GROUP ACTIONS

Let X be a simply connected topological space. Then the set of self homeomorphisms, $\text{homeo}(X)$, is a group with group operation composition.

1) $f, g \in \text{homeo}(X) \Rightarrow f \circ g \in \text{homeo}(X)$
2) If $id: X \to X$ is the identity map then $id \in \text{homeo}(X)$ & $f \circ id = id \circ f = f \quad \forall f \in \text{homeo}(X)$.
3) If $f \in \text{homeo}(X)$ then $f^{-1} \in \text{homeo}(X)$ & $f \circ f^{-1} = f^{-1} \circ f = id$.
4) Composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.

A subgroup $G \subset \text{homeo}(X)$ is a group action on X. Let

$$Gx = \{ y \in X \mid \exists g \in G \text{ with } g(x) = y \}$$

be the G-orbit of $x \in X$. We define $x \sim_G y$ if they are in the same G-orbit and let $X/G = X/\sim_G$ with the quotient topology.

1) $x \sim_G x$ since $id \in G$ and $id(x) = x$.
2) $x \sim_G y \Leftrightarrow y \sim_G x$ since if $y = g(x)$ for $g \in G$ then $g^{-1} \in G$ & $x = g^{-1}(y)$.
3) If $x \sim_G y$ & $y \sim_G z$ then $x \sim_G z$ since if $g, h \in G$ with $x = g(y)$ & $z = h(y)$ then

$h \circ g \in G$ & $z = h \circ g(x)$.

G is a deck action if every $x \in X$ has a nbd U such $U \cap g(U) = \emptyset$ if $g \neq id$.

LEMMA If $G \subset$ homeo(X) is a deck action the quotient map $q: X \to X/G$ is a covering map.

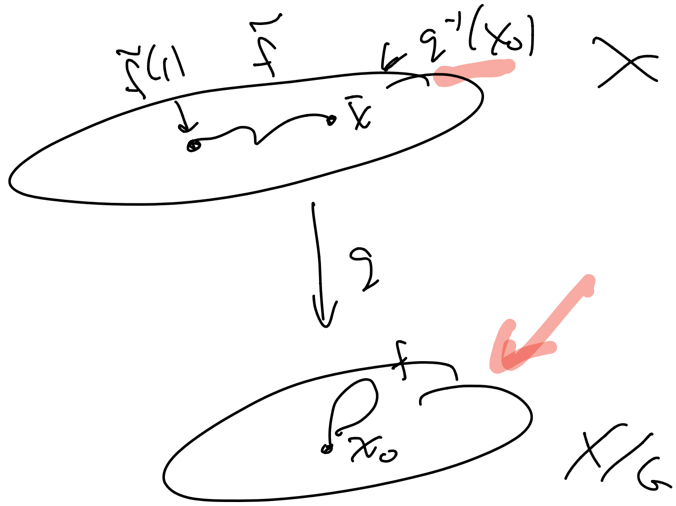
PROOF Since G is a deck action for every $x \in X$ $\exists$ a nbd U s.t. $U \cap gU \neq \emptyset$ if $g \neq id$. Note that U & gU are homeomorphic since any homeo restricted to a subspace is a homeomorphism onto its image. Furthermore $q(U)$ is a nbd of $q(x) \in X/G$ since quotient maps are open. Then $q(U)$ is an evenly covered nbd of $q(x)$ since

$$\coprod_{g \in G} g(U)$$

is a partition of $q^{-1}(q(U))$ into open sets & q restricted to $g(U)$ is a homeomorphism to $q(U)$. ∎

THEOREM If $G < \mathrm{homeo}(X)$ is a deck action, $\pi_1(X/G, x_0) \cong G$.

PROOF We define a homomorphism
$$\phi : \pi_1(X/G, x_0) \to G.$$
Fix an $\bar{x} \in q^{-1}(x_0) \subset X$. For $[f] \in \pi_1(X/G, x_0)$ let $\tilde{f} : [0,1] \to X$ be the unique lift with $\tilde{f}(0) = \bar{x}$. Then $\tilde{f}(1) \in q^{-1}(x_0) = G\bar{x}$ so $\exists\, g \in G$ with $g(\bar{x}) = \tilde{f}(1)$. Define
$$\phi([f]) = g.$$



$\tilde{f}(0) = \bar{x}$

$\tilde{f}(1) \sim_G \bar{x} \Rightarrow \exists\, g \in G$
$g\bar{x} = \tilde{f}(1)$

ϕ is well defined:

$[f] = [h]$

Need $\tilde{h}(1) = \tilde{f}(1)$

We saw if $f, h \in \pi_1(B, b_0)$ & $[f] = [h]$

$\Rightarrow \tilde{f}(1) = \tilde{h}(1)$.

$$g_0(\bar{x}) = \tilde{f}(1)$$

ϕ is a homomorphism:

If $[f], [h] \in \pi_1(X/G, x_0)$ with $\tilde{f}, \tilde{h}$ the lifts of f, h s.t. $\tilde{f}(0) = \tilde{h}(0) = \bar{x}$.

If $\phi([f]) = g_0$ then the lift $\widetilde{f * h}$ of $f * h$ with $\widetilde{f*h}(0) = \bar{x}$ is $\tilde{f} * (g_0 \circ \tilde{h})$.

If $\phi([h]) = g_1$ then $g_1(\bar{x}) = \tilde{h}(1)$.

Then $\tilde{f} * (g_0 \circ \tilde{h})(1) = g_0 \circ \tilde{h}(1) = g_0 \circ g_1(\bar{x})$ So

$\phi([f] \cdot [h]) = g_0 \circ g_1 = \phi([f]) \circ \phi([h])$.

ϕ is injective: $\boxed{HLL}$

If $\phi([f]) = id$ then $\tilde{f}(1) = \bar{x}$. But this happens iff $[f] = q_*(\pi_1(X, \bar{x})) = \{id\}$.

ϕ is surjective:

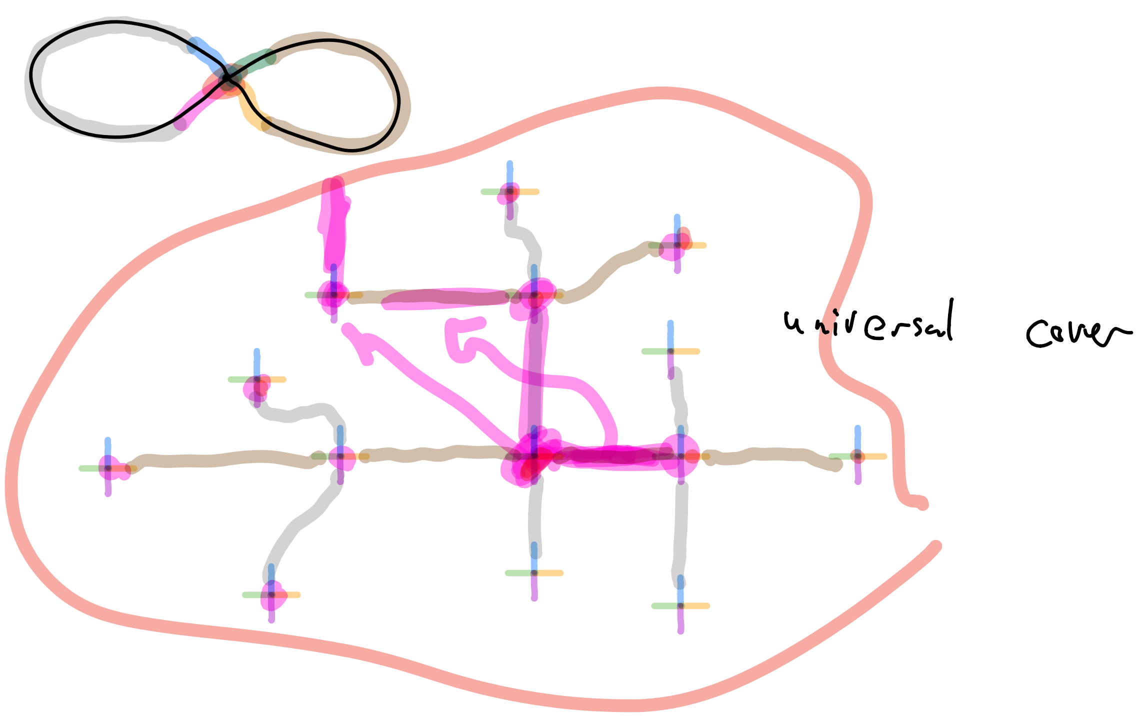
Given $g \in G$ let

$\tilde{f} : [0,1] \to X$

with $\tilde{f}(0) = \bar{x}$ & $\tilde{f}(1) = g(\bar{x})$.

Let $f = q \circ \tilde{f}$. Then $[f] \in \pi_1(G/X, x_0)$

with $\phi([f]) = g$. $\blacksquare$



Homework 4
Due Wednesday, Feb. 24th
Answers should be written in LaTeX.

Assume that

$$p\colon E \to B$$

is a covering space and E is simply connected. Let $b_0 \in B$ and $e_0 \in p^{-1}(b_0) \subset E$ be basepoints.

1. Let $e_1 \in p^{-1}(b_0)$. Show that that there is a lift of the map of pairs

   $$p\colon (E, e_1) \to (B, b_0).$$

   That is show that there exists a map

   $$p_1\colon (E, e_1) \to (E, e_0)$$

   with $p \circ p_1 = p$ and $p_1(e_1) = e_0$.

2. Show that p_1 is a homeomorphism.

3. Let $G \subset \mathsf{homeo}(E)$ the set of all such homeomorphisms (as we let e_1 vary of all points in $p^{-1}(b_0)$). Show that G is a subgroup.

4. Show that the action of G on E is a deck action.

5. Show that the quotient space is homeomorphic to B.