Homework 1, Math 5520
Spring 2018
Due 1/31/18

1. Let X be a topological space and $\sim$ an equivalence relation and let $Y = X/\sim$ be the quotient with $\pi\colon X \to Y$ the quotient map. Let $\mathcal{T}$ be the collection of sets $U \subset Y$ where $\pi^{-1}(U)$ is open. Show that $\mathcal{T}$ is a topology on Y and that π is continuous for this choice of topology.

2. Let X be a topological space and homeo(X) be the group of homeomorphisms of X. Assume that $G \subset$ homeo(X) is a subgroup such that every $x \in X$ has a neighborhood U with $g(U) \cap U \neq \emptyset$ if and only if $g = id$. Let $\sim$ be the equivalence relation given by $x_0 \sim x_1$ if and only if there exists a $g \in G$ with $g(x_0) = x_1$ and let $\pi\colon X \to Y = X/\sim$ be the quotient map. Show that $\pi(U)$ is a neighborhood of $\pi(x)$ (where x and U are as above) in Y that is homeomorphic to U. If X is a surface conclude that Y is a surface.

3. A polyhedron is regular if every face has the same number of sides and every vertex has the same valence. Show that the tetrahedron, cube, octahedron, dodecahedron and icosahedron are the only regular polyhedron.

4. Show that $T^2 \# P^2$ is homeomorphic to $P^2 \# P^2 \# P^2$. Here P^2 is the projective plane.

5. Prove that the Euler characteristic is topological invariant for non-orientable surfaces.

6. An *n-manifold* is a topological space M such that every point in M has a neighborhood homeomorphic to $\mathbb{R}^n$. Define a triangulation of a compact 3-manifold. One can then define the Euler characteristic $\chi(M)$ to be the sum of the number of even dimensional simplices minus the number of the odd-dimensional simplices. Show that $\chi(M) = 0$ for all compact 3-manifolds M. (Hint: For any vertex v in the triangulation take a small neighborhood of v that is homeomorphic to a ball. The boundary of this ball will a copy of the 2-sphere, S^2. This S^2 will inherit a triangulation from the triangulation of M; the intersection of S^2 with each 3-simplex, will be a 2-simplex on S^2, the intersection of a 2-simplex will be a 1-simplex (edge) and the the intersection of each edge will be a vertex. Now use the fact that $\chi(S^2) = 2$ to find $\chi(M)$.)

   This fact is part of a larger phenomena. Given any cell structure on an n-manifold there is a *dual structure* where each k-cell becomes an $n - k$-cell. Assuming this and that the Euler characteristic is a topological invariant one can also prove that $\chi(M) = 0$ for all odd-dimensional manifolds.