Cohomological Induction

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The setting for cohomological induction is the collection of pairs $(\mathfrak{g}, K)$ consisting of a complex Lie algebra and K an algebraic (or analytic) group over $\mathbb{C}$ (or $\mathbb{R}, \mathbb{Q}$) which acts on $\mathfrak{g}$ is a way compatible with the adjoint representation of $\mathfrak{g}$ on itself. It was shown by Vogan in **[?]** that understanding the category of pairs and the associated homological algebra thereof is a fruitful endeavor. We will mainly focus on the algebraic theory here, but will interject the analytic story when there is something to be gained from that view. Let us first formalize the definition we just hinted at:

Definition 0.0.1. An (algebraic) **Harish-Chandra pair** is a pair $(\mathfrak{g}, H)$ consisting of a complex Lie algebra $\mathfrak{g}$ and an algebraic group H defined over $\mathbb{C}$, such that

(a) There exists a morphism of algebraic groups $\varphi : H \to \mathrm{Aut}(\mathfrak{g})$

(b) There exists an injective Lie algebra homomorphism $\iota : \mathfrak{h} \to \mathfrak{g}$

(c) (Compatibility) $d\varphi = \mathrm{ad} \circ \iota$.

Let $(\mathfrak{g}, H)$ and $(\mathfrak{c}, L)$ be Harish-Chandra pairs. A morphism of Harish-Chandra pairs is a pair of maps $(\omega, \Omega) : (\mathfrak{g}, H) \to (\mathfrak{c}, L)$ where $\omega : \mathfrak{g} \to \mathfrak{c}$ is a Lie algebra homomorphism, $\Omega : H \to L$ is a morphism of algebraic groups, and $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}} d\Omega = \omega|_{\mathfrak{h}}$. Let **HCP** be the category of Harish-Chandra pairs.

Remark 0.0.2. In the analytic setting, we replace the algebraic assumption with K being a compact Lie group. Normally, K will be a maximal compact subgroup of a real reductive Lie group G, $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$.

Before we define modules for these pairs, we need to discuss representations in the algebraic setting. In particular, we will show that these representations form an abelian category as expected.

Definition 0.0.3. Let G be an algebraic group defined over $\mathbb{C}$. A representation (π, V) of G is **algebraic** if V is a union of finite dimensional representations $(V_i, i \in I)$ such that the representations of G on V_i are given by algebraic morphisms. Denote by $\mathcal{A}\,\mathrm{Rep}(G)$ the full subcategory of algebraic representations.

Lemma 0.0.4. *The category of algebraic representations is closed under subrepresentations, quotients, and direct sums. In particular, $\mathcal{A}\,\mathrm{Rep}(G)$ is abelian.*

Proof. (*Subrepresentations*) Let U be a subrepresentation of V. Then put $U_i = V_i \cap U$. These are clearly G-invariant. It suffices to show that the morphisms are algebraic. Fix $i \in I$. Let P_i be the subgroup of $GL(V_i)$ leaving U_i invariant. This is a closed subgroup and the morphism $G \to GL(V_i)$ factors through P_i. Whence, by restriction we get the morphism $G \to P_i \to GL(U_i)$ is algebraic.

(*Quotients*) Let (q, W) be the representation of G on $W = V/U$. Denote by ϕ the natural projection map. Set $W_i = \phi(V_i)$. Put $q_i : P_i \to GL(W_i)$ the natural morphisms. Then the representation of G on W_i is given by composition and thus this representation is algebraic.

(*Direct Sums*) Now let (π, V) and (σ, U) be two algebraic representations of G. Let the respective unions of finite dimensional representations be indexed by J and I. Then $V \oplus U$ is given as a union of the $V_j \oplus U_i$ and the morphisms are given by algebraic morphisms of G into $GL(V_j) \oplus GL(U_i) \subseteq GL(V_j \oplus U_i)$. Hence, the representation is algebraic and the proof is complete. $\square$

Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. Similar to analytic $(\mathfrak{g}, K)$-modules from before, we can now define them intrinsically without reference to a larger group.

Definition 0.0.5. (Lepowsky, Algebraic version) A $(\mathfrak{g}, K)$-**module** is a vector space V equipped with a representation of $\mathfrak{g}$ and an algebraic representation of K, denoted by $\pi : \mathfrak{g} \cup K \to \mathrm{End}(V)$, of $\mathfrak{g}$ and K, such that the following conditions are satisfied:

(a) The differential of $\pi|_K = (\pi|_{\mathfrak{g}})|_{\mathfrak{k}_0}$.

(b) For all $k \in K$ and $X \in \mathfrak{g}$, we have that $\pi(\varphi(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$.

We say that V is an **admissible** $(\mathfrak{g}, K)$-module if for every $\gamma \in \widehat{K}$, the γ-isotypic component $V(\gamma) = \bigcup_i V_i$, with $V_i \cong \gamma$ is a finite union.

Morphisms of $(\mathfrak{g}, K)$-modules are $\mathfrak{g}$ and K equivariant linear maps. Denote by $\mathcal{M}(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$-modules. An easy computation shows that $\mathcal{M}(\mathfrak{g}, K)$ is abelian. If $\mathfrak{g} = \mathfrak{k}$ then

$$\mathcal{M}(\mathfrak{k}, K) = \mathcal{A}\,\mathrm{Rep}(K)$$

and if $K = \{1\}$ then

$$\mathcal{M}(\mathfrak{g}, \{1\}) = \mathrm{Rep}(\mathfrak{g}) \simeq U(\mathfrak{g})\text{-Mod}$$

0.0.1 Induction, Coinduction, and the Zuckerman Functor

Now that we have the category $\mathcal{M}(\mathfrak{g}, K)$ associated to a Harish-Chandra pair, we can consider what happens when we have a morphism of pairs $(\omega, \Omega) : (\mathfrak{h}, L) \to (\mathfrak{g}, K)$. In the algebraic setting, we shall restrict ourselves to morphisms where at least one of the Lie algebra, or algebraic group morphism is injective. That is $(\mathfrak{h}, L)$ is a *Harish-Chandra subpair* of $(\mathfrak{g}, K)$.

Example 0.0.6. (a) Let $\mathfrak{h} = \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $L = (\mathbb{C}^\times)^n, K = GL(n, \mathbb{C})$. Then the morphism $(1, \iota) : (\mathfrak{g}, L) \to (\mathfrak{g}, K)$ realizes $(\mathfrak{gl}(n, \mathbb{C}), (\mathbb{C}^\times)^n)$ as a Harish-Chandra subpair of $(\mathfrak{gl}(n, \mathbb{C}), GL(n, \mathbb{C}))$.

(b) Let $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. If we put $L = SL(n, \mathbb{C})$ and $K = GL(n, \mathbb{C})$ then clearly $(\mathfrak{h}, L)$ is a subpair of $(\mathfrak{g}, K)$.

Induction for Harish-Chandra pairs

Definition 0.0.7 (Inducing Algebraically)**.** Let $(\mathfrak{g}, K) \in$ **HCP** and let (π, V) be a $(\mathfrak{h}, K)$-module for a subpair. Define

$$\operatorname{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V$$

to be the **Induced module** of (π, V).

Lemma 0.0.8. $\operatorname{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(V)$ *is a* $(\mathfrak{g}, K)$*-module.*

Proof. Let $\alpha : U(\mathfrak{g}) \times V \to U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$ be given by

$$\alpha(\xi, u) = \xi \otimes u$$

This map is bilinear and thus descends to a unique map on the tensor product $U(\mathfrak{g}) \otimes_{\mathbb{C}} V$. The map $\hat{\alpha}$ is clearly surjective and the kernel is a $(\mathfrak{g}, K)$-module. Combining this with the $\varphi \otimes \pi$-action makes this an algebraic representation of K. Define the representation of $U(\mathfrak{g})$ by left-multiplication in the first factor. Then differentiation of $\varphi \otimes \pi$ gives the following:

$$\xi \cdot (\eta \otimes u) = \mathrm{ad}(\xi)\eta \otimes u + \eta \otimes \xi u = [\xi, \eta] \otimes u + \eta\xi \otimes u = \xi\eta \otimes u$$

for all $\xi \in \mathfrak{k}$. Hence, $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$ is a $(\mathfrak{g}, K)$-module. $\square$

Lemma 0.0.9. $\operatorname{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \to \mathcal{M}(\mathfrak{g}, K)$ *is a functor.*

Proof. Let (π', V') be another $(\mathfrak{h}, K)$-module and $\Psi : V \to V'$ a $(\mathfrak{h}, K)$-homomorphism. Then the induced map is $1 \otimes \Psi$. This is clearly K-equivariant. Further, this is a $(\mathfrak{g}, K)$-homomorphism by the following computation:

$$\pi'(X)(\xi \otimes \Psi(u)) = X\xi \otimes \Psi(u) = (\pi(X)\xi) \otimes \Psi(u)$$

$\square$

Combining this with the Poincaré-Birkhoff-Witt theorem, we get the following result.

Theorem 0.0.10. $\operatorname{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *is exact.*

The existence of an exact functor begs the question: is $\operatorname{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ adjoint to another functor? The adjunction between extension and restriction of scalars implies this should be true. The following theorem makes this precise.

Proposition 0.0.11. $\operatorname{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *is left adjoint to the forgetful functor* $\operatorname{Res}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}, K)$.

Proof. As $U(\mathfrak{g})$ is a free $U(\mathfrak{h})$-module, it is clear that

$$\operatorname{Hom}_{U(\mathfrak{h})}(V, \operatorname{Res}_{\mathfrak{h}}^{\mathfrak{g}} U) = \operatorname{Hom}_{U(\mathfrak{g})}(V \otimes_{U(\mathfrak{h})} U(\mathfrak{g}), U)$$

as $U(\mathfrak{h})$-modules. The explicit bijection being given right to left by $\varphi \mapsto \varphi \circ i$ where $i : V \to V \otimes_{U(\mathfrak{h})} U(\mathfrak{g})$ is an injection by freeness. Now suppose V, U are in addition $(\mathfrak{h}, K)$ and $(\mathfrak{g}, K)$-modules respectively. Then using the bijection above, we know that the underlying $(\mathfrak{h}, K)$ and $(\mathfrak{g}, K)$ morphisms are $U(\mathfrak{h})$-submodules of $\mathrm{Hom}_{U(\mathfrak{h})}(V, \mathrm{Res}_{\mathfrak{h}}^{\mathfrak{g}} U)$ and $\mathrm{Hom}_{U(\mathfrak{g})}(V \otimes_{U(\mathfrak{h})} U(\mathfrak{g}), U)$ respectively. In particular, it suffices to show that the bijection constructed above is K-equivariant if we now assume the morphisms are K-morphisms additionally. To see this, let $k \in K$. Then

$$\varphi \circ i(kv) = \varphi(ki(v)) = k\varphi(i(v)) = k \cdot \varphi \circ i(v)$$

Hence, the bijections are K-equivariant and we get an $(\mathfrak{h}, K)$-module isomorphism

$$\mathrm{Hom}_{(h,K)}(V, \mathrm{Res}_{\mathfrak{h}}^{\mathfrak{g}} U) = \mathrm{Hom}_{(\mathfrak{g},K)}(V \otimes_{U(\mathfrak{h})} U(\mathfrak{g}), U)$$

□

Corollary 0.0.12. $\mathrm{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *maps projective objects to projective objects.*

Since $\mathrm{Ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ is exact, we also have:

Corollary 0.0.13. $\mathrm{Res}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *maps injective objects to injective objects.*

If we now assume K is reductive (say for example K is the complexification of a maximal compact subgroup of a real Lie group), then $\mathrm{Rep}(K)$ is semisimple and every object is projective. Whence, we see the following:

Corollary 0.0.14. *If K is reductive group, then $\mathcal{M}(\mathfrak{g}, K)$ has enough projectives.*

Proof. By use of the counit of the adjuction, we see that every $(\mathfrak{g}, K)$-module is a quotient of a projective module. □

In particular, this tells us that this functor will be nearly uninteresting in terms of homological techniques. Thus, we need to look for a different one that will yield a better view of these module categories. In particular, the induction functor corresponded to tensor products. Thus another option is to investigate Hom.

To do this properly however, we need to take a quick moment to discuss algebraic representations more thoroughly.

Algebraic Representations

Similar to passing to K-finite vectors in the analytic setting, we want to pass to the underlying algebraic representation of a given representation of K. To show that this is possible, we need some preliminary lemmata.

Definition 0.0.15. Let (π, V) be a representation of an algebraic group K. Then $v \in V$ is algebraic if there exists an algebraic subrepresentation of π containing v.

Lemma 0.0.16. *The set of all algebraic vectors in V is a K-invariant subspace. In particular, it is an algebraic representation.*

Proof. Let $v \in V$ be algebraic. By definition, there exists some (σ, U) an algebraic subrepresentation containing v. In particular, we see that $Kv \subseteq U$ and thus the set of algebraic vectors is a K-set.

Now let v_1, v_2 be two different algebraic vectors. To see that $v_1 + v_2$ is algebraic, let (σ_1, U_1) and (σ_2, U_2) be the algebraic subrepresentations containing v_1 and v_2 respectively. Then $v_1 + v_2 \in U_1 + U_2$ which is a quotient of an algebraic representation, namely $U_1 \oplus U_2$. Hence, the set of algebraic vectors is a vector subspace of V. □

Definition 0.0.17. The **algebraization** of V denoted $V_{[K]}$ is defined by

$$V_{[K]} := \{v \in V : v \text{ algebraic}\}$$

Clearly if $U \subseteq V$ is an algebraic subrepresentation then $U \subseteq V_{[K]}$. For this reason, $V_{[K]}$ is the *largest algebraic subrepresentation* of (π, V). Further, V is algebraic if and only if $V = V_{[K]}$.

Lemma 0.0.18. *Algebraization is an additive functor.*

Proof. It suffices to prove that given an algebraic representation (σ, U) and an arbitrary representation (π, V) that for any K-morphism $\phi : U \to V$, $\phi(u)$ is algebraic. This follows however from the first isomorphism theorem combined with Lemma 0.0.4. In particular, we get that $(-)_{[K]}$ is a functor. To show that it is additive, it suffices to show that $(U_1 \oplus U_2)_{[K]} = (U_1)_{[K]} \oplus (U_2)_{[K]}$.

For the reverse inclusion, let u_1, u_2 be algebraic in U_1 and U_2 respectively. Then $u_1 + u_2$ is algebraic in $U_1 \oplus U_2$ hence is contained in $(U_1 \oplus U_2)_{[K]}$. F

For the forward inclusion, let $(U_1)_{[K]} = \bigcup (U_{1i})_{[K]}$ and $(U_2)_{[K]} = \bigcup (U_{2j})_{[K]}$. Then $(U_1 \oplus U_2)_{[K]} = \bigcup (U_{1i} \oplus U_{2j})_{[K]}$. Now, we get a K-equivariant algebraic morphism

$$(U_1)_{[K]} \oplus (U_2)_{[K]} \to (U_1 \oplus U_2)_{[K]}$$

given by addition. This is injective clearly. In particular, it is injective on each of the finite dimensional pieces. Hence, an isomorphism by linear algebra.

□

Now we can prove the following theorem:

Theorem 0.0.19. *The algebraization functor is right adjoint to the forgetful functor from* $\mathcal{A}\,\mathrm{Rep}(K) \to \mathrm{Rep}(K)$.

Proof. Let U be an algebraic representation of K and V an arbitrary representation. Then for any morphism $\phi : U \to V$, $\phi(U)$ is an algebraic subrepresentation of V hence contained in $V_{[K]}$. Therefore, we get an injective map

$$\mathrm{Hom}_K(U, V) \to \mathrm{Hom}_{\mathcal{A}}(U, V_{[K]})$$

As $\mathcal{A}\,\mathrm{Rep}(K)$ is a full subcategory, this map is a bijection. Further, naturality of this bijection is immediate as it is K-equivariant. □

Corollary 0.0.20. *The algebraization functor is left exact.*

Now we shall consider the bi-functor $\mathrm{Hom}_{\mathbb{C}}(-,-)_{[K]}$. Given two representations (ν, U) and (π, V), the space $\mathrm{Hom}_{\mathbb{C}}(U, V)$ is a K representation given by

$$\rho(g)T = \pi(g)T\nu^{-1}(g)$$

This induces a left-exact (in both variables) bi-functor

$$\mathrm{Hom}_{\mathbb{C}}(-,-)_{[K]} : \mathcal{A}\,\mathrm{Rep}(K)^{op} \times \mathcal{A}\,\mathrm{Rep}(K) \to \mathcal{A}\,\mathrm{Rep}(K)$$

By differentiating the action of K, we get an action of $\mathfrak{k}$ on $\mathrm{Hom}_{\mathbb{C}}(-,-)$ given by

$$\rho(X)T = \pi(X)T - T\nu(X)$$

for all $X \in \mathfrak{k}$. By restricting we get a representation of $\mathfrak{k}$ on $\mathrm{Hom}_{\mathbb{C}}(-,-)_{[K]}$. Similarly we get a representation of $\mathfrak{k}$ on $\mathrm{Hom}_{\mathbb{C}}(-,-)_{[K]}$ given by differentiation of the K action. We want to know that these coincide.

Proposition 0.0.21. *The differential of the algebraic representation $\rho_{[K]}$ on $\mathrm{Hom}_{\mathbb{C}}(U,V)_{[K]}$ is given by*

$$\rho_{[K]}(X)T = \pi(X)T - T\nu(X)$$

Proof. Let $u \in U$, and $T \in \mathrm{Hom}_{\mathbb{C}}(U,V)_{[K]}$. Then the orbit map $\omega : K \to V$ given by

$$\omega(k) = \pi(k)T\nu^{-1}(k)u$$

is regular. Hence, we can apply the chain rule and get that

$$\rho_{[K]}(X)u = d\omega(X) = \pi(X)Tu - T\nu(X)u$$

for any $X \in \mathfrak{k}$. This completes the proof. □

In a similar fashion to before, assume K is now reductive. Then we have the following:

Lemma 0.0.22. *For K reductive, $\mathrm{Hom}_{\mathbb{C}}(-,-)_{[K]}$ is exact in both arguments.*

Proof. (Exactness is first argument) Let $0 \to U' \to U \to U'' \to 0$ be a short exact sequence of algebraic representations. This splits as G is reductive (say by a Zorn's lemma argument). Let $P : U \to U'$ be the splitting map. Therefore, the sequence

$$0 \to \mathrm{Hom}_{\mathbb{C}}(U'',V) \to \mathrm{Hom}_{\mathbb{C}}(U,V) \to \mathrm{Hom}_{\mathbb{C}}(U',V) \to 0$$

is exact and splits. Further, the sequence after passing to the algebraization is left exact. Now, let $A \in \mathrm{Hom}_{\mathbb{C}}(U',V)_{[K]}$. Then $A \circ P$ is in $\mathrm{Hom}_{\mathbb{C}}(U,V)_{[K]}$ and $i^*(A \circ P) = A$. Therefore

$$0 \to \mathrm{Hom}_{\mathbb{C}}(U'',V)_{[K]} \to \mathrm{Hom}_{\mathbb{C}}(U,V)_{[K]} \to \mathrm{Hom}_{\mathbb{C}}(U',V)_{[K]} \to 0$$

is exact.

Following this for the other argument, we arrive at the same result by way of splitting on the right. This completes the proof. □

Coinduction for Harish-Chandra pairs

Similar to coinduction of representations for finite groups, we consider how our representations can arise from Hom. This will be a bit more difficult than in the case of $\otimes$ simply because morphisms are slightly more complicated.

Nonetheless, define a $\mathfrak{g}$-module structure on $\mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}),U)$ for any representation (σ,U) of $\mathfrak{h}$ by

$$(\xi \cdot A)(u) = A(u\xi)$$

for $u \in U(\mathfrak{g})$.

Definition 0.0.23. Let V be a $(\mathfrak{h},K)$-module. The **coinduced module** of V is

$$\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(V) := \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}),V)_{[K]}$$

Proposition 0.0.24. $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(V)$ *is a* $(\mathfrak{g}, K)$*-module.*

Proof. Suppose first that we have an $\mathfrak{h}$-representation V. Then $\mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V)$ is a $\mathfrak{g}$-representation given by the action above. By the Poincaré-Birkoff-Witt theorem, $U(\mathfrak{g})$ is free over $U(\mathfrak{h})$. Hence, $\mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), -)$ is an exact functor from $\mathrm{Rep}(\mathfrak{h}) \to \mathrm{Rep}(\mathfrak{g})$.

Now, suppose V was indeed an $(\mathfrak{h}, K)$-module. Define a K action on $\mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V)$ by

$$(k \cdot A)(u) = \pi(k)A(\varphi(k^{-1})u)$$

it then follows that

$$(k \cdot A)(uv) = \pi(k)A(\varphi(k^{-1})(uv)) = \pi(k)\pi(\varphi(k^{-1})(u))A(\varphi(k^{-1})v) = \pi(u)(k \cdot A)(v)$$

for all $u \in U(\mathfrak{h}), k \in K$. Therefore, we have a representation of K on $\mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V)$.

Denote the action map as $\Psi : \mathfrak{g} \times \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V) \to \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V)$. Then for every $k \in K$, we have that

$$\Psi(k\xi, kA)(u) = \pi(k)A(\varphi(k^{-1})u\varphi(k)\xi) = \pi(k)(\xi \cdot A)(\varphi(k)^{-1}(u)) = k \cdot \xi \cdot A(u)$$

Thus the action is K-equivariant. Furthermore, differentiating this action gives

$$(\zeta \cdot A)(u) = \zeta A(u) - A((\mathrm{ad}\, \zeta)u) = A(\zeta u)$$

Thus, modulo the following result that the algebraization of this representation is $\mathfrak{g}$-stable, we have the proposition. $\square$

Lemma 0.0.25. *Let V be a vector space equipped with actions σ of $\mathfrak{g}$ and τ of K. Assume that the action map $\Psi : \mathfrak{g} \times V \to V$ is K-equivariant. Then $V_{[K]}$ is a $\mathfrak{g}$-submodule of V.*

Proof. The action map is bilinear and thus there exists $\hat{\Psi} : \mathfrak{g} \otimes V \to V$ which is K-equivariant. Since, φ is an algebraic representation of K on $\mathfrak{g}$, we have that for all algebraic vectors $\xi \in V$, the element $X \otimes \xi$ is algebraic in $\mathfrak{g} \otimes V$. By functoriality, $\hat{\Psi}(X \otimes \xi)$ is also algebraic. Hence, $\hat{\Psi}(\mathfrak{g} \otimes V_{[K]}) \subseteq V_{[K]}$ and $V_{[K]}$ is $\mathfrak{g}$-stable. $\square$

Corollary 0.0.26. $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *is a functor* $\mathcal{M}(\mathfrak{h}, K) \to \mathcal{M}(\mathfrak{g}, K)$.

Proof. $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(V)$ is a composition of functors

$$\mathcal{M}(\mathfrak{h}, K) \to \mathrm{Rep}^K(\mathfrak{g} \cup K) \to \mathcal{M}(\mathfrak{g}, K)$$

where $\mathrm{Rep}^K(\mathfrak{g} \cup K)$ is the category vector spaces equipped with a representation of $\mathfrak{g} \cup K$ such that the $\mathfrak{g}$ action is K-equivariant. $\square$

Again we have a version of Frobenius Reciprocity:

Theorem 0.0.27. *The functor* $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *is right adjoint to the forgetful functor.*

Proof. Let (σ, U) be an $(\mathfrak{h}, K)$-module and (π, V) be a $(\mathfrak{g}, K)$-module. Define

$$\alpha : \mathrm{Hom}_{U(\mathfrak{g})}(V, \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U)) \longrightarrow \mathrm{Hom}_{U(\mathfrak{h})}(V, U)$$

given by $f \mapsto \mathrm{ev}_1 \circ f$. This is clearly an abelian group homomorphism. We first need to check that ev_1 is an $(\mathfrak{h}, K)$-module map. It is clear that ev_1 is $U(\mathfrak{h})$ invariant. Further, for all $k \in K$, we have

$$\begin{aligned}
\alpha(k \cdot f)(v) &= \mathrm{ev}_1 \circ (k \cdot f)(v) \\
&= \mathrm{ev}_1(\pi(k) f(\varphi(k^{-1})v)) \\
&= \pi(k) f(\varphi(k^{-1})v)(1) \\
&= \pi(k)(\mathrm{ev}_1 \circ f)(\varphi(k^{-1})v) \\
&= k \cdot \alpha(f)(v)
\end{aligned}$$

With this established, we now need to show that α is a bijection. Let $\Psi \in \ker \alpha$. Then for any $v \in V$

$$0 = \alpha(\Psi)(\pi(\eta)v) = \mathrm{ev}_1(\Psi(\pi(\eta)v)) = \Psi(\pi(\eta)v)(1) = \rho(\eta)\Psi(v)(1) = \Psi(v)(\eta)$$

for all $\eta \in U(\mathfrak{g})$. Hence, $\Psi(v) = 0$ for all v and $\Psi = 0$. Thus α is injective.

Suppose now that $\Phi \in \mathrm{Hom}_{\mathfrak{h},K}(V, U)$. We want to construct Ψ such that $\alpha(\Psi) = \Phi$. For each $v \in V$ define $\Psi_v(\eta) = \Phi(\pi(\eta)v)$. Then for all $\zeta \in U(\mathfrak{h})$, we have that

$$\Psi_v(\zeta\eta) = \Phi(\pi(\zeta)\pi(\eta)v) = \sigma(\zeta)\Phi(\pi(\eta)v) = \sigma(\zeta)\Psi_v(\eta)$$

Thus $\Psi_v \in \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U)$. In addition, for any $k \in K$, we see

$$(k \cdot \Psi_v)(\eta) = \sigma(k)\Phi(\pi(\varphi(k^{-1})\eta)v) = \Phi(\pi(\eta)\pi(k)v) = \Psi_{\pi(k)v}(\eta)$$

Define $\Omega : V \to \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U)$ by $\Omega(v) = \Psi_v$. The previous computation shows that Ω is K-equivariant. To show it is a $(\mathfrak{g}, K)$-module map, let $\xi \in U(\mathfrak{g})$. Then

$$(\rho(\xi)\Omega(v))(\eta) = \Psi_v(\eta\xi) = \Phi(\pi(\eta)\pi(\xi)v) = \Psi_{\pi(\xi)v}(\eta) = \Omega(\pi(\xi)v)(\eta)$$

whence, Ω is a $(\mathfrak{g}, K)$-module morphism. As V is an algebraic representation of K, we see that $\Omega(v) = \Psi_v \in \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U)_{[K]}$.

Now set $\beta(\Phi) = \Omega$. Then

$$(\alpha \circ \beta)(\Phi)(v) = \alpha(\Omega)(v) = \Omega(v)(1) = \Phi(v)$$

Hence, β is a right inverse for α and α is bijective. $\square$

We now have the following formal categorical consequences:

Corollary 0.0.28. $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *is left exact.*

Corollary 0.0.29. $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ *maps injectives to injectives.*

If we assume K is reductive, we additionally have the following:

Corollary 0.0.30. *Assume K is reductive, then $\mathcal{M}(\mathfrak{g}, K)$ has enough injectives.*

Proof. Algebraic representations of K are semisimple and thus every object is injective. Combining this with the unit of the adjunction, we have the result. $\square$

Also similar to induction, we can deduce more about $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$.

Theorem 0.0.31. *If K is reductive, the functor $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ is exact.*

Unlike the case of induction, coinduction requires much more setup to prove this theorem. The setup begins with the following observation:

Observation 0.0.32. There exists a linear subspace $\mathfrak{s} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ where this decomposition is as a K-representation.

This follows from reductivity. Note that these actions extend to K-representations on $U(\mathfrak{h})$ and $S(\mathfrak{s})$. In this way, we have a K-action and left $U(\mathfrak{h})$-module structure on $U(\mathfrak{g})$. We can relate $S(\mathfrak{s})$ to $U(\mathfrak{g})$ by the symmetrization map:

$$\lambda(\xi_1 ... \xi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} ... \xi_{\sigma(n)}$$

By the Poincaré-Birkhoff-Witt theorem, this map is an injective vector space morphism. Now we can define the key function for the proof of Theorem 0.0.31. Let

$$\Lambda : U(\mathfrak{h}) \otimes S(\mathfrak{s}) \to U(\mathfrak{g})$$

be defined by $\Lambda(\eta \otimes \xi) = \eta \lambda(\xi)$.

Lemma 0.0.33. *The map Λ is a vector space isomorphism. Further, it is $U(\mathfrak{h})$ and K-equivariant.*

Proof. We will define two filtrations, one on each space, and then set up an isomorphism between the associated graded modules via Λ.

Define a filtration on $U(\mathfrak{g})$ via $U(\mathfrak{h})$ submodules by $F_p U(\mathfrak{g}) = U(\mathfrak{h}) \cdot U_p(\mathfrak{g})$ where $U_p(\mathfrak{g})$ is the standard filtration on $U(\mathfrak{g})$ by total degree of an element. Now define a filtration on $U(\mathfrak{h}) \otimes S(\mathfrak{s})$ by $F_p(U(\mathfrak{h}) \otimes S(\mathfrak{s})) = U(\mathfrak{h}) \otimes S_p(\mathfrak{s})$ where $S_p(\mathfrak{s})$ is the standard filtration on the symmetric algebra.

Under Λ it is clear that we obtain maps

$$\Lambda_p : U(\mathfrak{h}) \otimes S_p(\mathfrak{s}) \to F_p U(\mathfrak{g})$$

The Poincaré-birkoff-Witt theorem implies that $\mathrm{gr}(U(\mathfrak{g}), F_p) = U(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s})$ and that $\mathrm{gr}\,\Lambda$ is the identity. Therefore Λ is an isomorphism. The additional equivariance conditions follow immediately. $\square$

We can now proceed to the proof of the theorem.

Proof of Theorem 0.0.31. By the lemma, we get a K-equivariant isomorphism

$$\mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U) \to \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{h}) \otimes S(\mathfrak{s}), U) \to \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U)$$

where the second map is given by restriction.

Let $0 \to U' \to U \to U'' \to 0$ be an exact sequence of $(\mathfrak{h}, K)$-modules. Using the above, we get a corresponding commuting diagram

$$\begin{tikzcd} \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U'') \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U') \arrow[d] \\ \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U'') \arrow[r, hook] & \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U) \arrow[r, two heads] & \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U') \end{tikzcd}$$

with the vertical arrows isomorphisms and the horizontal rows are exact. Applying the algebraization functor and appealing to the exactness of $\mathrm{Hom}_{\mathbb{C}}(-,-)_{[K]}$ for reductive K, implies that we get

$$\begin{tikzcd} \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U'')_{[K]} \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U)_{[K]} \arrow[r] \arrow[d] & \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), U')_{[K]} \arrow[d] \\ \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U'')_{[K]} \arrow[r, hook] & \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U)_{[K]} \arrow[r, two heads] & \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U')_{[K]} \end{tikzcd}$$

Using the vertical isomorphisms, we see that the top row is exact. Hence, $\mathrm{Coind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ is exact. □

The Zuckerman Functor

As we have seen, the change of Lie algebras functors are exact when we assume reductivity. In particular, in the same theme as with the analytic theory, these are then uninteresting for the purposes of generating new representations from old. Therefore, we now consider changing the group! Given an algebraic subgroup T of K (this takes the place of the maximal torus of a compact Lie group), we clearly have that $(\mathfrak{g}, T)$ is a Harish-Chandra subpair of $(\mathfrak{g}, K)$. Thus, we obtain a forgetful functor:

$$\mathrm{For}_T^K : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}, T)$$

As these categories are abelian and For_T^K preserves all colimits, we know that there exists a right adjoint to it.

Definition 0.0.34. The **Zuckerman functor**, denoted Γ_T^K, is the right adjoint to the forgetful functor For_T^K.

This description of the Zuckerman functor is not helpful for explicit computations. For this reason, we will now describe a more explicit construction of this functor which will be useful.

Let $\mathcal{O}(K)$ denote the ring of regular functions. For any vector space V, consider $\mathcal{O}(K) \otimes V$ to be the space of regular functions on K with values in V. Denote this space by $\mathcal{O}(K, V)$. There are two natural representations of K on $\mathcal{O}(K, V)$ given by the left and right regular action.

Convention 0.0.35. For the remainder of this section, we will always assume $\mathcal{O}(K, V)$ is equipped with the *left regular representation* tensored with the trivial representation unless otherwise noted. For the sake of clarity, let the right regular action be denoted by R_k and the left regular action be denoted by L_k.

Assume now that (π, V) is an algebraic representation of K. Then there is a natural map

$$c : V \to \mathcal{O}(K, V)$$

$$k \mapsto \pi(k)v$$

Definition 0.0.36. The map c is the **matrix coefficient map**.

It follows immediately that $c(\pi(k)v)(h) = R_k(c(v))(h)$. That is to say that c is an injective intertwining operator between the two representations. If in addition V is a $(\mathfrak{g}, K)$-module, then $c(\pi(\xi)v)(k) = \pi(\varphi(k)\xi)c(v)(k)$ for all $\xi \in U(\mathfrak{g})$. Thus, define a representation ν of $\mathfrak{g}$ on $\mathcal{O}(K, V)$ by

$$\nu(\xi)F = \pi(\varphi(k)\xi)F$$

Whence c intertwines the $U(\mathfrak{g})$-module structure on V and $\mathcal{O}(K, V)$.

Remark 0.0.37. Note that we have not yet shown that $\mathcal{O}(K, V)$ is a $(\mathfrak{g}, K)$-module. The actions we have defined have not yet been shown to be compatible. As we will see later, this is actually more than we require. We will end up showing that a certain subspace is indeed a $(\mathfrak{g}, K)$-module.

The representation we are about to define will be the central player in determining a concrete realization of the Zuckerman functor. Set

$$L^\pi := L \otimes \pi : K \to \mathcal{O}(K, V)$$

to be another representation of K on $\mathcal{O}(K, V)$.

Lemma 0.0.38. *The image of the matrix coefficient map is contained in the space of all L^π invariant functions on K. In particular, it is an isomorphism between V and $\mathcal{O}(K, V)^{L^\pi}$.*

Proof. The following computation shows the first result:

$$L^\pi(k)c(v)(h) = \pi(k)c(v)(k^{-1}h) = \pi(h)v = c(v)(h)$$

To obtain the second statement, let $\Psi \in \mathcal{O}(K, V)^{L^\pi}$. Then

$$\Psi(k) = L^\pi(h)\Psi(k) = \pi(k)\Psi(k^{-1}h)$$

for all $h, k \in K$. Therefore $\Psi(kh) = \pi(k)\Psi(h)$. In particular, $\Psi(k) = \pi(k)\Psi(1)$. Whence $\Psi = c(\Psi(1))$. As c is a linear isomorphism which intertwines the actions of $\mathfrak{g}$ and K, it is a $(\mathfrak{g}, K)$-module isomorphism. This completes the proof. $\square$

Assume that (π, V) is only a $(\mathfrak{g}, T)$-module (again for $T \subseteq K$ algebraic). The structure of a $U(\mathfrak{g})$-module on $\mathcal{O}(K, V)$ was already defined. Equip $\mathcal{O}(K, V)$ with the right regular representation $R_k F(h) = F(hk)$ and let

$$L^\pi_T := L_T \otimes \pi$$

where L_T is the left regular representation of $\mathfrak{k}$ and T on $\mathcal{O}(K)$. This defines a $(\mathfrak{k}, T)$-module structure on $\mathcal{O}(K, V)$.

Lemma 0.0.39. *The representation L^π_T commutes with R and ν.*

Proof. It is clear that L_T^π commutes with R as the left and right regular representations commute. Thus we need to show ν commutes with L_T^π. Let us first show that for $t \in T$ and $\xi \in \mathfrak{g}$ this commutes:

$$\begin{aligned} L_T^\pi(t)\nu(\xi)F(k) &= \pi(t)(\nu(\xi)F)(t^{-1}k) \\ &= \pi(\varphi(k)\xi)\pi(t)F(t^{-1}k) \\ &= \nu(\xi)(L_T^\pi T(t)F)(k) \end{aligned}$$

On the other hand, the differential of L_T^π of $\mathfrak{k}$ on $\mathcal{O}(K,V)$ is given by

$$L_T^\pi(X)F(k) = (L(X)F)(k) + \pi(X)F(k)$$

Thus, for $h \in K$, we have that

$$(\nu(X)F)(h^{-1}k) = \pi(\varphi(h^{-1}k)X)F(h^{-1}k) = \pi(\varphi(h^{-1})\varphi(k)X)F(h^{-1}k)$$

Differentiating this at $h = 1$, we get

$$\begin{aligned} (L(X)\nu(\xi)F)(k) &= -\pi(\mathrm{ad}(X)(\varphi(k)\xi))F(k) + \pi(\varphi(k)\xi)(L(X)F)(k) \\ &= -\pi(X)(\nu(\xi)F)(k) + \pi(\varphi(k)\xi)((\lambda(X)F)(k) + \pi(X)F(k)) \\ &= -\pi(X)(\nu(\xi)F)(k) + (\nu(\xi)L_T^\pi(X)F)(k) \end{aligned}$$

for all $\xi \in \mathfrak{g}, X \in \mathfrak{k}$. Combining this with what we have from above, we see then that

$$(L_T^\pi(X)\nu(\xi)F)(k) = (\nu(\xi)L_T^\pi(X)F)(k)$$

Hence, L_T^π commutes with ν. $\square$

Definition 0.0.40. Let V be a $(\mathfrak{g}, T)$-module. Then the **Zuckerman module** of V is

$$\Gamma_T^K(V) := \mathcal{O}(K,V)^{L_T^\pi}$$

This is $\mathfrak{g}$ and K invariant by the above construction.

As with every functor we have defined thusfar, we have the following lemma:

Lemma 0.0.41. *$\Gamma_T^K(V)$ is a $(\mathfrak{g}, K)$-module.*

Proof. For $k \in K, \xi \in \mathfrak{g}$ and $F \in \mathcal{O}(K,V)$, we have

$$(\nu(\varphi(k)\xi)F)(h) = (R_k\nu(\xi)R_{k^{-1}}F)(h)$$

by a simple calculation. Further, for $F \in \Gamma_T^K(V)$ we have that

$$F(kh) = F(\mathrm{Int}(k)(h)k)$$

Since F is invariant for the L^π-action of K, by differentiating with respect to h, we get

$$(R_\xi F)(k) = -(L^\pi(\varphi(k)\xi)F)(k) + \pi(\varphi(k)\xi)F(k) = (\nu(\xi)F)(k)$$

Hence, $\Gamma_T^K(V)$ is a $(\mathfrak{g}, K)$-module. $\square$

Lemma 0.0.42. *Γ_T^K is a functor.*

Proof. For another $(\mathfrak{g}, T)$-module U and a morphism $\alpha : V \to U$ we have a natural candidate for the intertwining map: $1 \otimes \alpha : \mathcal{O}(K, V) \to \mathcal{O}(K, U)$. It clearly intertwines all of the actions and thus descends to the invariants. $\square$

We will now prove the key theorem which motivated this entire construction:

Theorem 0.0.43. *Γ_T^K is right adjoint to the forgetful functor $\mathrm{For}_T^K : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}, T)$.*

Before proving this, we need some setup involving the matrix coefficient map. If for each V we denote the matric coefficient map as c_V, then it is clear that $c_{(-)}$ is a natural transformation

$$\mathbb{1}_{\mathcal{M}(\mathfrak{g},K)} \xRightarrow{c_{(-)}} \Gamma_T^K \circ \mathrm{For}_T^K$$

between functors on $\mathcal{M}(\mathfrak{g}, K)$. In a similar fashion, the evaluation at 1 morphism $\mathrm{ev}_V : \Gamma_T^K(V) \to V$ gives a natural transformation:

$$\mathrm{For}_T^K \circ \Gamma_T^K \xRightarrow{\mathrm{ev}_{(-)}} \mathbb{1}_{\mathcal{M}(\mathfrak{g},T)}$$

Now we can prove the theorem:

Proof of Theorem 0.0.43. Let V be a $(\mathfrak{g}, K)$-module and U a $(\mathfrak{g}, T)$-module. For any $\alpha \in \mathrm{Hom}_{\mathfrak{g},T}(V, U)$ put $\bar{\alpha} = \Gamma_T^K(\alpha) \circ c_V$. This is a $(\mathfrak{g}, K)$-module morphism. Additionally, for any $\beta \in \mathrm{Hom}_{\mathfrak{g},K}(V, \Gamma_T^K(U))$, put $\hat{\beta} = \mathrm{ev}_U \circ \beta$. We claim these maps are mutually inverse. Let $v \in V$. Then

$$\begin{aligned}
(\bar{\hat{\beta}})(v)(k) &= (1 \otimes \hat{\beta}) \circ c_V(v)(k) \\
&= \hat{\beta}(c_V(v)(k)) \\
&= \mathrm{ev}_U(\beta(\pi(k)v)) \\
&= \beta(\pi(k)v)(1) \\
&= R_k\beta(v)(1) \\
&= \beta(v)(k)
\end{aligned}$$

Thus, $\bar{\hat{\beta}} = \beta$.

Similarly, for $\alpha \in \mathrm{Hom}_{\mathfrak{g},T}(V, U)$ we have

$$\hat{\bar{\alpha}}(v) = \bar{\alpha}(v)(1) = ((1 \otimes \alpha) \circ c_V)(v)(1) = \alpha(c_V(v))(1) = \alpha(v)$$

Hence, the result is proved. $\square$

We now have the following formal corollaries.

Corollary 0.0.44. *Γ_T^K is the Zuckerman functor as defined abstractly.*

Corollary 0.0.45. *Γ_T^K is left exact and maps injectives to injectives.*

Corollary 0.0.46. *The category $\mathcal{M}(\mathfrak{g}, K)$ has enough injectives.*

Proof. Consider Γ_1^K and the fact that the unit of the adjunction gives a monomorphism $V \to \Gamma_1^K(V) \to \Gamma_1^K(I)$ for I an injective object in $\mathrm{Rep}(\mathfrak{g})$. $\square$

Note that this result does not depend on K being reductive. Therefore, we can always define a cohomology theory for $(\mathfrak{g}, K)$-modules by way of the Zuckerman functor. To be precise about this, we will need the language of derived categories. One huge upshot of this approach is that we arrive at a quick proof of the Borel-Weil-Bott theorem.

0.0.2 Derived Categories of $(\mathfrak{g}, K)$-modules and vanishing of some Total Derived Functors

We begin, as always, with a definition.

Definition 0.0.47. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. The **unbound derived category** of $(\mathfrak{g}, K)$-modules is

$$D(\mathfrak{g}, K) := W^{-1}\mathcal{K}(\mathcal{M}(\mathfrak{g}, K))$$

where W is the localizing class of quasi-isomorphisms and $\mathcal{K}(-)$ denotes the homotopy category of chain complexes of $(\mathfrak{g}, K)$-modules. If we care to only consider semi-bounded complexes then we shall amend the notation to be $D^*(\mathfrak{g}, K)$ for $* \in \{b, +, -, \varnothing\}$ for bounded, bounded below, bounded above, and trivial complexes (complexes concentrated in a single degree).

For any Lie subalgebra with $\mathfrak{k} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$, we have a forgetful functor corresponding to the inclusion of pairs $(\mathfrak{h}, K) \hookrightarrow (\mathfrak{g}, K)$ which we (by an abuse of notation) will denote as

$$\operatorname{Res}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)} : D^*(\mathfrak{g}, K) \to D^*(\mathfrak{h}, K)$$

We saw previously that $\operatorname{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$ is exact. Combining this result with [**?**, Ch. 5, 1.7.1], we obtain the following result:

Theorem 0.0.48. *The functor* $\operatorname{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)} : D^*(\mathfrak{h}, K) \to D^*(\mathfrak{g}, K)$ *is left adjoint to the forgetful functor* $\operatorname{Res}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$.

Now, the category $\mathcal{M}(\mathfrak{h}, K)$ has enough injectives and $\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$ is left exact as proven in the previous section. By [**?**, Ch. 5, 3.1.3], we have that the total derived functor $\mathbf{R}\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$: $D^+(\mathfrak{h}, K) \to D^+(\mathfrak{g}, K)$ exists and by [**?**, Ch. 5, 1.7.1] we have:

Theorem 0.0.49. *The functor* $\boldsymbol{R}\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$ *is right adjoint to the forgetful functor.*

If K is assumed to be reductive, then we saw that $\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$ is exact. Thus, $\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$ lifts to a functor $\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)} : D^*(\mathfrak{h}, K) \to D^*(\mathfrak{g}, K)$. Whence, we obtain the following:

Theorem 0.0.50. $\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)} : D^*(\mathfrak{h}, K) \to D^*(\mathfrak{g}, K)$ *is right adjoint to the forgetful functor* $\operatorname{Res}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$.

That is to say that for K reductive, we have that $\mathbf{R}\operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)} = \operatorname{Coind}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}$.

Now assuming $T \subseteq K$ is a closed subgroup, we saw in the previous section we can define the Zuckerman functor Γ^K_T and it is left exact. Again as the category $\mathcal{M}(\mathfrak{g}, T)$ has enough injectives, we see that $\mathbf{R}\Gamma^K_T$ exists. Further, as the forgetful functor preserves injectives, we see that the following diagram commutes:

$$\begin{tikzcd} D^+(\mathfrak{g},T) \arrow[r, "\mathbf{R}\Gamma_T^K"] \arrow[d, "\mathrm{Res}^{(\mathfrak{g},T)}_{(\mathfrak{h},T)}"'] & D^+(\mathfrak{g},K) \arrow[d, "\mathrm{Res}^{(\mathfrak{g},K)}_{(\mathfrak{h},K)}"] \\ D^+(\mathfrak{h},T) \arrow[r, "\mathbf{R}\Gamma_T^K"'] & D^+(\mathfrak{g},K) \end{tikzcd}$$

Moreover, in a similar theme to above, [**?**, Ch. 5, 1.7.1] gives the following:

Theorem 0.0.51. *The functor $\boldsymbol{R}\Gamma_T^K$ is right adjoint to the forgetful functor $\mathrm{Res}^{(\mathfrak{g},K)}_{(\mathfrak{g},T)} : D^+(\mathfrak{g},K) \to D^+(\mathfrak{g},T)$.*

We can now combine the group induction with Lie algebra induction to obtain the so called *cohomological induction* functor.

Definition 0.0.52. Let $(\mathfrak{g},K)$ be a Harish-Chandra pair, $T \subseteq K$ a closed subgroup, and $\mathfrak{h}$ a lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$.. Then we have the following inclusions of pairs

$$(\mathfrak{h},T) \hookrightarrow (\mathfrak{g},T) \hookrightarrow (\mathfrak{g},K)$$

which induces a forgetful functor

$$\mathrm{Res}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} : D^+(\mathfrak{g},K) \to D^+(\mathfrak{h},T)$$

that factors through $D^+(\mathfrak{g},T)$. Whence, by the above theorems, has a right adjoint given by $\mathbf{R}\Gamma_T^K \circ \mathbf{R}\,\mathrm{Coind}^{(\mathfrak{g},T)}_{(\mathfrak{h},T)}$. The **cohomological induction functor** is defined as this composition and is denoted

$$\mathbf{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} := \mathbf{R}\Gamma_T^K \circ \mathbf{R}\,\mathrm{Coind}^{(\mathfrak{g},T)}_{(\mathfrak{h},T)}$$

Corollary 0.0.53. *The cohomological induction functor $\boldsymbol{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} : D^+(\mathfrak{h},T) \to D^+(\mathfrak{g},K)$ is right adjoint to the forgetful functor $\mathrm{Res}^{(\mathfrak{g},K)}_{(\mathfrak{h},L)}$.*

Corollary 0.0.54. *If T is reductive then*

$$\boldsymbol{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} = \boldsymbol{R}\Gamma_T^K \circ \mathrm{Coind}^{(\mathfrak{g},T)}_{(\mathfrak{h},T)}$$

Suppose now that we have yet another Harish-Chandra subpair $(\mathfrak{l},S) \hookrightarrow (\mathfrak{h},T)$. We get three cohomolgical induction functors $\mathbf{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)}, \mathbf{R}\,\mathrm{I}^{(\mathfrak{h},T)}_{(\mathfrak{l},S)}$ and $\mathbf{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{l},S)}$. We want to relate these to each other. A result from basic category theory tells us the following:

Theorem 0.0.55 (Cohomological Induction in Stages)**.** *The cohomological induction functor $\boldsymbol{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{l},S)} : D^+(\mathfrak{l},S) \to D^+(\mathfrak{g},K)$ is isomorphic to $\boldsymbol{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} \circ \boldsymbol{R}\,\mathrm{I}^{(\mathfrak{h},T)}_{(\mathfrak{l},S)}$.*

Remark 0.0.56. Notice that for induction in stages classically is more technical and requires additional components to be true. In particular, such a simple composition is not present in the classical picture.

Algebraic Representations for Reductive Groups, revisited

For K an algebraic group and T a closed subgroup, we have the categories of algebraic representations $\mathcal{A}\,\mathrm{Rep}(K)$ (resp. T) which we will now denote by $\mathcal{M}(K)$ (resp. T). As we have seen prior, there is a natural identification

$$\mathcal{M}(K) \simeq \mathcal{M}(\mathfrak{k}, K)$$

Therefore, using the above we have a cohomological induction functor

$$\mathbf{R}\,\mathrm{I}_T^K : D^+(T) \to D^+(K)$$

which is right adjoint to the forgetful functor.

If we assume however that T, K are reductive, then we obtain an even more interesting statement. We recall that in this situation $\mathrm{Coind} = \mathbf{R}\,\mathrm{Coind}$. Thus, $\mathbf{R}\,\mathrm{I}_T^K = \mathbf{R}\Gamma_T^K \circ \mathrm{Coind}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}$. On the other hand we have that Γ_T^K is right adjoint to the restriction functor and thus we obtain the following chain of bijections for an irreducible algebraic representation of K:

$$\mathrm{Hom}_K(V, \Gamma_T^K(\mathrm{Coind}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}(U))) = \mathrm{Hom}_{\mathfrak{k},T}(V, \mathrm{Coind}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}(U)) = \mathrm{Hom}_T(V, U)$$

Since H is reductive, we have that Hom (resp. $\otimes$) is exact. Therefore $U \mapsto \mathrm{Hom}_T(V, U)$ is exact. Whence, the functor

$$U \mapsto \mathrm{Hom}_K(V, \Gamma_T^K\,\mathrm{Coind}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}(U))$$

is exact.

Since K is reductive, every algebraic representation is a direct sum of irreducibles. Therefore, the functor $I_T^K = \Gamma_T^K \circ \mathrm{Coind}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}$ is exact. It is right adjoint to the forgetful functor and we have classical Frobenius reciprocity:

Theorem 0.0.57. *Let V be an irreducible representation of K. Then*

$$\mathrm{Hom}_K(V, I_T^K(U)) = \mathrm{Hom}_T(V, U)$$

Since I_T^K is exact, it lifts to a functor on the corresponding derived categories. Thus, we obtain the following result:

Theorem 0.0.58. *The functors I_T^K and $\boldsymbol{R}\,\mathrm{I}_{(\mathfrak{t},T)}^{(\mathfrak{k},K)}$ are isomorphic.*

A Vanishing Theorem

Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair, T a subgroup of K, and $\mathfrak{h}$ a T-invariant subalgebra of $\mathfrak{g}$. Assume that

(a) $\mathfrak{t} = \mathfrak{k} \cap \mathfrak{h}$

(b) $\mathfrak{g} = \mathfrak{k} + \mathfrak{h}$.

Then $(\mathfrak{h}, T)$ is a Harish-Chandra subpair of $(\mathfrak{g}, T)$. By the Poincaré-Birkhoff-Witt theorem we obtain an isomorphism:

Lemma 0.0.59. *$U(\mathfrak{h}) \otimes_{U(\mathfrak{t})} U(\mathfrak{k}) \cong U(\mathfrak{g})$ as $U(\mathfrak{h})$-modules.*

Further, in combination with the natural inclusion $i : U(\mathfrak{t}) \to U(\mathfrak{g})$ we obtain an isomorphism:

$$\mathrm{Hom}_{\mathfrak{h}}(U(\mathfrak{g}), U) \to \mathrm{Hom}_{\mathfrak{t}}(U(\mathfrak{k}), U)$$

for all $U \in \mathrm{Rep}(\mathfrak{g})$.

If in addition U is a $(\mathfrak{h}, T)$-module, then the isomorphism intertwines the T action and thus induces an isomorphism of algebraic representations

$$\mathrm{Hom}_{\mathfrak{h}}(U(\mathfrak{g}), U)_{[T]} \to \mathrm{Hom}_{\mathfrak{t}}(U(\mathfrak{k}), U)_{[T]}$$

Henceforth, we obtain the following lemma:

Lemma 0.0.60. *The following diagrams commute:*

$$\begin{tikzcd}
\mathcal{M}(\mathfrak{h}, T) \arrow[r, "\mathrm{Coind}_{(\mathfrak{h},T)}^{(\mathfrak{g},T)}"] \arrow[d, "\mathrm{Res}_{(\mathfrak{t},T)}^{(\mathfrak{h},T)}"'] & \mathcal{M}(\mathfrak{g}, T) \arrow[d, "\mathrm{Res}_{(\mathfrak{t},T)}^{(\mathfrak{g},T)}"] \\
\mathcal{M}(\mathfrak{t}, T) \arrow[r, "\mathrm{Coind}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}"'] & \mathcal{M}(\mathfrak{k}, T)
\end{tikzcd}$$

$$\begin{tikzcd}
D^+(\mathfrak{h}, T) \arrow[r, "\boldsymbol{R}\,\mathrm{Coind}_{(\mathfrak{h},T)}^{(\mathfrak{g},T)}"] \arrow[d, "\mathrm{Res}_{(\mathfrak{t},T)}^{(\mathfrak{h},T)}"'] & D^+(\mathfrak{g}, T) \arrow[d, "\mathrm{Res}_{(\mathfrak{t},T)}^{(\mathfrak{g},T)}"] \\
D^+(\mathfrak{t}, T) \arrow[r, "\boldsymbol{R}\,\mathrm{Coind}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}"'] & D^+(\mathfrak{k}, T)
\end{tikzcd}$$

By now applying the functor $\mathbf{R}\Gamma_T^K$ to the diagram and assuming K and T reductive, we obtain the following diagram:

$$\begin{tikzcd}
D^+(\mathfrak{h}, T) \arrow[r, "\mathbf{R}\,\mathrm{I}_{(\mathfrak{h},T)}^{(\mathfrak{g},K)}"] \arrow[d, "\mathrm{Res}_{(\mathfrak{t},T)}^{(\mathfrak{h},T)}"'] & D^+(\mathfrak{g}, T) \arrow[d, "\mathrm{Res}_{(\mathfrak{t},T)}^{(\mathfrak{g},T)}"] \\
D^+(\mathfrak{t}, T) \arrow[r, "I_T^K"'] & D^+(\mathfrak{k}, K)
\end{tikzcd}$$

Since, I_T^K is exact, we conclude that the higher cohomologies of $\mathbf{R}\,\mathrm{I}_{(\mathfrak{h},T)}^{(\mathfrak{g},K)}$ vanish. Thus, we obtain the following theorem:

Theorem 0.0.61. *Under the assumptions above, $I_{(\mathfrak{h},T)}^{(\mathfrak{g},K)}$ is exact.*

0.0.3 $\mathfrak{n}$-homology and the Borel-Weil-Bott theorem

Using everything we have done, we can now prove a cohomological induction version of a theorem of Borel and Weil and extended by Bott:

Theorem 0.0.62. *Let G be a connected semisimple algebraic group, T a Cartan subgroup, and λ an anti-dominant weight of T. For each $w \in W(\mathfrak{g}, \mathfrak{t})$ we have*

$$H^p(G/T, V_{w(\lambda-\rho)+\rho} = F_\lambda$$

for $p = \ell(w)$ and 0 otherwise.

We will do this by making use of $\mathfrak{n}$-homology and a theorem of Kostant.

$\mathfrak{n}$-homology as an adjoint

Let G be a semisimple algebraic group, T a Cartan subgroup with Lie algebra $\mathfrak{t}$, and $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ the set of roots. With respect to a choice of positivity, we can define Δ^+ and a subalgebra

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha := \mathfrak{t} \oplus \mathfrak{n}$$

It then follows that $\mathrm{Ad}(T)$ normalizes $\mathfrak{n}$ and for any $(\mathfrak{b}, T)$-module V we obtain a $(\mathfrak{b}, T)$-module $V_\mathfrak{n} = V/\mathfrak{n}V$ where $\mathfrak{n}$ acts trivially. Therefore, using the identification $\mathcal{M}(\mathfrak{t}, T) = \mathcal{M}(T)$, we see that $V \mapsto V_\mathfrak{n}$ defines an additive functor

$$\mathcal{M}(\mathfrak{b}, T) \to \mathcal{M}(T)$$

In the reverse direction, we can treat any $(\mathfrak{t}, T)$-module as a $(\mathfrak{b}, T)$-module by extending the action trivially to $\mathfrak{n}$. This defines a functor

$$F : \mathcal{M}(T) \to \mathcal{M}(\mathfrak{b}, T)$$

Lemma 0.0.63. *The functor $(-)_\mathfrak{n}$ is left adjoint to the functor F.*

Proof. For V a $(\mathfrak{b}, T)$-module and U an algebraic representation of T, we see that

$$\mathrm{Hom}_{(\mathfrak{b},T)}(V, F(U)) = \mathrm{Hom}_{(\mathfrak{b},T)}(F(V_\mathfrak{n}), F(U)) = \mathrm{Hom}_T(V_\mathfrak{n}, U)$$

□

Now as T is reductive, the category $\mathcal{M}(\mathfrak{b}, T)$ has enough projectives by the previous section. Therefore the functor $(-)_\mathfrak{n}$ has a total left derived functor $D^-(\mathfrak{b}, T) \to D^-(T)$.

If we forget the T action and restrict the $\mathfrak{b}$ action to $\mathfrak{n}$ this functor will be the derived functor of Lie algebra homology. Thus, denote by

$$H_\bullet(\mathfrak{n}, -) := \mathbf{L}(-)_\mathfrak{n}$$

From the standard complex, we see that the homological dimension $\mathrm{Hdim}(-)_\mathfrak{n}$ is finite. Thus the cohomological dimension is also finite and the functor $H_\bullet(\mathfrak{n}, -)$ extends to a functor on the entire unbound derived category. By restricting to bounded below complexes, we obtain the following:

Theorem 0.0.64. *The functor $H_\bullet(\mathfrak{n}, -) : D^+(\mathfrak{b}, T) \to D^+(T)$ is left adjoint to the functor $F : D^+(T) \to D^+(\mathfrak{b}, T)$.*

Combining this adjunction with the adjunction for the cohomological induction functors implies the following:

Corollary 0.0.65. *The functor $\boldsymbol{R}\,\mathrm{I}_{(\mathfrak{b},T)}^{(\mathfrak{g},G)} : D^b(T) \to D^b(G)$ is right adjoint to the functor $H_\bullet(\mathfrak{n}, -)$.*

The proof of the main theorem

Let μ be a weight of T and denote by $\mathbb{C}_\mu$ the one dimensional representation of T corresponding to μ. Let F_λ be the irreducible finite dimensional representation of G with lowest weight λ. Kostant proved the following elegant theorem:

Theorem 0.0.66 (Kostant)**.** *In the setting above, denote by $W(p)$ the elements of the Weyl group of length p. Then we have*

$$H_p(\mathfrak{n}, F_\lambda) = \bigoplus_{w \in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho}$$

for any $0 \leq p \leq \dim \mathfrak{n}$.

Applying this theorem to the above, we know that

$$H_\bullet(\mathfrak{n}, F_\lambda) = \bigoplus_{p=0}^{\dim \mathfrak{n}} \left(\bigoplus_{w \in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho} \right) [p]$$

By the adjointness we just proved, we have

$$\begin{aligned} \operatorname{Hom}_{D^+(G)}(F_\lambda, \mathbf{R}\,\mathrm{I}_{(\mathfrak{b},T)}^{(\mathfrak{g},G)}(\mathbb{C}_\mu)[q]) &= \operatorname{Hom}_{D^+(T)}(H_\bullet(\mathfrak{n}, F_\lambda), \mathbb{C}_\mu[q]) \\ &= \operatorname{Hom}_{D^+(T)} \left(\bigoplus_{w \in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho}, C_\mu[q] \right) \end{aligned}$$

Thus, $\operatorname{Hom}_{D^+(G)}(F_\lambda, \mathbf{R}\,\mathrm{I}_{(\mathfrak{b},T)}^{(\mathfrak{g},G)}(\mathbb{C}_\mu)[q]) = 0$ for $\mu \neq w(\lambda - \rho) + \rho$ for some $w \in W$. If μ is of this form and w has length p, then

$$\operatorname{Hom}_{D^+(G)}(F_\lambda, \mathbf{R}\,\mathrm{I}_{(\mathfrak{b},T)}^{(\mathfrak{g},G)}(\mathbb{C}_\mu)[q]) = \mathbb{C}$$

if $p = q$ and is 0 otherwise.

Applying Schur's lemma, we arrive at the result:

Theorem 0.0.67 (Cohomological Induction Borel-Weil-Bott)**.** *Let λ be an anti-dominant weight of T and $w \in W$. Then*

$$\boldsymbol{R}^p I_{(\mathfrak{b},T)}^{(\mathfrak{g},G)}(\mathbb{C}_{w(\lambda-\rho)+\rho}) = F_\lambda$$

for $p = \ell(w)$ and 0 otherwise.