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) \bigcup \{x\}$$

is closed set so its complement, which we denote U, is open. Let $\mathcal{O}$ be the collection of balls $B_{d(x_i,x)/2}(x_i)$ and the set U. Then $\mathcal{O}$ is an open cover of A. We will show that $\mathcal{O}$ has not finite subcover.

Let $\mathcal{O}'$ be a finite subcollection of the open sets in $\mathcal{O}$. Since $\mathcal{O}'$ contains only finite many sets there exists an N such that if $i > N$ then $B_{d(x_i,x)/2}(x_i)$ is not in $\mathcal{O}'$. Let $\epsilon = \min\{d(x_1, x)/2, \dots, d(x_N, x)/2\}$. Since $x_i \to x$ there exists an n_0 such that $d(x_{n_0}, x) < \epsilon$. By the triangle inequality $d(x_i, x) \le d(x_i, x_{n_0}) + d(x_{n_0}, x)$ and after rearranging this becomes $d(x_i, x_{n_0}) \ge d(x_i, x) - d(x_{n_0}, x)$. If $i \le N$ then $d(x_i, x) \ge 2\epsilon$ so we have $d(x_i, x_{n_0}) > 2\epsilon - \epsilon = \epsilon$. In particular $x_{n_0} \notin B_\epsilon(x_i) \subset B_{d(x_i,x)/2}(x_i)$. Since x_{n_0} is also not in U the open sets in $\mathcal{O}'$ cannot cover A and $\mathcal{O}$ has no finite subcover. $\boxed{1}$

Theorem 2 *If A is a subset of K, A is closed and K is compact then A is compact.*

Proof. Let $\mathcal{O}$ be an open cover of A. Let $\mathcal{O}'$ be all of the open sets in $\mathcal{O}$ and the open set A^c. Then $\mathcal{O}'$ is an open cover of K and therefore there are finitely many open sets $U_1, \dots, U_n$, each in $\mathcal{O}'$, that cover K. If A^c is not one of the U_i then all of the U_i are in $\mathcal{O}$ and they are a finite subcover. If A^c is one of the U_i, say U_n, then $U_1, \dots, U_{n-1}$ are all in $\mathcal{O}$. But $U_1, \dots, U_{n-1}$ are also a finite subcover of A because if $x \in A \subset K$ then $x \in U_i$ for some i since the U_i cover K. Since $x \notin U_n = A^c$ we must have $x \in U_i$ for some $i \le n - 1$ and therefore the $U_1, \dots, U_{n-1}$ cover A. $\boxed{2}$

Theorem 3 *Let K_i be non-empty compact sets with $K_{i+1} \subset K_i$. Then*

$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset.$$

Proof. We assume the intersection is empty and we will obtain a contradiction. The sets K_i are closed and hence compact so the sets $U_i = K_i^c$ are open. Since

$$\bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} K_i^c = \left(\bigcap_{i=1}^{\infty} K_i\right)^c = \emptyset^c = \mathbb{R}^n \supset K_1,$$

the collection $\{U_i\}$ is an open cover of K_1. Since

$$\bigcup_{i=1}^{n} U_i = U_n = (K_n)^c$$

no finite subcollection of the U_i covers K_1. This contradicts the compactness of K_1 so the intersection must be non-empty. $\boxed{3}$

Theorem 4 *Let $I_n = [a_n, b_n]$ be a sequence of nested intervals, i.e. $I_{n+1} \subset I_n$ for all n. Show that*

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof. Let n and m be positive integers with $n \leq m$. Then $I_n \subset I_m$ so $a_n \leq a_m \leq b_m \leq b_n$. In particular $a_i < b_j$ for all i and j. This implies that

$$a = \sup\{a_i\} \leq b_i$$

for all i. By the definition of the supremum we also have $a \geq a_i$ for all i so $a \in I_i$ for all i and the intersection is non-empty. $\boxed{4}$

A *closed n-cell* is a product of closed intervals. That is

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

is a closed n-cell.

Problem 1 Show that a nested family of closed n-cells has a non-empty intersection.

Theorem 5 *A closed n-cell Q is compact.*

Proof. We will assume Q is not compact. Then there exists an open cover, $\mathcal{O}$, of Q that contains no finite subcover. We will construct a sequence of nested, closed n-cells $Q_0 \supset Q_1 \supset Q_2 \dots$ with the property that for each Q_i the collection $\mathcal{O}$ is a cover with no finite subcover and such that $\mathrm{diam}(Q_i) \to 0$.[1]

Assuming we have constructed the Q_i we can finish the proof. By Problem 1 the intersection

$$Q_\infty = \bigcap_{i=0}^{\infty} Q_i$$

is non-empty. We claim that Q_∞ contains only one point. Let x and y be points in Q_∞. Since $\mathrm{diam}(Q_i) \to 0$ for $\epsilon > 0$ there exists an k such that $\mathrm{diam}(Q_k) < \epsilon$. Both x and y are in Q_k so $d(x, y) < \epsilon$ and as ϵ is arbitrary we must have $d(x, y) = 0$. Therefore $x = y$ and Q_∞ contains only one point which we label q.

Let U be an open set in the collection $\mathcal{O}$ with $q \in U$. Since U is open there exists a $\delta > 0$ such that $B_\delta(q) \subset U$. Again using the fact that $\mathrm{diam}(Q_i) \to 0$ we can find an m such that $\mathrm{diam}(Q_m) < \delta$. By the definition of diameter, if A is a set with $d > \mathrm{diam}(A)$ and $x \in A$ then $A \subset B_d(x)$. In particular, $Q_m \subset B_\delta(q) \subset U$. This gives us a contradiction since $\{U\}$ is a finite subcover of Q_m.

Now we need to construct the Q_i. We will do so inductively. We begin by setting $Q_0 = Q$. By assumption $\mathcal{O}$ has no finite subcover of Q_0. The n-cell Q_0 is the product of n-intervals. We can assume the longest interval has length ℓ.

Now assume we have constructed nested, closed n-cells $Q_0 \subset Q_1 \subset \dots \subset Q_{k-1}$ such that $\mathcal{O}$ has no finite subcover on any of the Q_i and the length of the longest side of Q_i is $2^{-i}\ell$. To choose Q_n we subdivide Q_{k-1} into 2^n closed n-cells which we label $Q_{k,1}, \dots, Q_{k,2^n}$. The $Q_{k,i}$ are of the following form. The n-cell Q_{n-1} is the product of n intervals, $[a_1, b_1], \dots, [a_n, b_n]$. Let c_i be the midpoint of $[a_i, b_i]$. Then each $Q_{k,i}$ is a product $I_1 \times \dots \times I_n$ with each I_j either the interval $[a_j, c_j]$ or the interval $[c_j, b_j]$. For each I_j there are two choices of intervals and there are n intevrals I_j so there are exactly 2^n possible $Q_{k,i}$. Note that $Q_{k-1} = \bigcup Q_{k,i}$ so if $\mathcal{O}$ has a finite subcover for each the $Q_{k,i}$ then $\mathcal{O}$ has a finite subcover on Q_{k-1}. Since we are assuming this is not true there is some Q_{k,i_k} such that $\mathcal{O}$ doesn't have a finite subcover on Q_{k,i_k}. Let $Q_k = Q_{k,i_k}$.

To finish the construct of the Q_i we need to calculate the length of the longest interval in product Q_k. This is easy to do since the length of the intervals in the product that forms Q_k are exactly half the length of the intervals in Q_{k-1}. Therefore the length of the longest interval is $2^{-1} \times 2^{-(k-1)}\ell = 2^{-k}\ell$ and we have inductively found nested, closed n-cells Q_i with the length of the longest interval in each Q_i exactly $2^{-i}\ell$. An application

[1] The *diameter* of a set A is defined to be $\mathrm{diam}(A) = \inf\{d | \text{if } x, y \in A \text{ then } d(x, y) \le d\}$.

of the triangle inequality shows that $\operatorname{diam}(Q_i) \leq n2^{-i}\ell$ so $\operatorname{diam}(Q_i) \to 0$ as $i \to \infty$. $\boxed{5}$

Problem 2 Let x_n be a sequence with no convergent subsequence. Show that the set $\{x_1, x_2, \dots\}$ is closed

Problem 3 A point x is *isolated* in a set $A \subset \mathbb{R}^n$ if there exists an $\epsilon > 0$ such that $B_\epsilon(x) \bigcap A = \{x\}$. Show that x is isolated if and only if there doesn't exists a sequence of distinct points $x_i \in A$ with $x_i \to x$.

Theorem 6 *Let K be a subset of $\mathbb{R}^n$. The following are equivalent:*

1. *K is closed and bounded;*
2. *K is compact;*
3. *Every sequence in K has a subsequence that converges in K.*

Proof. (1 ⇒ 2) A bounded set is contained in some closed n-cell Q. By Theorem 5. Since K is a closed subset of a compact set K is compact by Theorem 2.

(2 ⇒ 3) Let x_n be a sequence in K. If the sequence has a convergent subsequence then the limit is in K since K is compact and therefore closed. In this case we are done.

Now we assume the sequence has no convergent subsequence and we will obtain a contradiction. Then by Problem 2 the set $C = \{x_1, x_2, \dots\}$ is closed. By Theorem 2, C is also compact. Problem 3 implies that every point in C is isolated. In particular, for each x_i there is an ϵ_i such that $B_{\epsilon_i}(x_i) \bigcap C = \{x_i\}$. The collection

$$\mathcal{O} = \{B_{\epsilon_1}(x_1), B_{\epsilon_2}(x_2), \dots\}$$

is an open cover of C. However if we remove any of the $B_{\epsilon_i}(x_i)$ from $\mathcal{O}$ we no longer have an open cover since x_i is not in any of the open subsets. Therefore $\mathcal{O}$ has no finite subcover, contradicting the compactness of C.

(3 ⇒ 1) We will prove the contrapositive. If K is not closed there exists a sequence $\{x_i\}$ in K such that $x_i \to x$ but $x \notin K$. Every subsequence $\{x_i\}$ will then also converge to x so $\{x_i\}$ has no subsequence that converges in K.

If K is not bounded, for each i we can find an $x_i \in K$ such that $d(x_i, 0) > i$. Given an i choose j such that $j_0 > d(x_i, 0) + 1$. Then for all $j > j_0$, $d(x_i, x_j) \geq d(x_j, 0) - d(x_i, 0) > j_0 - d(x_i, 0) > 1$. This implies that $\{x_i\}$ has no Cauchy, and therefore no convergent, subsequence. $\boxed{6}$

We now define the Cantor set, C, in a way somewhat different than was done in class. Define

$$C = \left\{ x \in [0,1] | x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \text{ where } a_i \in \{0,2\} \right\}.$$

Some examples of points in C are $2/3$ and $2/9$. It is less obvious, but $1/3$ is also in C since $1/3 = \sum_{i=2}^{\infty} 2/3^i$.

Problem 4 Show that the Cantor set is:

1. closed;
2. has no interior;
3. has no isolated points;
4. is uncountable.