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by 11 pm.

$\boxed{\text{The fundamental group.}}$

(X, x_0) X is a topological space

$x_0 \in X$ base point

We will give a group structure to the set of homotopy classes

$([0,1], \{0,1\}) \to (X, \{x_0\})$.

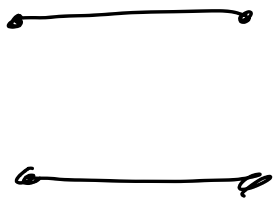
First we need to define the operation:

$f, g : ([0,1], \{0,1\}) \to (X, \{x_0\})$

$$f * g(t) = \begin{cases} f(2t) & t \in [0, \tfrac{1}{2}] \\ g(2t-1) & t \in (\tfrac{1}{2}, 1] \end{cases}$$

when $t = \frac{1}{2}$
$f(2(\frac{1}{2}))$
$= g(2(\frac{1}{2})-1)$
$\Leftarrow$ continuous

$*$ defines a binary operation on paths



X
x_0

$\boxed{\text{GROUPS}}$ A group is a pair $(G, \cdot)$ where G is a set and $\cdot$ is a binary operation satisfying:

$\boxed{\text{closure}}$ If $a, b \in G \Rightarrow a \cdot b \in G$.

$\boxed{\text{identity}}$ There exists an $e \in G$ s.t

$$a \cdot e = e \cdot a = a \quad \forall\, a \in G.$$

$\boxed{\text{inverse}}$ $\forall\, a \in G$, $\exists\, a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

$\boxed{\text{associativity}}$ $\forall\, a, b, c \in G$ we have

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

EXAMPLES

1. $\mathbb{Z}$ with operation of addition.

closure $n, m \in \mathbb{Z} \Rightarrow n+m \in \mathbb{Z}$

identity $0 \in \mathbb{Z}$ is the identity since $0+n = n+0 = n \quad \forall\, n \in \mathbb{Z}$

inverses If $n \in \mathbb{Z}$ then $-n \in \mathbb{Z}$ is the inverse since

$$n + (-n) = (-n) + n = 0.$$

ASSOC $(n+m) + s = n + (m+s)$.

2. X is a topological space.

$G = \text{Homeo}(X) =$ homeomorphisms of X to itself

binary operation is composition

CLOSURE

Given $f, g \in \text{Homeo}(X) \Rightarrow f \circ g \in \text{Homeo}(X)$.

NOTE: $f \circ g$ is not necessarily $=$ to $g \circ f$.

Identity Define $e: X \to X$ by $e(x) = x$ $\forall\, x \in X$.

$$\Rightarrow f \circ e = e \circ f = f \quad \forall\, x \in X.$$

$$f \circ e(x) = f(x)$$

Inverses Since $f \in \text{Homeo}(X) \Rightarrow \exists\, f^{-1} \in \text{Homeo}(X)$ with $f \circ f^{-1} = f^{-1} \circ f = e$.

ASSOC $(f \circ g) \circ h = f \circ (g \circ h)$

Given $f : ([0,1], \{0,1\}) \to (X, \{x_0\})$

we let $[f]$ be the equivalence class

of maps homotopic to f as pairs.

(path homotopic)

We've defined $f * g$.

We need to show that

$[f] * [g] = [f * g]$ is well defined.

That is we need to show

if $f_0 \simeq_p f_1$ & $g_0 \simeq_p g_1$

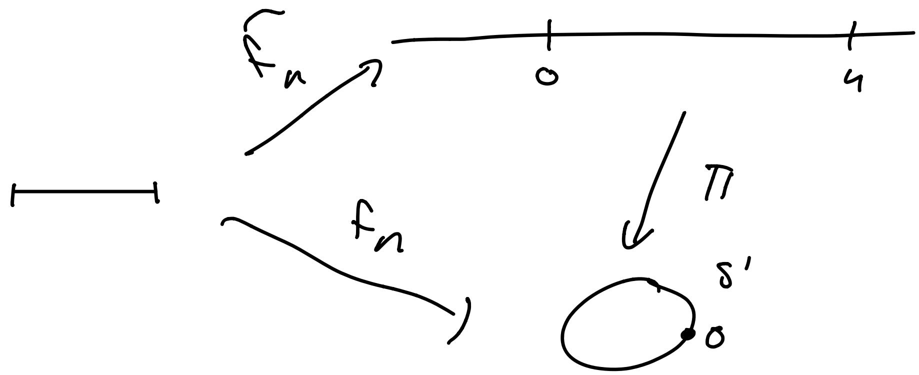
then $[f_0 * g_0] = [f_1 * g_1]$.

EXAMPLE

$\tilde{f}_n(t) = nt$

Recall our maps

$$f_n : ([0,1], \{0,1\}) \to (S^1, [0])$$



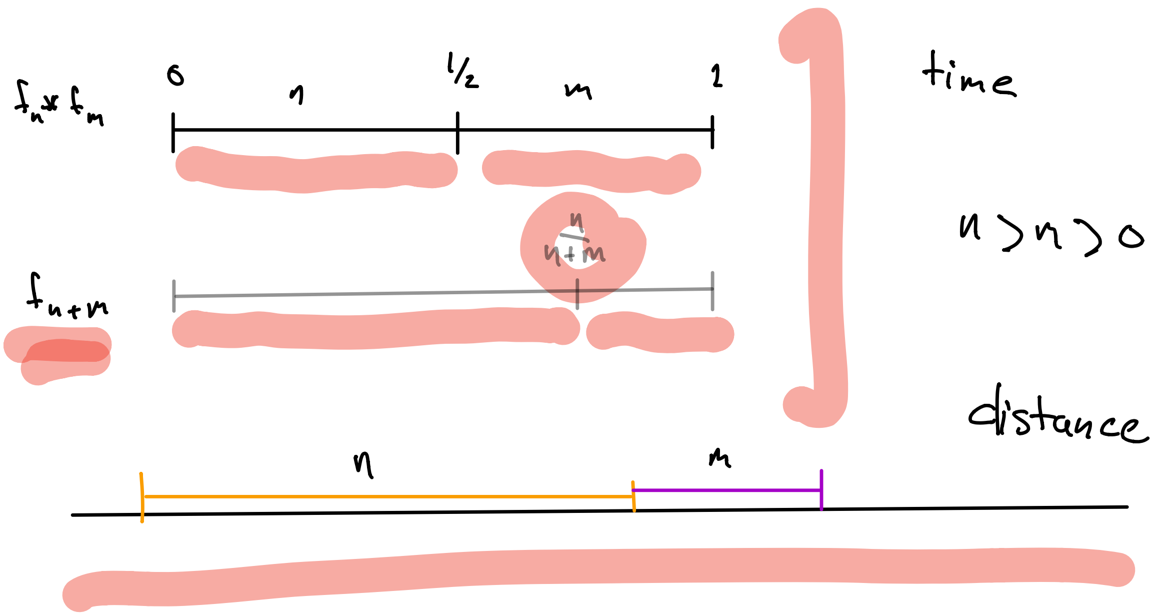
Then $f_n * f_m \simeq_p f_{n+m}$.

If $n=m$ then $f_n * f_n = f_{2n}$

but otherwise they are only homotopic.

To see $f_n * f_m \simeq_p f_{n+m}$

we need to re parameterize

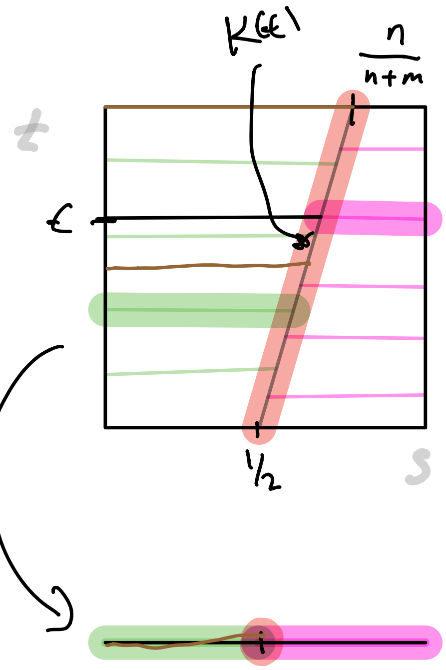


First define homotopy of

id: $[0,1] \to [0,1]$

that "stretches" $[0, \frac{1}{2}]$ to $[0, \frac{n}{n+m}]$

and shrink $[\frac{1}{2}, 0]$ to $[\frac{n}{n+m}, 1]$.



$$G(s,\epsilon)=\begin{cases}\dfrac{s}{2K(\epsilon)} & \text{if } \quad s\le K(\epsilon)\\[1em] & \quad\;\; s> K(\epsilon)\end{cases}$$

$$(1-\epsilon)\frac{1}{2}+\epsilon\frac{n}{n+m}=K(\epsilon)=s$$

$$\frac{(s-K(\epsilon))}{1-K(\epsilon)}\cdot 1+\frac{1}{2}\frac{(s-1)}{(K(\epsilon)-1)}$$

$f_n * f_m$

Define

$$h_\epsilon(s)=H(s,\epsilon)=\left(f_n * f_m\right)\circ G(s,\epsilon).$$

Then $\qquad h_0 = f_n * f_m \;\;\&\;\; h_1 = f_{n+m}$