Two-bead microrheology: Modeling protocols

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Microbead rheology maps the fluctuations of beads immersed in soft matter to viscoelastic properties of the surrounding medium. In this paper, we present modeling extensions of the seminal results of Mason and Weitz [Phys. Rev. Lett. **74**, 1250 (1995)] for a single bead and of Crocker *et al.* [Phys. Rev. Lett. **85**, 888 (2000)] and Levine and Lubensky [Phys. Rev. Lett. **85**, 1774 (2000)] for two beads. We formulate the linear response analysis for two beads so that the model equations retain the local diffusive properties of each bead (through the memory kernel of the shell or depletion zone surrounding each bead) and the nonlocal dynamic moduli of the medium separating the beads (through the memory kernel that transmits fluctuations of one bead to the other). We then derive a 3×3 invertible system of equations relating: an isolated bead's autocorrelations, the autocorrelations and cross-correlations of two coupled beads; and the shell radius surrounding each bead, the memory kernels of the shell, and of the medium between the two beads.

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I. INTRODUCTION

In passive microrheology, fluctuations of beads are experimentally recorded. In the one-bead protocol, fluctuations from neighboring beads are ignored, assuming the beads are sufficiently far away from one another, and an inference is made using the Mason and Weitz modeling formalism about the viscoelastic properties of the surrounding medium (see Ref. [1]). The generalized Langevin equation (GLE) model for the fluctuations assumes a memory drag law, with a kernel that physically represents some unknown combination of the bulk viscoelastic modulus of the fluid and the bead-fluid surface chemical potential. Following Levine and Lubensky [2–4], we call this kernel the inner modulus G_i. The one-bead protocol from Mason and Weitz [1] offers a way to recover, in frequency space, the inner modulus through mean square displacement data. For sufficiently long time scales, a Brownian sphere in a viscoelastic fluid approaches viscous diffusion with a Stokes drag coefficient. This limiting behavior is determined by an effective viscosity which takes into account the depletion zone surrounding the bead as derived by Fan *et al.* [5]. Another approach taken by Santamaria-Holek and Rubi [6] is to analyze the Fokker-Planck equation for the probability distribution associated with the GLE. They obtain short-time power law behavior in the mean-squared displacement of a bead, depending on the finite size of the Brownian particle relative to the polymer network and the high frequency behavior of the loss modulus.

The inability to screen or explicitly account for the effects of bead-fluid interactions through surface chemistry led to the development of two-bead microrheology. By careful spacing of the beads, two-bead microrheology as presented by Crocker *et al.* [7], Levine and Lubensky [2], and Valentine *et al.* [8] allows for the determination of the bulk viscoelastic modulus of the medium between the two beads. We call this modulus the outer shear modulus G_o. In Refs. [7,8], the bead-bead correlations are dominated by G_o at leading order in the ratio of the shell thickness to the bead radius.

Our primary aim in this paper is to extend previous one-bead and two-bead models so that local diffusive (the inner modulus) and nonlocal bulk (the outer modulus) properties are coupled in such a way that they can both be inferred from experimental data. This aim is achieved by carrying the analysis of Levine and Lubensky [3] and Chen *et al.* [9] to linear order in the asymptotics of the shell radius surrounding each bead. By doing so, we derive an invertible 3×3 system of equations relating single-bead and two-bead fluctuation measurements to the inner (diffusive) modulus G_i, outer (shear) modulus G_o, and γ, the ratio of the shell thickness to the bead radius.

The motivation for this work arises from the Virtual Lung Project at UNC, which aims to model hydrodynamics of pulmonary liquids and transport of diverse Brownian particles within them. These challenges require fundamental understanding of both the diffusion of pathogens and particulates in biological complex liquids such as mucus and their flow transport properties on scales small relative to typical rheometric probes of dynamic moduli. Another anticipated application of this work is to passage time of foreign particles through biological barriers (see Hansen and McDonald [10]).

Following Levine *et al.* [3] we describe, in Sec. II, a generalization of an elastic problem to include both inner and outer moduli and the shell thickness in the case where the second sphere is considered a passive point source of force. In Sec. III, we present inverse characterization tools inherent to the two-bead coupled GLEs with application for the determination of the local (G_i) and nonlocal (G_o) kernels, as well as the thickness of the chemically modified layer (or shell [4] or depletion zone [5]). We formulate and analyze the particular limit where the bead separation distance is approximately 5 to 10 bead radii as in Ref. [3], retaining the terms that are linear in the ratio of shell thickness to the bead

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radius. These results, coupled with the single bead results of Mason and Weitz, yield the aforementioned 3×3 invertible system relating one-bead and two-bead experimental data with local and nonlocal kernels and the shell radius.

II. TWO-BEAD GENERALIZED LANGEVIN EQUATIONS

We consider two spherical beads separated by a radial distance R and we use spherical coordinates (r,θ,ϕ) with origin centered at bead 1. Let v_r^1, v_r^2 denote the velocity of beads 1,2, respectively, in the radial direction, v_θ^1, v_θ^2 the velocities in the polar direction and let $\zeta_{11},\zeta_{22},\zeta_{12},\zeta_{21}$ denote the components of the memory kernel tensor. ζ_{12} describes the response of bead 2 to the displacement of bead 1 and has been discussed by Levine and Lubensky [4]. The inclusion of the displacement of the second bead as a force on the first bead leads to the following coupled Langevin system (see Starrs and Bartlett [11]) for $i=r,\theta$:

$$m\frac{dv_i^1(t)}{dt} = -\zeta_{11}(t)\star v_i^1(t) - \zeta_{12,i}(t)\star v_i^2(t) + f_i^1(t), \qquad (1)$$

$$m\frac{dv_i^2(t)}{dt} = -\zeta_{22}(t)\star v_i^2(t) - \zeta_{21,i}(t)\star v_i^1(t) + f_i^2(t), \qquad (2)$$

where the covariance matrix of the random forces is $\langle f_i^j(t) f_i^{j'}(t')\rangle = k_B T \zeta_{jj',i}(t-t')$, with k_B the Boltzmann constant and T temperature in accordance with the fluctuation dissipation theorem (see Bonet Avalos *et al.* [12]) and $\star$ denotes convolution. We will follow standard practice and assume $\zeta_{11}=\zeta_{22}$ and $\zeta_{12,i}=\zeta_{21,i}$, although these conditions could be relaxed for applications to heterogeneous materials and/or differently coated beads.

The memory kernel tensor reflects medium viscoelasticity and local bead surface chemistry. If the system were purely viscous and the bead chemically neutral, then $\zeta_{11}=\zeta_{22}=6\pi\mu a$ and $\zeta_{12}=-9\pi\mu a/\rho$, where a is the radius of the bead, $\rho=\frac{R}{a}$, R is the bead separation distance, and μ is the viscosity of the fluid. ζ_{12} for a viscous fluid corresponds to the lower order term in the mutual friction coefficient derived by Batchelor [13]. The diagonal memory kernels reflect that the bead modifies its local environment leading to a local modulus G_i, while the viscoelastic properties on the separation length scale R are given by G_o, as first discussed by Levine and Lubensky [2,4]. We assume that each bead is surrounded by a sphere of radius b which includes the bead and a shell of radius s, and that the viscoelastic properties inside this shell are characterized by G_i. Since viscoelastic relations can be obtained by generalizing either viscous or elastic relations, we study the equivalent elastic problem as in Ref. [3] (see Fig. 1).

A. Response functions for a two-fluid elastic fluid

We consider the problem of two spheres of radius a embedded in a shell of radius b in a two-medium elastic fluid: the Lame constants are λ_i,μ_i in the inner shell and λ_o,μ_o in the outer shell (Fig. 1). We remark that the Lame constants λ,μ allow for the general compressible case. In the incompressible limit, we have $\lambda\to\infty$ and $\nu=\frac{1}{2}$, where ν is the Poisson ratio. We define s so that $b=a+s$ and therefore $\beta=\frac{a}{b}=\frac{1}{1+s/a}=1-\gamma+\gamma^2+O(\gamma^3)$, with $\gamma=\frac{s}{a}$. Solving the elastic Navier-Stokes equation in the inner and outer shells with an azimuthal symmetric solution vanishing at infinity, initial displacement in the $\hat{z}$ direction (given as displacement or force on the sphere), continuity boundary condition and keeping only linear terms in $\hat{z}$, the inner and outer displacement fields are given by (see also Ref. [4])

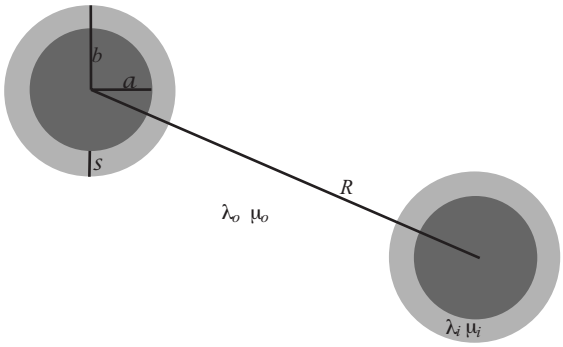

FIG. 1. Two elastic spheres model with shells following Levine and Lubensky [3].

$$u_i(r,\theta) = \frac{aC_{1,i}}{r}[(\gamma_{1,i}+1)\cos\theta\hat{r} - \sin\theta\hat{\theta}] + \frac{a^3C_{2,i}}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) + C_{3,i}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) + \frac{C_{4,i}r^2}{a^2}[(\gamma_{2,i}-1)\cos\theta\hat{r} + \sin\theta\hat{\theta}], \qquad (3)$$

$$u_o(r,\theta) = \frac{bC_{1,o}}{r}[(\gamma_{1,o}+1)\cos\theta\hat{r} - \sin\theta\hat{\theta}] + \frac{b^3C_{2,o}}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}). \qquad (4)$$

Here $\gamma_{1,o/i}=\frac{1}{3-4\nu_{o/i}}$, $\gamma_{2,o/i}=\frac{2(2-3\nu_{o/i})}{1-2\nu_{o/i}}$, and $\hat{r},\hat{\theta}$ are the unit vectors in spherical coordinates. In the incompressible limit $\gamma_{1,o/i}=1$ and $\gamma_{2,o/i}=\frac{1}{2}$. The initial condition is $u_i(a,\theta)=\epsilon\hat{z}$. The force in the $\hat{z}$ direction is found by integrating the stress $\sigma_{i,rz}=\sigma_{i,rr}\cos\theta-\sigma_{i,r\theta}\sin\theta$ over the surface of a sphere of radius r (see Appendix A). In the incompressible case, we have $F_{i,z}=8\pi aC_{1,i}\mu_i$ and $F_{o,z}=8\pi bC_{1,o}\mu_o$. This means that the lower order coefficient of the approximation (i.e., the $1/r$ term) only depends on the applied force in the $\hat{z}$ direction, a result that is extensively used by Crocker *et al.* [7]. We set $\kappa=\frac{\mu_o}{\mu_i}$. In Appendix B, we give the formulas for the coefficients $C_{j,i}$, $j=1,\ldots,4$ and $C_{j,o}$, $j=1,2$ as rational functions in β and κ. We define

$$p_1(\beta,\kappa) = 2\kappa\beta^5 - 2\beta^5 - 3 - 2\kappa$$

and

$$p_2(\beta,\kappa) = -10\beta^3\kappa + 3\beta\kappa + 6\beta + 4\beta^6 - 8\beta^6\kappa + 4\beta^6\kappa^2 - 9\beta\kappa^2 + 10\beta^3\kappa^2 - 9\beta^5\kappa^2 + 9\kappa\beta^5 + 4\kappa^2 + 6\kappa.$$

Then, since $\kappa C_{1,o}=\beta C_{1,i}$, we find that the force exerted by the bead on the medium at both interfaces is $F_z = \frac{12\epsilon\pi\mu_o a p_1(\beta,\kappa)}{p_2(\beta,\kappa)}$ (see also Levine and Lubensky [2]).

The response of a bead to a force applied by the medium is assumed to be linear in the displacement of the form $\alpha\vec{u} = \vec{F}$, where $\vec{F}$ is the applied force, $\vec{u}$ is the resulting displacement, and α is the compliance/resistance tensor. Since F_z is the force exerted by the medium when the initial displacement of the first bead is ϵ in the Cartesian $\hat{z}$ direction, we find that the self-resistance tensor in the incompressible limit is $\alpha^{1,1}=-\frac{12\pi a\mu_o p_1(\beta,\kappa)}{p_2(\beta,\kappa)}$, where the index 1,1 indicates that the displacement of the first bead is related to the force on the first bead. Developing β as a Taylor series in γ and using $b=a(1+\gamma)$ we find

$$\alpha^{1,1} = 6\pi a\mu_o[1 + (1-\kappa)\gamma - 3(1-\kappa)\gamma^2 + O(\gamma^3)].$$

In the one-medium limit $\kappa=1$ ($\mu_o=\mu_i=\mu$), the compliance tensor reduces to the Stokes coefficient $\alpha^{1,1}=6\pi a\mu$. Similarly, if $\gamma \ll 1$, $\alpha^{1,1}=6\pi a\mu_o$.

In order to find $\alpha^{1,2}$, the compliance tensor of the second bead due to the displacement of bead one, we compute as a first approximation the displacement at the center of the second bead, reducing the second bead to a point source without a shell. As shown by Levine and Lubensky [3], correction terms induced by the point source approximation are of higher order in R^{-1}. Prescribing the displacement or the force on the first bead is totally equivalent, so that we might assume that a constant force $F_z\hat{z}$ is applied on the first bead resulting in an initial displacement ϵ in the $\hat{z}$ direction. Since $F_{o,z}=8\pi bC_{1,o}\mu_o$ and $F_{i,z}=8\pi aC_{1,i}\mu_i$, the coefficients in Eqs. (3) and (4) and the added unknown ϵ can be fully determined in terms of F_z. The resulting displacement in spherical coordinates is then linearly proportional to the force F_z in each spherical direction $\hat{r}$ and $\hat{\theta}$. Therefore we obtain the compliance tensor in spherical coordinates in the incompressible limit

$$\frac{1}{\alpha_{rr}^{1,2}} = \frac{1}{4\pi\mu_o R} - \frac{1}{12}\frac{b^2(-5\kappa\beta^2 - 3 + 3\kappa - 2\beta^5 + 2\kappa\beta^4)}{\pi\mu_o p_1(\beta,\kappa)R^3},$$

$$\frac{1}{\alpha_{\theta\theta}^{1,2}} = \frac{1}{8\pi\mu_o R} + \frac{1}{24}\frac{b^2(-5\kappa\beta^2 - 3 + 3\kappa - 2\beta^5 + 2\kappa\beta^5)}{\pi\mu_o p_1(\beta,\kappa)R^3},$$

which is implicitly available in Ref. [3]. Since for an incompressible fluid the Poisson ratio is 1/2, it follows that $\alpha_{\phi\phi}^{1,2} = \alpha_{\theta\theta}^{1,2}$ and $\alpha_{ij}^{1,2}=0$ if $i\neq j$. We remark that the lower order term only depends on the separation distance R between the bead and on the outer viscosity between the beads.

Since the constant force F_z is related to the displacement field of the first bead by $\alpha^{1,1}$, we substitute $F_z=\alpha^{1,1}\epsilon$ into the solution (4) evaluated at the separation distance R, again assuming the second bead to be a point source. Developing β as a Taylor series in γ, setting $b=a(1+\gamma)$, and defining $\rho = \frac{R}{a}$ we find that the radial and polar components of the displacement field of the second passive bead are

$$u_r^2 = \left[\frac{3}{2\rho} - \frac{1}{2\rho^3} + \frac{3}{2}(\kappa-1)\left(-\frac{1}{\rho} + \frac{1}{\rho^3}\right)\gamma + \left(-\frac{3}{2\rho^3} - \frac{9\kappa}{2\rho} + \frac{9\kappa^2}{2\rho} + \frac{6\kappa}{\rho^3} - \frac{9\kappa^2}{2\rho^3}\right)\gamma^2 + O(\gamma^3)\right]u_r^1, \quad (5)$$

$$u_\theta^2 = \left[\frac{3}{4\rho} + \frac{1}{4\rho^3} - \frac{3}{4}(\kappa-1)\left(\frac{1}{\rho} + \frac{1}{\rho^3}\right)\gamma + \left(\frac{3}{4\rho^3} - \frac{9\kappa}{4\rho} + \frac{9\kappa^2}{4\rho} - \frac{3\kappa}{\rho^3} + \frac{9\kappa^2}{4\rho^3}\right)\gamma^2 + O(\gamma^3)\right]u_\theta^1, \quad (6)$$

where we used $\epsilon=u_r^1=u_\theta^1$. Thus $\vec{u}_2=u_r^2\cos\theta\hat{r}-u_\theta^2\sin\theta\hat{\theta}$ can be expressed in terms of $\vec{u}_1=u_r^1\cos\theta\hat{r}-u_\theta^1\sin\theta\hat{\theta}$. We remark that Levine and Lubensky [2] and Crocker *et al.* [7] only use the first order term in $\frac{1}{\rho}$ and zeroth order term in γ in the previous approximation, which suffices for their goals.

In this paper, we consider the limit in which $O(R^{-3})$ and $O(\gamma^3)$ terms are neglected in (5) and (6). This limit arises in experiments, where the beads are far enough away from each other to neglect terms higher order in R^{-1} and the thickness of the shell created by the effect of the chemical coating is small, but not negligibly so, relative to the bead radius. Moreover, we assume (as in Levine and Lubensky [2], Crocker *et al.* [7]) that R is constant (fluctuations are small compared to the separation distance). We find

$$u_r^2 = [1 + (1-\kappa)\gamma]\frac{3}{2\rho}u_r^1, \quad u_\theta^2 = [1 + (1-\kappa)\gamma]\frac{3}{4\rho}u_\theta^1. \quad (7)$$

Equation (7) will be used to derive a modeling protocol for the determination of both viscosities μ_o and μ_i and the shell thickness γ.

B. Generalization to a viscoelastic liquid

To generalize the results obtained in the elastic case we replace the elastic shear modulus $\mu_{o/i}$ by $G_{o/i}^*(\omega)$ the complex shear modulus, as in Ref. [4]. We define $G^*(\omega)=i\omega\eta^*(\omega) = G'(\omega)+i\omega G''(\omega)$, where η^* is the complex viscosity, G' the loss modulus, and G'' the storage modulus. For simplicity of notation we consider only the equations in radial direction r and we drop the corresponding subscript. Similar results hold for the angular coordinates. Let $\hat{u}^1(\omega)$ and $\hat{u}^2(\omega)$ be the Fourier transform of the radial coordinates of the displacement of each bead.

The viscoelastic generalization of Eq. (7) is

$$\hat{u}^2(\omega) = [1 + (1-\kappa^*)\gamma]\frac{3}{2\rho}\hat{u}^1(\omega) \quad (8)$$

with $\kappa^*=\frac{G_o^*(\omega)}{G_i^*(\omega)}$. In the limit considered here, the self-compliance coefficient $\alpha^{1,1}$ is $6\pi aG_o^*(\omega)[1+(1-\kappa^*)\gamma]$, so that the generalized Stokes-Einstein relation can be written as $\hat{F}^{1,1}(\omega)=\alpha^{1,1}(\omega)\hat{u}^1(\omega)$. The drag force on the second bead due to the force on the first bead is $\hat{F}^{2,1}(\omega)=-\alpha^{1,1}(\omega)\hat{u}^2(\omega)$. With Eq. (8) we find

$$\hat{F}^{2,1}(\omega) = \frac{-9\pi a G_o^*(\omega)}{\rho}[1+(1-\kappa^*)\gamma]^2\hat{u}^1(\omega).$$

Let $\hat{v}^1$ and $\hat{v}^2$ be the corresponding velocities. In frequency space, we have $\hat{v}^j(\omega)=i\omega\hat{u}^j(\omega)$, so that the force can be expressed with the complex viscosity and the Fourier transform of the velocities. We define the kernels in frequency space in the following way, keeping only linear terms in γ:

$$\hat{\zeta}_{11}(\omega) = 6\pi a\,\eta_o^*(\omega)\left[1+\left(1-\frac{\eta_o^*(\omega)}{\eta_i^*(\omega)}\right)\gamma\right], \tag{9}$$

$$\hat{\zeta}_{21}(\omega) = \frac{-9\pi a\,\eta_o^*(\omega)}{\rho}\left[1+2\left(1-\frac{\eta_o^*(\omega)}{\eta_i^*(\omega)}\right)\gamma\right]. \tag{10}$$

Finally, since multiplication in Fourier space corresponds to convolution in real space, we obtain

$$F^{1,1}(t) = \zeta_{11}(t)\star v^1(t), \quad F^{2,1}(t) = \zeta_{21}(t)\star v^2(t), \tag{11}$$

as in the force equations (1) and (2). We note that there is an intrinsic separation of time scales, since $\rho=\frac{R}{a}$ is constant while determining the relaxation kernel.

C. "Decoupling" of the Langevin system

We now transform to "normal coordinates" where the coupled generalized Langevin equations (GLEs) (1) and (2) are almost diagonalized. We define normal coordinates: $V^1=v^1+v^2$ and $V^2=v^1-v^2$. Then the GLEs (1) and (2) decouple as

$$m\frac{dV^j(t)}{dt} = -\zeta^j(t)\star V^j(t) + G^j(t), \quad j=1,2, \tag{12}$$

where $\zeta^j=\zeta_{11}+(2\delta_{j1}-1)\zeta_{21}$, and $G^j=f^1+(2\delta_{j1}-1)f^2$.

Since $\langle f^j(t)f^{j'}(t')\rangle=k_BT\zeta_{jj'}(t-t')$, and with the previous definitions of ζ^1,ζ^2, the covariance matrix becomes

$$\langle G^1(t)G^1(t')\rangle = 2k_BT\zeta^1(t-t'),$$

$$\langle G^2(t)G^2(t')\rangle = 2k_BT\zeta^2(t-t'),$$

$$\langle G^1(t)G^2(t')\rangle = 0.$$

This decoupling of the random contributions is only valid in the usual approximation that R is constant.

III. INVERSE CHARACTERIZATION

Based on the Langevin description of a single bead in a viscoelastic fluid, Mason and Weitz [1] developed a one-bead microrheology protocol whose idea is summarized in Appendix C. Neglecting inertia, the Mason and Weitz one-bead protocol [see Eq. (C2)] in one coordinate direction is

$$\langle[\Delta\hat{u}^j(\omega)]^2\rangle = \frac{k_BT}{3\pi a i\omega G_i^*(\omega)}. \tag{13}$$

In other words, measurements of the mean square displacement allow for the determination of the transform of the inner kernel G_i. Equation (13) is a transform equation and thus does not directly give access to the physical parameters characterizing the memory kernel in the time domain. Fricks *et al.* [14] present a maximum likelihood method applied to single bead time series data aimed at the reconstruction of kernel parameters of the time-dependent representation $G_i(t)$.

In the GLEs (1) and (2), the displacement of one bead influences the displacement of the other bead as in the formula (5) and (6). The goal is to find a formula analogous to Eq. (13) using the average of the cross-correlated displacements $\langle\Delta\hat{u}^1\Delta\hat{u}^2\rangle$. Crocker *et al.* [7] ignore higher order terms and assume that R is constant to conclude that $\langle\Delta\hat{u}^2(\omega)\Delta\hat{u}^1(\omega)\rangle=\frac{k_BT}{2\pi a\rho i\omega G_o^*(\omega)}$.

We derive instead a formula based on the "decoupled" Langevin equations (12). We write the kernels as $\hat{\zeta}^j(\omega)=6\pi\eta_o^*(\omega)p^j(\rho,\kappa^*,\gamma)$, where p^j are polynomials in three variables $p^j(\rho^{-1},\kappa^*,\gamma)=1+(1-\kappa^*)\gamma+(1-2\delta_{j1})\frac{3}{2\rho}[1+2(1-\kappa^*)\gamma]$. Transforming Eqs. (12) into Fourier space, multiplying by $V^j(0)$, and taking the ensemble average we find

$$mi\omega\langle\hat{V}^j(\omega)V^j(0)\rangle + 6\pi a\langle p^j(\rho,\kappa^*,\gamma)\hat{V}^j(\omega)V^j(0)\rangle\,\eta_o^*(\omega) = m\langle V^j(0)V^j(0)\rangle + \langle G^j(\omega)V^j(0)\rangle.$$

In the limit defined above, we assume that R is constant, so that p^j is a constant in the ensemble average

$$mi\omega\langle\hat{V}^j(\omega)V^j(0)\rangle + 6\pi a\,\eta_o^*(\omega)p^j(\rho,\kappa^*,\gamma)\langle\hat{V}^j(\omega)V^j(0)\rangle = 2mk_BT.$$

Here we remark that equipartition of energy reads $m\langle V^j(t)V^j(0)\rangle=2k_BT$, because of the two-dimensional displacement (r,θ). We set $\Delta U^j(t)=U^j(t)-U^j(0)$. It is straightforward to show that $2\langle\hat{V}^j(\omega)V^j(0)\rangle=-\omega^2\langle[\Delta\hat{U}^j(\omega)]^2\rangle$. Then we obtain for $j=1,2$

$$\langle[\Delta\hat{U}^j(\omega)]^2\rangle = \frac{4k_BT}{-mi\omega^3-\omega^2 6\pi a\,\eta_o^*(\omega)p^j(\rho,\kappa^*,\gamma)}.$$

By definition we have

$$\langle(\Delta U^j)^2\rangle = \langle(\Delta u^1)^2\rangle + 2(2\delta_{j1}-1)\langle\Delta u^1\Delta u^2\rangle + \langle(\Delta u^2)^2\rangle$$

so that, neglecting inertia and using the definition of the complex viscosity we find

$$\langle[\Delta\hat{u}^1(\omega)]^2\rangle = \frac{k_BT}{3\pi a i\omega G_o^*(\omega)}\frac{1+(1-\kappa^*)\gamma}{p^1(\rho,\kappa^*,\omega)p^2(\rho,\kappa^*,\omega)},$$

$$\langle\Delta\hat{u}^1(\omega)\Delta\hat{u}^2(\omega)\rangle = \frac{k_BT}{2\pi a i\omega G_o^*(\omega)\rho}\frac{1+2(1-\kappa^*)\gamma}{p^1(\rho,\kappa^*,\omega)p^2(\rho,\kappa^*,\omega)}.$$

Expanding the above equations in a series in γ and ρ^{-1} gives the general formula for the two-point autocorrelation up to error terms $O(\gamma^2)$ and $O(\rho^{-2})$

$$\langle[\Delta\hat{u}_r^1(\omega)]^2\rangle = \frac{k_BT}{3\pi a i\omega G_o^*(\omega)}\left[1+\left(1-\frac{G_o^*(\omega)}{G_i^*(\omega)}\right)\gamma\right], \tag{14}$$

$$\langle \Delta \hat{u}_r^1(\omega) \Delta \hat{u}_r^2(\omega) \rangle = \frac{k_B T}{2\pi \rho i \omega G_o^*(\omega)}. \quad (15)$$

Combining Eqs. (13)–(15), we arrive at a 3×3 system of equations to determine the local (inner) complex modulus G_i^*, the outer shear modulus G_o^*, and the normalized thickness s of the shell surrounding the bead, given the standard autocorrelation and cross-correlation data from independent one-bead and two-bead experiments. [We note that the Mason-Weitz formula (13) is recovered by removing the distinction between the inner and outer kernels, $G_i^* = G_o^*$.]

We summarize the consequences of these results. We propose a two-step experiment, first with an isolated bead and then with coupled beads. Tracking one bead which does not interact with any neighbor, we extract $G_i^*(\omega)$ from Eq. (13). Since the displacement of one bead is constrained to a plane of coordinates (r,θ), Eq. (13) has to be multiplied by 2, to reflect the proper equipartition of energy formula. Tracking two beads interacting with each other in a range where their separation distance R remains constant and $O(R^{-2})$ is negligible, we find $G_o^*(\omega)$ from Eq. (15). Finally with the same two-bead data set, $\gamma = \frac{s}{a}$ is given from Eq. (14). This protocol assumes that each bead modifies its local environment in the same way so that one-bead data can be combined with two-bead data.

The protocol derived by Chen *et al.* [9] for determining the bulk modulus and the thickness of the shell is similar, although based on a different logic and use of experimental data. They determine an implicit formula for the ratio of G_i^* and G_o^* containing high order terms in $\beta = \frac{b}{a}$ as derived by Levine and Lubensky [3]. They then perform a series of three experiments with different bead radii, followed by numerical regression from their formula to determine the shell radius and the outer modulus. The inner modulus for their particular experimental system was purely viscous and known *a priori*. Our asymptotic ordering of the same underlying linear response equations achieves simplicity in the relations between experimental and model information: an explicit and one-to-one correspondence between (one-bead mean square displacement data, two-bead autocorrelation data, two-bead cross-correlation data) and (the inner modulus, the outer modulus, and the shell thickness). The implied protocol requires experimental data to be collected for one– and two-bead experiments and beads with identical size and surface chemistry.

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APPENDIX A: DETERMINATION OF THE STRESS TENSOR

The quasisteady state compressible Navier-Stokes is

$$0 = \mu \nabla^2 \vec{u} + (\mu + \lambda) \vec{\nabla}(\vec{\nabla} \cdot \vec{u}).$$

We make the ansatz of an azimuthally symmetric and linear in $\hat{z}$ solution. In spherical coordinates the general quasisteady state solution is

$$\vec{u}(\vec{x}) = -B_{-2} r^2 (\gamma_2 \cos\theta \hat{r} - \hat{z}) + B_0 \hat{z} + \frac{B_1}{r}(\gamma_1 \cos\theta \hat{r} + \hat{z}) - \frac{B_3}{r^3}(3\cos\theta \hat{r} - \hat{z}).$$

In the outer shell the solution vanishes at infinity, so that $B_0 = B_{-2} = 0$.

Since $\vec{u}$ is the displacement field, we define the strain tensor as E in the usual way. The stress is assumed to be linear and given by Hooke's law $\sigma = 2\mu E + \lambda \,\mathrm{tr}(E) I$. In spherical coordinates the stress tensor has the form

$$\sigma = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{pmatrix}.$$

The force acting in $\hat{z}$ through a sphere of radius r is the opposite of the force exerted by the medium obtained by integrating $-\sigma\hat{z}$ over the surface of the sphere $\sigma\hat{z} = \sigma_{rr}\cos\theta\hat{r} + \sigma_{r\theta}(\cos\theta - \sin\theta)\hat{\theta} - \sigma_{\theta\theta}\sin\theta\hat{\theta}$. The normal vector to the surface is $\vec{r}$ so that $\sigma\hat{z}\cdot\hat{r} = \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta = \sigma_{rz}$ and the force in $\hat{z}$ through the sphere of radius r becomes

$$F_z = -2\pi r^2 \int_0^\pi (\sigma_{rr}\cos\theta\sin\theta - \sigma_{r\theta}\sin^2\theta) d\theta.$$

For the inner and outer shell solution it turns out that the force in the z direction is independent of the radius of the sphere, $a < r < b$ or $r > b$, but only depends on the elastic properties of the medium; in other words,

$$F_{z,i} = \frac{16\pi a C_{1,i} \mu_i (\nu_i - 1)}{-3 + 4\nu_i}, \quad F_{z,o} = \frac{16\pi b C_{1,o} \mu_o (\nu_o - 1)}{-3 + 4\nu_o},$$

where $aC_{1,i} = B_1$ and $bC_{1,i} = B_1$, respectively, for $a < r < b$ and $b < r$.

APPENDIX B: DETERMINATION OF THE COEFFICIENTS $C_{j,o}, j=1,2$ AND $C_{j,i}, j=1,\ldots,4$

In the incompressible limit the constants in the inner and outer displacements are

$$C_{1,o} = -\frac{3}{2}\frac{\beta\epsilon p_1(\beta,\kappa)}{p_2(\beta,\kappa)}, \quad C_{1,i} = -\frac{3}{2}\frac{\kappa\epsilon p_1(\beta,\kappa)}{p_2(\beta,\kappa)},$$

$$C_{2,o} = \frac{1}{2}\frac{\beta\epsilon(-2\beta^5 + 2\kappa\beta^5 - 3 + 3\kappa - 5\kappa\beta^2)}{p_2(\beta,\kappa)},$$

$$C_{2,i} = \frac{1}{2}\frac{\kappa\epsilon(-2\beta^3 - 3 + 2\beta^3\kappa - 2\kappa)}{p_2(\beta,\kappa)},$$

$$C_{3,i} = \frac{-\epsilon\beta(-5\kappa\beta^2(1-\kappa)+3\kappa+6+4\beta^5(1-\kappa)^2-9\kappa^2)}{p_2(\beta,\kappa)},$$

$$C_{4,i} = \frac{6\epsilon\beta^3\kappa(1-\kappa+\kappa\beta^2-\beta^2)}{p_2(\beta,\kappa)},$$

where $p_1(\beta,\kappa)=2\kappa\beta^5-2\beta^5-3-2\kappa$ and $p_2(\beta,\kappa)=-10\beta^3\kappa+3\beta\kappa+6\beta+4\beta^6-8\beta^6\kappa+4\beta^6\kappa^2-9\beta\kappa^2+10\beta^3\kappa^2-9\beta^5\kappa^2+9\kappa\beta^5+4\kappa^2+6\kappa$. These formulas are mentioned, but not explicitly given, by Levine *et al.* [3].

APPENDIX C: ONE-POINT MICRORHEOLOGY

The Langevin equation for a single bead is

$$m\frac{dv_i(t)}{dt} = -\zeta(t)\star v_i(t) + f_i(t), \quad i = r,\theta. \tag{C1}$$

The memory kernel ζ is defined for $t \geqslant 0$, but can be extended as $\zeta(t)=0$ if $t<0$. This implies that the convolution term can be changed from $(0,t)$ to $(0,\infty)$ and the Langevin equation (C1) transformed into Fourier space retaining the initial conditions for the velocity. Multiplying by $v_i(0)$ and ensemble averaging we obtain

$$mi\omega\langle\hat{v}_i(\omega)v_i(0)\rangle + \hat{\zeta}(\omega)\langle\hat{v}_i(\omega)v_i(0)\rangle = m\langle v_i(0)v_i(0)\rangle + \langle\hat{f}_i(\omega)v_i(0)\rangle.$$

If we consider each displacement coordinate $i=r,\theta$ independently, then the equipartition of energy says that $m\langle v_i(0)v_i(0)\rangle = k_BT$. Moreover, we have $\langle f_i(t)v_i(0)\rangle = 0$ (see Duffy [15]), so that we can write

$$\langle\hat{v}_i(s)v_i(0)\rangle = \frac{k_BT}{mi\omega+\hat{\zeta}(s)}.$$

We set $\Delta u_i(t) = u_i(t) - u_i(0)$ and with $2\langle\hat{v}_i(\omega)v_i(0)\rangle = -\omega^2\langle(\Delta\hat{u}_i(\omega))^2\rangle$ we find

$$\langle[\Delta\hat{u}_i(\omega)]^2\rangle = \frac{2k_BT}{-mi\omega^3-\omega^2\hat{\zeta}(\omega)}.$$

Since $\hat{\zeta}(\omega)=6\pi a\eta_o^*=6\pi a\frac{G^*(\omega)}{i\omega}$ [see Sec. II B Eq. (9) with $\kappa=1$], the previous Eq. becomes

$$\langle[\Delta\hat{u}_i(\omega)]^2\rangle = \frac{2k_BT}{-mi\omega^3+6\pi ai\omega G_0^*(\omega)}. \tag{C2}$$

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