$\boxed{\text{ASSOCIATIVITY}}$ Need to show

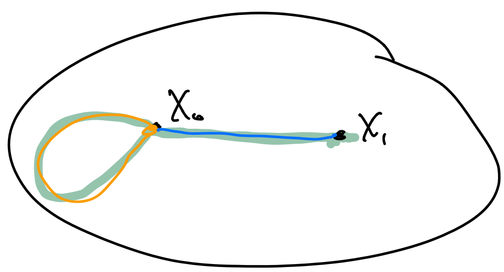
$(f * g) * h \simeq_p f * (g * h)$

$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{2} \qquad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{4}$

How do these differ?

$f, g, h : [0,1] \to X$

reparametrize



X path connected

How does $\pi_1(X, x_0)$ & $\pi_1(X, x_1)$ compare?

$\boxed{\text{Basepoint dependence}}$ How does $\pi_1(X, x_0)$ compare $\pi_1(X, x_1)$?

For our example S^1 clearly $\pi_1(S^1, [0])$ is isomorphic to $\pi_1(S^1, [t])$ for any point $[t] \in S^1$.

What if $X = S^1 \sqcup pt$? That is X is the disjoint union of a circle and a point.

Then $\pi_1(X, [0]) = \pi_1(S^1, [0]) \simeq \mathbb{Z}$ but $\pi_1(X, pt) \cong \pi_1(pt, pt) \cong \{e\}$.

$\pi_1(X, x_0)$ only depends on the path component of X that contains x_0.

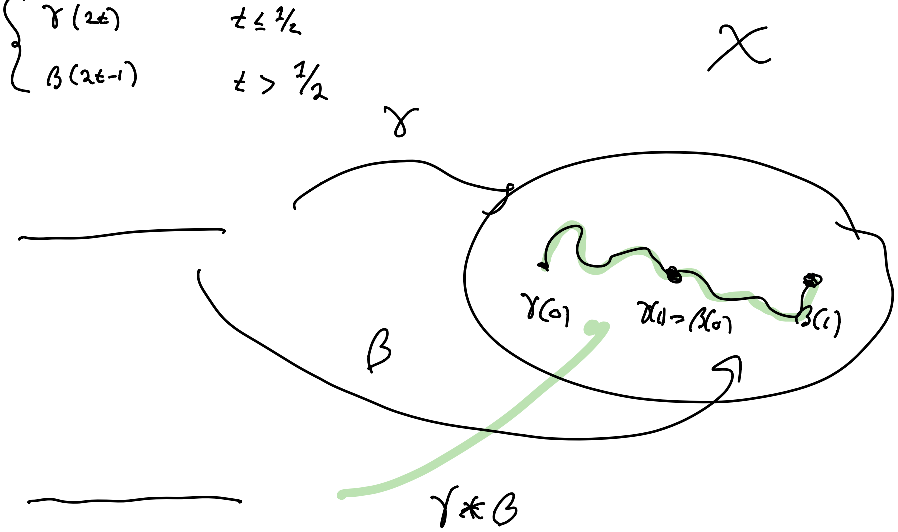
What if x_0 & x_1 are in the same path component?

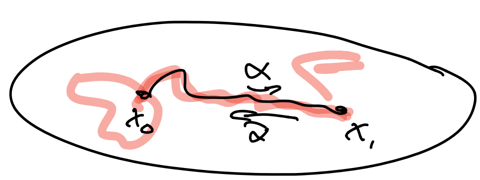
Then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ & every path $\alpha : [0,1] \to X$ with $\alpha(0) = x_0$ & $\alpha(1) = x_1$ determines an explicit isomorphism:

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

To define $\hat{\alpha}$ we first need to generalize the concatenation operation. Let $\gamma : [0,1] \to X$ & $\beta : [0,1] \to X$ be paths with $\gamma(1) = \beta(0)$. Then

$$\gamma * \beta(t) = \begin{cases} \gamma(2t) & t \leq 1/2 \\ \beta(2t-1) & t > 1/2 \end{cases}$$





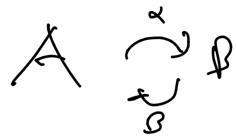
Back to α:

Define $\bar{\alpha}(t) = \alpha(1-t)$ & $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$\hat{\alpha}([f]) = [\bar{\alpha} * f * \alpha]$

We need to check:

1. $\hat{\alpha}$ is well defined: If $f_0 \simeq_p f_1$ then $\bar{\alpha} * f_0 * \alpha \simeq_p \bar{\alpha} * f_1 * \alpha$.
2. $\hat{\alpha}$ is a homomorphism: $\hat{\alpha}([f_0] \cdot [f_1]) = \hat{\alpha}([f_0]) \cdot \hat{\alpha}([f_1])$.
3. $\hat{\alpha}$ is a isomorphism: Let $\beta(t) = \bar{\alpha}(t)$. Then $\hat{\alpha} \circ \hat{\beta} = id$ & $\hat{\beta} \circ \hat{\alpha} = id$.



$\alpha \circ \beta = id$

$\beta \circ \alpha = id$

$\Rightarrow \alpha, \beta$ are bijections

1. $\boxed{\text{EXERCISE}}$

2. $\hat{\alpha}([f_0] \cdot [f_1]) = \bar{\alpha} * f_0 * f_1 * \alpha$ but $\hat{\alpha}([f_0]) * \hat{\alpha}([f_1]) = \bar{\alpha} * f_0 * \alpha * \bar{\alpha} * f_1 \, \alpha$.

General fact $\gamma * \alpha * \bar{\alpha} * \beta \simeq_p \gamma * \beta$. Pf is similar to the existence of inverses.

3. Use general fact again: $f \in \pi_1(X, x)$ then
$$\hat{\beta}(\hat{\alpha}([f])) = \hat{\beta}([\bar{\alpha} * f * \alpha]) = [\bar{\beta} * \bar{\alpha} * f * \alpha * \beta] = [\alpha * \bar{\alpha} * f * \alpha * \bar{\alpha}] = [f].$$

FACTS ABOUT CONCATENATION

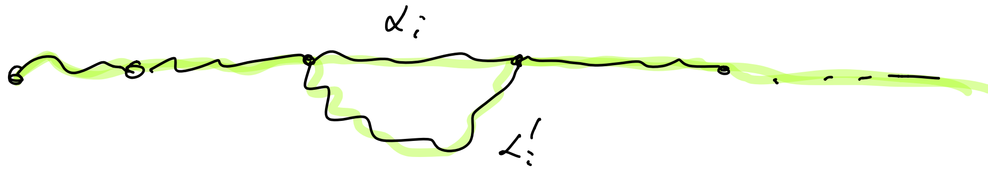
$\alpha_1 * \alpha_2 * \dots\dots * \alpha_n$

1. order of concatenation isn't important.

   Order doesn't path homotopy class

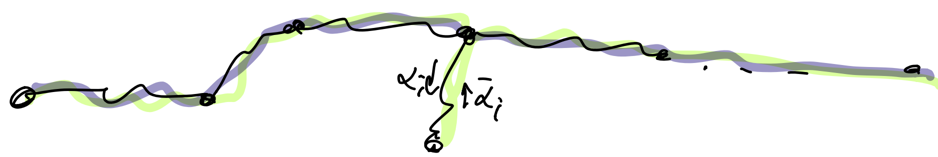
2. If $\alpha_i \simeq_p \alpha_i'$ the

$\alpha_1 * \dots \alpha_i * \dots * \alpha_n \simeq_p \alpha_1 * \dots \alpha_i' * \dots \alpha_n$



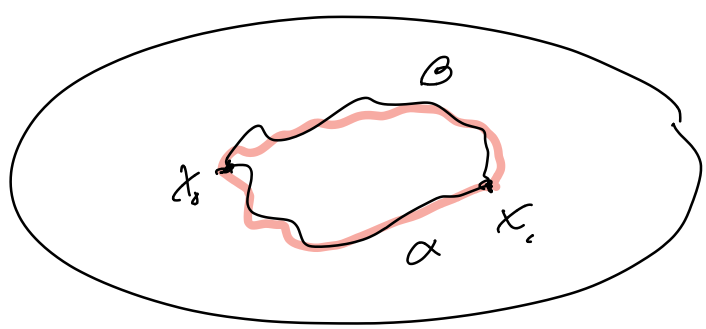
3. If $\alpha_{i+1} = \bar{\alpha}_i$ then

$\alpha_1 * \dots \alpha_i * \alpha_{i+1} * \dots \alpha_n \simeq_p \alpha_1 * \dots * \alpha_{i-1} * \alpha_{i+2} * \dots \alpha_n$



If $\alpha \simeq_p \beta$

$\Rightarrow \hat{\alpha} = \hat{\beta}$



INDUCED HOMOMORPHISMS Let $h:(X,x_0)\to(Y,y_0)$ be continuous.

We define a homomorphism $h_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$ by

$$h_*([f]) = [h\circ f].$$

Again need to check that h_* is a well defined map and that h_* is a homomorphism. Again use a general fact: If $f_0\simeq_p f_1 \Rightarrow h\circ f_0 \simeq_p h\circ f_1$. This implies that h_* is a well defined map.

To see that h_* is a homomorphism we observe that

$$[h_*(f_0)]\cdot[h_*(f_1)] = [h\circ f_0 * h\circ f_1] = [h\circ(f_0 * f_1)] = h_*([f_0]\cdot[f_1])$$

LEMMA Given $h:(X,x_0)\to(Y,y_0)$ & $g:(Y,y_0)\to(Z,z_0)$ we have

$$g_*\circ h_* = (g\circ h)_*.$$

PROOF The proof is formal:

$$g_*\circ h_*([f]) = g_*([h\circ f]) = [g\circ h\circ f] = (g\circ h)_*([f]). \blacksquare$$

By def'n

COR If $h:(X,x_0)\to(Y,y_0)$ is a homeomorphism then h_* is an isomorphism.

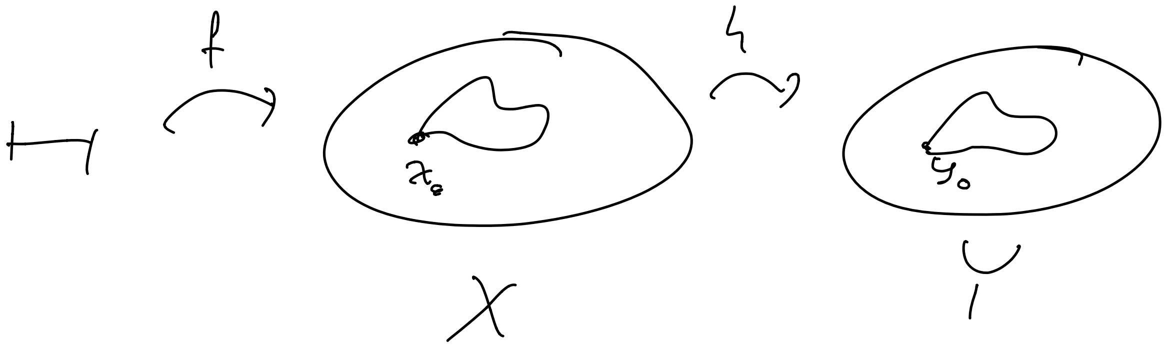
PROOF Since h is a homeomorphism there is a continuous inverse

$$h^{-1}:(Y,y_0)\to(X,x_0).$$

In particular $h^{-1}\circ h = id_X$ & $h\circ h^{-1} = id_Y$.

For the identity map the induced map on π_1 is also the identity since $(id_X)_*([f]) = [id_X\circ f] = [f]$.

Therefore $(h^{-1})_*\circ h_* = id$ & similarly $h_*\circ(h^{-1})_* = id \Rightarrow h_*$ is an isomorphism. $\blacksquare$


I
f
X
x₀
h
Y
y₀