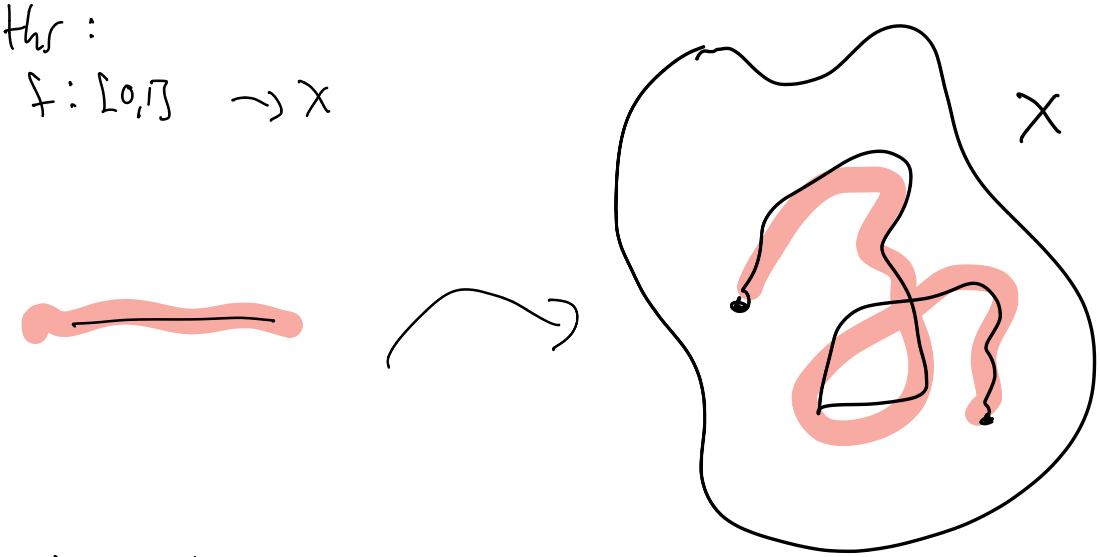
Paths:

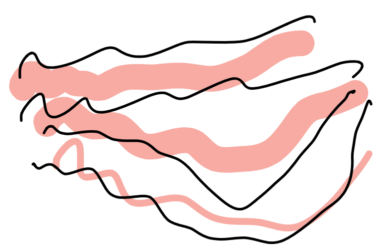
$f : [0,1] \to X$



X topological space

- will usually assume X is path connected.

Homotopy of paths.

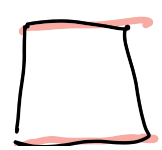


Perturbation of path

DEFN 2 paths $f : [0,1] \to X$ and $g : [0,1] \to X$ are homotopic

if $\exists\ F : [0,1] \times [0,1] \to X$

s.t $f = f_0$ & $g = f_1$



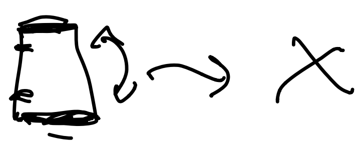
with $f_t(s) = F(s,t)$.

Notation If f and g are homotopic we write $f \simeq g$.

LEMMA Homotopy is an equivalence relation:

1. • $f \simeq g \iff g \simeq f$
2. • $f \simeq f$
3. • $f \simeq g$ & $g \simeq h \Rightarrow f \simeq h$

Proof

1. $f \simeq g \Rightarrow \exists$ 

$F : [0,1] \times [0,1] \longrightarrow X$ with $f = f_0$ & $g = f_1$.

Let $G : [0,1] \times [0,1] \longrightarrow X$ be defined by

$$G(s,t) = F(s, 1-t).$$

$\Rightarrow \quad g \simeq f$

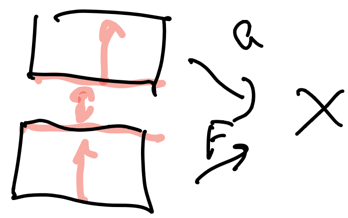
② For $f \sim f$ we define the homotopy by $F(s,t) = f(s)$. (constant homotopy).

③ Transitivity: $f \simeq g$ & $g \simeq h$ so we have

$F(s,t)$ & $G(s,t)$ with

$f(s) = F(s,0)$, $g(s) = \underline{F(s,1) = G(s,0)}$

& $h(s) = G(s,1)$.



Define

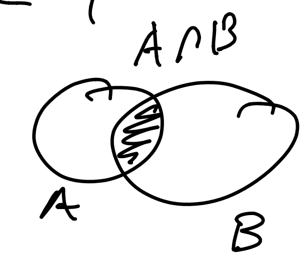
$$H(s,t) = \begin{cases} F(s, 2t) & 0 \le t \le 1/2 \\ G(s, 2t-1) & 1/2 < t \le 1 \end{cases}$$

As H is <u>continuous</u> this defines a homotopy from f to h. $\Rightarrow f \simeq h$ ∎

EXERCISE

Here we're using the following

fact: $A, B \subset Y$



$f : A \to X$

$g : B \to X$

continuous functions with

$f = g$ on $A \cap B$.

Then $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$

is continuous on $A \cup B$.

PROP If X is path connected then $\forall$ $f, g : [0,1] \to X$ we have $f \simeq g$.

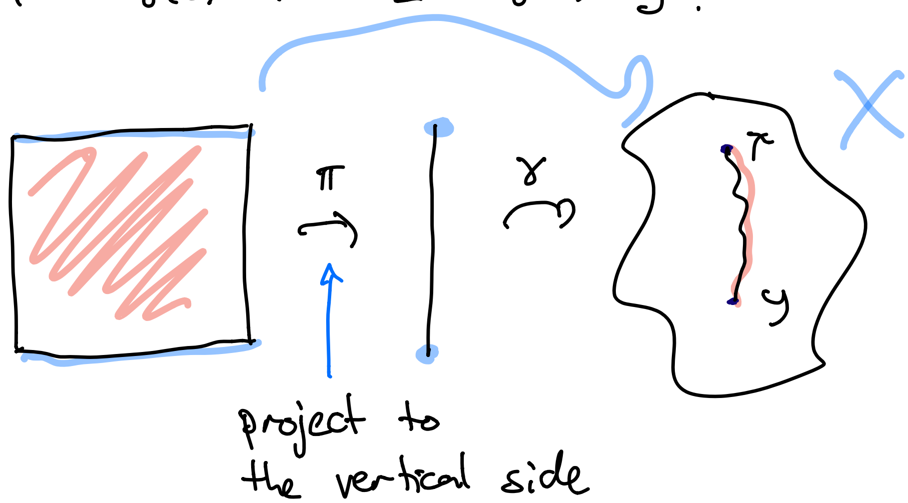
PROOF We first assume that f & g are constant maps:

$$f([0,1]) = x \in X \quad \& \quad g([0,1]) = y \in X$$

Since X is path connected we have

$$\gamma : [0,1] \to X$$

with $\gamma(0) = x$ & $\gamma(1) = y$.



Define $\pi(s,t) = t$ &

$$F(s,t) = \gamma \circ \pi(s,t) = \gamma(t).$$

$$\Rightarrow \quad f \simeq g$$

Need to show that an arbitrary

$f: [0,1] \to X$ is

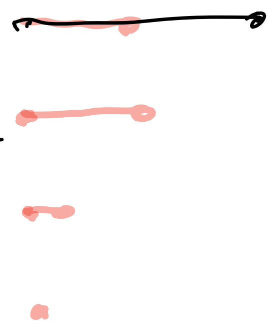
homotopic to a point.

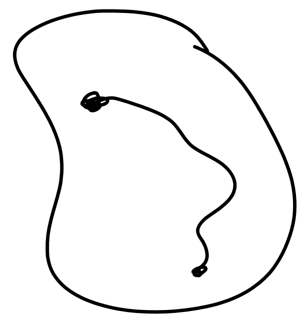
$t=0$

$t=1/2$

$t=3/4$

$t=1$





$$F(s,t) = f(s(1-t))$$

gives a homotopy of f to a constant map.

Given $f, g: [0,1] \to X$

$f \simeq \text{const.}$ $\quad g \simeq \text{const}$

& 2 const maps are homotopic

$\Rightarrow f \simeq g$.

$\boxed{\text{Examples}}$ Let $X = [0,1]$ &

$f : [0,1] \to [0,1]$ with

$f(0) = 0$ & $f(1) = 1$.

note that F fixes 0 & 1.

Then $f \simeq id$.

Define

straight line homotopy

$$F(s,t) = (1-t)\, f(s) + ts$$

F is continuous and $f_0(s) = F(s,0) = f(s)$ and $f_1(s) = F(s,1) = s$.

We need to check that the image of F is $[0,1]$.

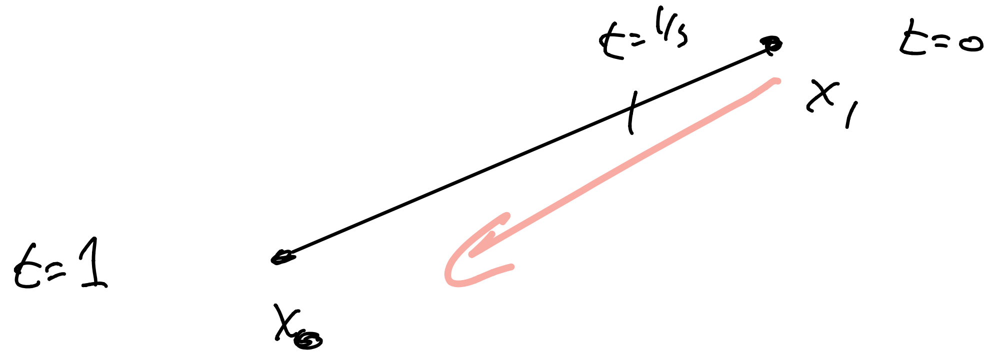
Note that for any $a < b \in \mathbb{R}$

$a \le (1-t)a + tb \le b$ if $t \in [0,1]$

since $(1-t)a + tb \geq (1-t)a + ta = a$

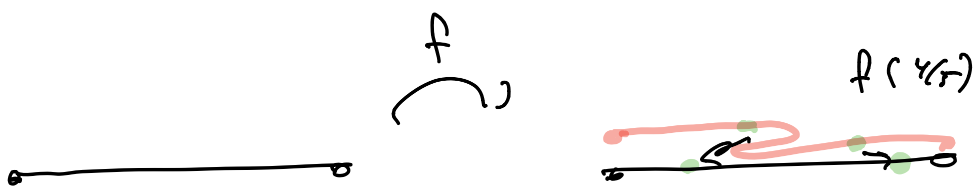
& $(1-t)a + tb \le (1-t)b + tb = b$

$\mathbb{R}^2$



$t x_0 + (1-t) x_1$

$t \in [0,1]$



More generally if $f, g : [0,1] \to [0,1]$

we have the homotopy

$$F(s,t) = (1-t) f(s) + t g(s)$$

COMPOSITION OF MAPS

$$\begin{tikzcd} {[0,1]} \arrow[r, shift left, "f"] \arrow[r, shift right, bend right, "g"'] & X \arrow[r, "h"] & Y \end{tikzcd}$$

If $f \simeq g$ then $h \circ f \simeq h \circ g$.

$F(s,t)$ is the homotopy between f and g.

Define $H = h \circ F$ & then H define a homotopy between $h \circ f$ and $h \circ g$.

We can combine: & reparameterization of a path is homotopic to the original path.

If $\sigma : [0,1] \to [0,1]$ is a homeomorphism

& $f : [0,1] \to X$ is a path

then $f \simeq f \circ \sigma$.

We have seen that if X is path connected then all paths are homotopic.

That is the equivalence relation of homotopy has only one equivalence class.

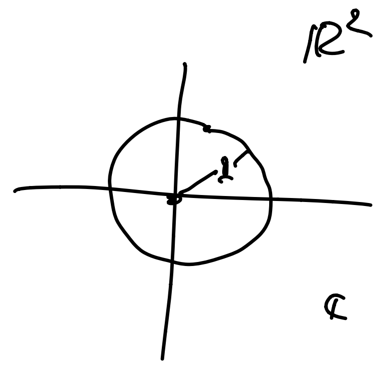
This is not very interesting!

To make things more interesting we can replace $[0,1]$ with any topological space.

We can also look at maps of pairs.

In general it is much harder to show that maps are not homotopic.

The circle S^1.



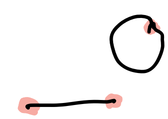
$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$

why homeo?

$S^1 = \{ z \in \mathbb{C} \mid |z| = 1\}$ $\quad |z| = x^2+y^2$

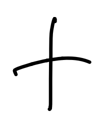
$z = x+iy$

$S^1 = [0,1]/\sim$ where $\{0\} \sim \{1\}$



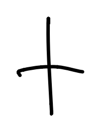
$\boxed{S^1 = \mathbb{R}/\sim \text{ where } t \sim t+n \ \forall\, n \in \mathbb{Z}.}$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$

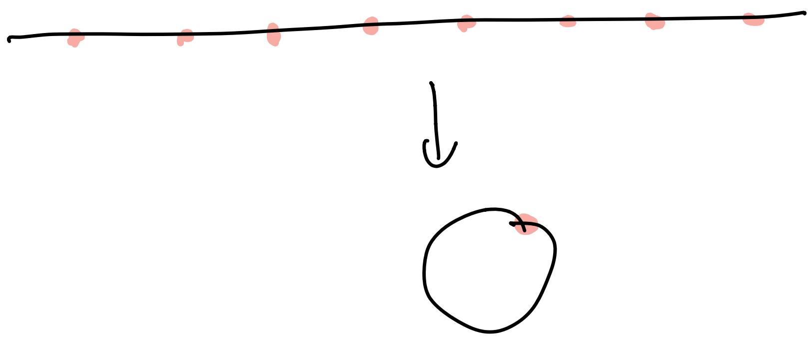


or

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid \max\{|x|, |y|\} = 1\}$



$S^1 = \mathbb{R}/\sim$ where $t \sim t+n \quad \forall n \in \mathbb{Z}$.



General definition of homotopy

$f, g : X \to Y$

are homotopic ($f \simeq g$) if $\exists$

before $X = [0,1]$

$F : X \times [0,1] \to Y$

s.t. $f = f_0$ & $g = f_1$

$\boxed{\text{EXERCISE}}$ This is (still) an equivalence relation.

How can we define maps from S^1 to S^1?

$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$

Define

$f_n : S^1 \to S^1$

by

$f_n(z) = z^n$.

Since $|z^n| = |z|^n = 1^n = 1$ this is a map from S^1 to S^1.

Alternative construction.

Define

$$\tilde{f}_n : \mathbb{R} \to \mathbb{R}$$

by

$$\tilde{f}_n(t) = nt.$$

$\tilde{f}_n(t)$

$\|$

Note that $\tilde{f}_n(t+m) = n(t+m) = nt + nm$ $\quad \in \mathbb{Z}$

so in our equivalence relation on $\mathbb{R}$ we have $\tilde{f}_n(t) \sim \tilde{f}_n(t+m)$.

$\tilde{f}_n$ takes equivalence classes to equivalence classes

$$\begin{tikzcd}
 & \mathbb{R} \arrow[r, "\tilde{f}_n"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] & \\
x \in & S^1 \arrow[r, "f_n"'] \arrow[r, bend right=60] & S^1 & f_n(x)
\end{tikzcd}$$

choose some $t \in \mathbb{R}$ s.t. $\pi(t) = x$

Define $f_n(x) = \pi \circ \tilde{f}_n(t)$.