Problem #1 is from section 1.2, #10 which was on our first homework.

#2 This problem is incorrect as $K = \mathbb{Q}$. To see this let $x \in \mathbb{Q}$ and we will show that $x \in K$.

~~Since $L \neq \mathbb{Q}$ $\exists$ $x \notin L$.~~

Since L is a Dedekind cut, $L \neq \emptyset$ so $\exists\, y \in L$. If $x+y < 0$ then $x \in K$ and we are done. If $x+y \geq 0$ then $x+y+1 > 0$ so $y-(x+y+1) < y$ & therefore $y-(x+y+1) = -x-1 \in L$. Since $x + (-x-1) = -1 < 0$ this implies that $x \in K$.

#3 Let $r = \frac{p}{q}$ where p & q are relatively prime. Then

$$2\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \quad + a_0 = 0$$

so $2\left(\frac{p}{q}\right)^n = -\left(a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_0\right)$.

If we multiply both sides by q^n we have

$$2p^n = -q^n\left(a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_0\right).$$

Note that

$$K = -q^{n-1}\left(a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_0\right)$$

$$= -\left(a_{n-1}p^{n-1} + a_{n-2}\,q\,p^{n-2} + \dots + q^{n-1}a_0\right)$$

is an integer so

$$2p^n = qK.$$

Assume that $q \neq 1$.
Let q' be a prime s.t $q' \mid q$.
Then $q' \mid 2p^n \Rightarrow q' \mid 2$ or $q' \mid p^n$.
If $q' \mid p^n \Rightarrow q' \mid p$ but since q & p are relatively prime $q' \nmid p$ & thus $q' \mid 2$. This implies that $q' = 2$.

Divide both sides of $2p^n = qK$ by 2 we see that $p^n = \frac{q}{2}K$. Where $\frac{q}{2}$ is an integer. If $\frac{q}{2} \neq 1$ then there exist another prime q'' s.t $q'' \mid \frac{q}{2}$. But then $q'' \mid q$ (since $2q'' = q$) & $q'' \mid p^n \Rightarrow q'' \mid p$. Again since p & q are relatively prime this is impossible so we must have $\frac{q}{2} = 1$.

We have shown that $q=1$ or $q=2$. Therefore $2\frac{p}{q}$ is an integer.