BOUNDS FOR DETECTABILITY OF MATERIAL DAMAGE BY NOISY ELECTRICAL MEASUREMENTS

Elena CHERKAEVA and Andrej CHERKAEV

Department of Geophysics Department of Mathematics,
The University of Utah
Salt Lake City, Utah, 84112, U.S.A.

ABSTRACT

The problem of detectability of inclusions by boundary measurements is considered. The inclusions differ from the background material by their conductivity, the location of inclusions is unknown. The detectability is equal to the difference of energies needed to inject currents in the model body without inclusion and in the sample under investigation. We consider a game where inclusions hide in the domain and current distribution seeks them; this is either max-min or a min-max problem for detectability. These problems have different solutions and different physical meaning, both are solved and the bounds of detectability are established. The bounds are independent of the shape of the domain and have a simple analytical form.

KEYWORDS

Inverse problem, technical diagnostics, tomography, optimization, composites, eigenvalue problem, variational problem, min-max problem.

DESCRIPTION OF THE PROBLEM

We deal with the problem of detection of a damage by nondestructive boundary electrical measurements. We answer the question: whether or not a conducting body contains damaged regions inside. We are not interested in finding the location of these regions, their shape, etc., only in detecting the possible presence of damage. A damaged part of the sample is called inclusion; it is supposed that the conductivity σ_I of inclusions is constant and is less than constant conductivity σ_B of a basic material. The problem of detection naturally arises in technical diagnostics, where one is interested in detecting cracks, corrosion, or any other defect in a sample.

We apply arbitrary currents J on the surface $\partial\Omega$ of the body Ω and measure the energy required to inject these currents. To detect an inclusion we compare energies spent for current injection in the undamaged body (which we call a model body) and in a real sample. Clearly, that the energy spent to inject a current into a body with inclusions is greater than the energy of the same current for the body without inclusion; so the presence of the inclusion is easy to find. The next assumption makes the problem nontrivial: we assume that the sensitivity (resolution) of measurements is limited. An inclusion is detected if the difference of responses for the body with and without inclusion is greater then a sensitivity of measurements.

Nothing supposed to be known about inclusions location; we deal with the problem of finding an injected current which detects the 'worst' position of inclusions. It can be formulated as a 'hide and seek' game: the inclusion hides itself somewhere in the domain Ω and the injected current detects its presence. We determine the best strategies for both players; one of them controls the position and shape of inclusions and the other controls the injected currents. The abilities of both sides are restricted by fixing the total volume of inclusions and the total energy of currents.

We find in particular that the best policy is to inject currents with uniform density in the model body, and that it is better to inject two orthogonal currents with fixed total energy.

Also we find that the best policy for the inclusion is to be dispersed into infinitely small pieces which are distributed uniformly throughout the body; the optimal inclusions form a composite material! Shapes of inclusions are found using the theory of the composite materials with extremal properties or the theory of bounds, see for example (Lurie and Cherkaev, 1984).

Solving the problem we find an a priori bound of the sensitivity of measurements which is needed for detection of inclusions no matter how they are located. This bound has an explicit and simple analytical form which makes it convenient for use, it does not depend on the shape of the body.

Equations

The steady state conductance of the model body is described by the interior Neumann problem

$$\nabla \cdot \ (\sigma_B \nabla \ w_B) = 0 \ \text{ in } \Omega, \quad \sigma_B \frac{\partial w_B}{\partial n} \ = \ J \ \text{ on } \partial\Omega, \quad \int_{\partial\Omega} J \ ds = 0 \tag{1}$$

where w_B is the electrical potential inside the model body, σ_B is its conductivity. The energy $E_B(J)$ associated with the process is equal to

$$E_B(J) = \int_{\partial\Omega} w_B \ J \ ds. \tag{2}$$

The tested sample differs by its conductivity from the model one, its conductivity is

$$\sigma(x) = \sigma_I \ \chi(x) + \sigma_B \ (1 - \chi(x)), \tag{3}$$

where $\chi(x)$ is the index function of a domain ω occupied by the inclusions:

$$\chi(x) \ = \ \begin{cases} 1, & \text{if } x \in \omega \\ 0, & \text{if } x \in \Omega - \omega \end{cases} . \tag{4}$$

The conductance process in the real body is also described by (1) where one must change σ_B by σ and w_B by w, with the energy $E(J)$ of the process, $E(J) > E_B(J) \ \ \forall J$.

The normalized difference d between $E(J)$ and $E_B(J)$ distinguishes the body with and without inclusions. Inclusions are detected if this difference is greater than a given parameter ϵ of the sensitivity of the measurements:

$$d \ = \ \frac{E(J) - E_B(J)}{E_B(J)} = \frac{\int_{\partial\Omega} (w - w_B) J \ ds}{\int_{\partial\Omega} w_B J \ ds} > \epsilon. \tag{5}$$

d is a homogeneous functional of J, it depends on an injected current J and on a position χ of the inclusion:

$$d \; = \; d\,(\chi, J). \tag{6}$$

Both arguments are subject to integral constraints. The set τ of admissible currents J is described as:

$$\tau = \{J: \; j \cdot n \; = \; J \;\; \text{on } \partial\Omega, \;\; \nabla \cdot j = 0 \;\; \text{in } \Omega\}. \tag{7}$$

The set $\mathcal{X}$ of χ is restricted by fixing the total volume of inclusion (damaged material):

$$\mathcal{X} = \{ \; \chi(x): \quad \int_\Omega \chi(x)\, dx = m_0 \; \}. \tag{8}$$

Extremal problem

Dealing with the problem of ensured detection of unknown inclusions, we consider two different questions. The problem

$$P_1 = \max_{J \in \tau} \min_{\chi \in \mathcal{X}} d(\chi, J) \tag{9}$$

describes the following situation. For each applied current J we find the most inconvenient for the detection position of inclusions and the corresponding lower bound for detectability, then the current J is chosen to maximize this bound.

The problem

$$P_2 = \min_{\chi \in \mathcal{X}} \max_{J \in \tau} d(\chi, J) \tag{10}$$

answers the question: what inclusions location is the most difficult to detect by any boundary current?

For each particular location of the inclusion χ one can determine the current wich maximizes the response difference $d(\chi, J)$

$$D^*(\chi) = \max_{J \in \tau} \; d(\chi, J) = d(\chi, J_*), \quad J_* = \text{Argmax } d(\chi, J) \tag{11}$$

upon all injected currents which provide a unit energy for the standard body. A similar problem of maximization of $L^2(\partial\Omega)$ norm of voltage response difference was considered in (Isaacson, 1986).

The parameter $D^*(\chi)$ depends on the shape and position of the inclusion χ and shows the maximal ability of the boundary measurements to detect it. $D^*(\chi)$ is equal to the maximal eigenvalue of the problem

$$D^*(\chi) = \max_{J \in \tau} \frac{\int_\Omega (\sigma(\chi(x))^{-1} j^2 - \sigma_B^{-1} j_B^2)\; dx}{\int_{\partial\Omega} w_B J \; ds} \tag{12}$$

where $\sigma(\chi(x))$ is a conductivity distribution inside the real body. The problem became of the form of minimization of the maximal eigenvalue of a linear operator by a control χ. The question of multimodality is to be addressed to this problem.

The problem (10) asks for the lower bound for the first (maximal) eigenvalue $\mu_1 = D^*$ of the spectral problem (12) for an arbitrary location of inclusions.

The value of the functional P_1 is never greater than the value of the functional P_2; solving the first problem we obtain a lower bound for the solution of the second problem. Interesting, that $P_1 < P_2$ and these two problems have different solutions which means that the problem does not possess a saddle point solution.

ONE APPLIED CURRENT

Location of a small inclusion. Optimal currents

Let us consider a small inclusion, and let us concentrate on the dependence of the increment of the functional on the current density. The value of the functional d in the right hand side of (9) is proportional to the square of current density: $d = \psi \max_{J \in \mathcal{T}} \min_{x \in \Omega} j^2(x)$, the coefficient ψ depends on the shape of an inclusion.

Clearly, the optimal location of an infinitesimal small inclusion is in the point where the modulus of the current density reaches its minimal value.

Hence the best strategy of the electrical current is to maximize this minimum upon all admissible boundary currents. This requirement, together with differential restrictions (7) implies that the best current should have a constant density within the sample body,

$$j_*(x) = \theta \mathbf{o} = \text{const(x)} \tag{13}$$

where θ is the magnitude of the constant field and $\mathbf{o} : \ |\mathbf{o}| = 1$ is its ort.

To find injected current J which generates a constant current density inside a homogeneous body it is enough to solve an elementary problem: the domain Ω is disposed to the constant vector field $\theta \mathbf{o}$ which penetrates it. We set the normal component of the injected current equal to the scalar product of this field and the normal n to the boundary of the domain: $J(s) = \theta n(s) \cdot \mathbf{o}$.

Note that the problem possesses infinitely many optimal solutions: the direction of the optimal current $\mathbf{o}$ is arbitrary.

The optimal location of a pair of inclusions

Here we determine the optimal shape of an infinitely small inclusion.

To do this we consider the problem with a spherical inclusion placed in an uniform field j_x and determine the position of the next small inclusion which minimizes the measured energy difference.

Doing standard analysis of the perturbed fields we conclude that the most preferable location of the next inclusion is the area near the first inclusion. The center of the "best" second inclusion lies on the line of the direction of the applied field which passes through the center of the first inclusion. Physically speaking, the second inclusion tries to hide itself in the "shadow" of the first one.

Continuation of this process leads to a chain of inclusions distributed along a line of an exterior field or to a strip elongated in that direction.

We can continue this procedure and take an ellipsoidal inclusion, and again find regions of minimal values of current density, which are the most preferable for the inclusion intending to hide itself. We end up with a needlelike inclusions which disturb the field to the minimal extent being placed along the lines of the electric field. The location of this strip is arbitrary.

Composites and the optimal type of microstructures

The general problem of the inclusion of the most hidden shape may not have a classical solution unless additional geometrical parameters of inclusion are not restricted. The possible degeneration of the problem may be caused by the sequence of the inclusions which increase in the number unrestricted while the size of each inclusion tends to zero. Such a behaviour of the material properties is known to be common in the optimization problems. The reason for this is that the class of all possible inclusions is not closed until all distributions of infinitely small inclusions are included in it as well. The last distributions can be described in terms of the effective properties of the corresponding structures.

The set of effective properties of composites which contains the initial materials of the fixed volume fraction (denoted by m) is called the G_m-closure of the initial set of material properties.

We compare the energy of the standard body with the energy of the sample made of a composite material. The last one can have different concentration of the inclusion material in its different parts $m = m(x)$, but the total amount of the material must be obviously prescribed:

$$\int_\Omega m(x)\ dx = m_0. \tag{14}$$

The optimal microstructures belong to the upper boundary of the G_m-closure restrictions on the effective properties tensor σ_*:

$$\begin{aligned} \mathrm{Tr}(\sigma_* - \sigma)^{-1} &\le \frac{1}{(1-m_0)}[\frac{n}{\sigma_B - \sigma} + \frac{m_0}{\sigma_B}]^{-1} \\ \sigma_* &\le m_0\,\sigma_I + (1-m_0)\,\sigma_B \end{aligned} \tag{15}$$

where $n = 2$ or 3 is the space dimension.

The initial problem of the best detection is reformulated as following: Find an applied current J with fixed energy E_B which maximizes the detectability of a composite distribution in the body assuming that the parameters of a composite $\sigma_* \in G_m$-closure minimize the detectability.

It is easy to show now that properly oriented laminates are the best maximally hidden structures. Indeed, the energy of a composite body reachs its minimum if the eigendirection of the minimal eigenvalue $\sigma_{\min}$ of that tensor is codirected with the applied current; the energy difference is equal to:

$$\frac{m_0\,(\sigma_B - \sigma_I)}{\sigma_B - m_0\,(\sigma_B - \sigma_I)}. \tag{16}$$

It takes especially simple form if we denote as κ a ratio of conductivity differences to the background conductivity:

$$d = \frac{m_0\,\kappa}{1 - m_0\,\kappa}\ \ ,\quad \kappa = \frac{\sigma_B - \sigma_I}{\sigma_B}. \tag{17}$$

Thus we obtain a first answer to the question under consideration - what the minimal value of the detectability parameter is. If we have only one current at our disposal to apply on the boundary then the optimal policy is to apply the current which generates the constant current density distribution inside the body. The most unfortunate position of inclusions then is a laminated composite with laminates oriented along the field.

SEVERAL APPLIED CURRENTS

Reformulation of the problem

Now we come to the second problem under consideration. The problem P_2 asks for the location of inclusions the most hidden for any applied boundary currents. The found laminated inclusion distribution is not optimal for this problem: it can be easily detected by applying the current in the direction perpendicular to the laminates.

This observation shows how to reformulate the problem $P2$. Suppose one can apply several currents $J_i(x)$ with smaller magnitude instead of applying one current J_*. The total energy of the system of currents is kept constant. This corresponds to the situation when either a constant current with random orientation is applied and the mean of detectability is maximized, or when two orthogonal currents are injected in the same sample.

Now laminated composites are not optimal, because a current in the orthogonal direction gives a higher value for detectability. Instead laminates of the second rank can be used to minimize the sum of the energy differences due to two currents in orthogonal directions because these structures minimize any convex combination of stored energies caused by different sources.

Still the optimal currents densities must be constant throughout the body to prevent the position of inclusions in the less 'enlightened' parts of the sample where the magnitude of the current densities is minimal. Locally, one have to minimize the functional

$$I_{minmax} = \min_{\sigma_*} \int_\Omega \sum_{(i)} j_i \cdot \sigma_*^{-1} \cdot j_i \, dx \tag{18}$$

where σ_* belongs to (15), then choose the currents j_i subject to the total energy restriction

$$\sum_{(i)} j_i^2 = 1. \tag{19}$$

One can check that it is enough to consider two or three orthogonal currents j_i (their number is equal to the dimension of the space). Also, the detectability in this problem is a saddle function of parameter of composite family (15) and of magnitudes of orthogonal currents. Hence the reformulated problem possesses a saddle point solution which is the following: The best applied currents have equal and constant density in each point of the domain. The inclusions form an isotropic effective medium; the optimal conductivity is

$$\sigma_* = \sigma_B \; + \; m \;\; (\frac{1}{\sigma_I - \sigma_B} \; + \; \frac{1-m}{n\,\sigma_B})^{-1}. \tag{20}$$

Detectability P_2 now can be easily calculated as:

$$P_2 = \frac{\sigma_B - \sigma_*}{\sigma_*} = \frac{m_0\,\kappa}{1 - m_0\,\kappa - (1-m_0)\,\kappa\, n^{-1}} \tag{21}$$

where n is dimension, and κ is defined earlier in (17).

REFERENCES

Isaacson, D. (1986). Distinguishability of conductivities by electric current computed tomography, *IEEE Trans. Med. Imaging*, MI-5, 91-95.

Lurie, K. and Cherkaev A. (1984) Exact estimates of conductivity of composites formed by two isotropically conducting media taken in prescribed proportion *Proc. Royal Soc. of Edinbourgh, sect.A,* 99 (P1-2), pp. 71-87.