## Groups - A cheat sheet

A *group* is a pair $(G, \cdot)$ where $G$ is a set and $\cdot$ is a binary operation with:

- Closure: for all $a, b \in G$, $a \cdot b \in G$.
- Identity: there exists an $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$.
- Inverses: for all $a \in G$ there exists a $a^{-1} \in G$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$.
- Associativity: for all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

If $a \cdot b = a \cdot c$ then $b = c$. Similarly if $b \cdot a = c \cdot a$ then $b = c$.

### Examples

**Trivial group:** Here $G$ is a single element which is necessarily the identity element $e$.

**$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$:** These are all groups with the usual operation of addition.

**Matrix groups:** The set of $n \times n$ matrices with non-zero determinant and group operation matrix multiplication is a group. This is usually denoted $GL_n(\mathbb{F})$ where $\mathbb{F}$ is the field of entries of the matrices. The most common example is $\mathbb{F} = \mathbb{R}$.

**Bijections:** Let $X$ be a set and $\mathbf{Bij}(X)$ the set of bijections of $X$ to itself. This set is a group with group operation composition. If $X$ is a topological space the $\mathbf{Homeo}(X)$ is the set of homeomorphisms of $X$ to itself and this is also a group with group operation composition.

### Homorphisms

A map

$$\phi\colon G_0 \to G_1$$

between groups $G_0$ and $G_1$ is a homomorphism if for all $a, b \in G_0$ we have

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b).$$

- If $k \in \mathbb{Z}$ then
  $$\phi_k\colon \mathbb{Z} \to \mathbb{Z}$$
  given by
  $$\phi_k(n) = kn$$
  is a homomorphism.
- The determinant map
  $$\det\colon GL_n(\mathbb{R}) \to \mathbb{R}$$
  is a homorphism.

### Subgroups

A subset $H \subset G$ is a subgroup of $(G, \cdot)$ if $(H, \cdot)$ is a group. The inclusions

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

are all subgroup inclusions.

If $\phi : G_0 \to G_1$ is a homomorphism and $H_0 \subset G_0$ is a subgroup then the image $\phi(H_0) \subset G_1$ is a subgroup of $G_1$. If $H_1 \subset G_1$ then $\phi^{-1}(H_1) \subset G_0$ is a subgroup. An important special case is when $H_0 = \phi^{-1}(\{e\})$ is the trivial subgroup. In this case $\phi^{-1}(e)$ is the *kernel* of the homomorphism.

### Cosets

If $H$ is a subgroup of $G$ and $g$ is an element of $G$ then the a *left coset* is

$$gH = \{g \cdot h | h \in H\}.$$

We similarly define right cosets $Hg$. If $g_0H \cap g_1H \neq \emptyset$ then $g_0H = g_1H$ with a similar statement for right cosets. For any coset the map

$$h \mapsto g \cdot h$$

is a bijection from $H$ to $gH$. Therefore the left (or right) cosets partition $G$ into sets that have the same cardinality.

### Normal subgroups

The left coset $gH$ and the right coset $Hg$ both contain $g$ so they will always have a non-empty intersection. If the cosets are always equal then the set of cosets has a natural group structure. In this case $H$ is a *normal subgroup*. That is $H$ is a normal subgroup if for all $g \in G$, $gH = Hg$.

We define an operation on the set of cosets by

$$g_0H \cdot g_1H = (g_0 \cdot g_1)H.$$

One needs to check that this is well defined and for this one needs to use that the left and right cosets are equal.

This group is a *quotient group* and is denoted $G/H$. The map from $G$ to $G/H$ given by

$$g \mapsto gH$$

is a homomorphism. The identity of $G/H$ is the coset $eH = H$ so the kernel of this homomorphism is the original subgroup $H$.

Conversely given any homomorphism

$$\phi : G_0 \to G_1$$

is a normal subgroup of $G_0$.

If the group operation is commutative ($a \cdot b = b \cdot a$ for all $a, b \in G$) then every subgroup is normal. Such groups are usually called *abelian*. One basic example comes from subgroups of $\mathbb{Z}$ where subgroups are multiples of a fixed integer $k$. In this case the quotient group is $\mathbb{Z}_k$, the intgers modulo $k$.