Math 1321 Full Course Knowledge Checklist and Practice

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Chapter 8

1. ☐ (8.1) Sequences

   (a) Know the definition of a sequence as an infinite-sized ordered list of numbers $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$. A sequence is indexed by a whole number n that indicates the point in the sequence.

   (b) Sequences can be any list of numbers, but there are several types that are of specific interest: (11)

      (i) Functions of n: $a_n = f(n)$ defined by a specified mathematical operation on an input n; e.g., $a_n = 1/n^2$, or $a_n = e^{-n}\sin(n)$.

      (ii) Recursively defined sequences: a_{n+1} is defined in terms of a_n; e.g., $a_1 = 2$ and $a_{n+1} = 0.5(3 - a_n)$.

      (iii) Geometric: given a, r, define $a_n = ar^n$.

      (iv) Alternating sign: $(-1)^n$ or $(-1)^{n-1}$.

   (c) Be able to recognize when two differently-indexed or represented sequences are equivalent, or not. (12)

   (d) Know the definition of convergence for sequences: If $\lim\limits_{n\to\infty} a_n = L$ exists, then the sequence is convergent.

   (e) Be able to compute limits of sequences when possible; particularly, if $a_n = f(n)$, and $f(x)$ is an indeterminate form, you can apply l'Hospital's rule when needed.

   (f) Know how to utilize the Monotone Sequence Theorem (MST): If the sequence is monotonic increasing/decreasing, and the sequence is bounded above/below, respectively, then the sequence is convergent. When applying the MST be sure to verify/assert/prove that the hypotheses of the theorem are met in order to assert convergence.

2. ☐ (8.2) Series

   (a) An *infinite series* s is a sum of a sequence: $s = \sum\limits_{n=1}^{\infty} a_n$.

   (b) A *partial sum* s_N is the sum of the first N terms: $s_N = \sum\limits_{n=1}^{N} a_n$.

   (c) Know the definition of convergence and divergence of a series: s converges when the sequence of partial sums converges, that is $\lim\limits_{N\to\infty} s_N = s$ is a finite number; s diverges when it is undefined or diverges to infinity. (13)

   (d) Be able to recognize special categories of series and their rules for convergence/divergence:

(i) Geometric series: a. r numbers, then $s = \sum_{n=0}^{\infty} ar^n$ converges when $|r| < 1$, equal to $\frac{a}{1-r}$. (14)

(ii) p-series: p any number, $s = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \geq 1$, by the integral test (see below).

(e) Know the basic series manipulation identities. If $\sum a_n$ and $\sum b_n$ are convergent and c any number, then (15)

(i) $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$ is convergent.

(ii) $\sum ca_n = c \sum a_n$.

3. ☐ (8.2-4) Series convergence tests for $s = \sum_{n=h}^{\infty} a_n$, where h is some integer.

(a) Test for divergence: A series is divergent if $\lim_{n\to\infty} a_n \neq 0$ or undefined. (16)

(b) Integral test: If $a_n = f(n)$, and $f(x) > 0$ for x in $[h, \infty)$, and $f(x)$ is decreasing, then $\int_h^{\infty} f(x)dx$ has the same convergence/divergence property as s. (17)

(c) Comparison test (bounding by a known series): Suppose $a_n, b_n > 0$, and suppose $\sum b_n$ is known to converge, and if $a_n \leq b_n$ then s converges; conversely, if $\sum b_n$ is known to diverge, and if $a_n \geq b_n$ then s diverges. (18)

(d) Limit comparison test: suppose $\sum b_n$ is known to converge or diverge, and $a_n,\ b_n > 0$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, where c is finite, then s shares the same convergence property. (19)

(e) Alternating series test: Suppose $a_n = (-1)^{n-1} b_n$, where $b_n > 0$, $b_{n+1} \leq b_n$, and $\lim_{n\to\infty} b_n = 0$, then s converges. (20)

(f) Know the definition of absolute convergence: $\sum a_n$ converges absolutely if $\sum |a_n|$ is convergent. An absolutely convergent series is convergent, but a convergent series does not necessarily converge absolutely.

(g) The ratio test: Compute $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$. The scenarios are as follows (21)

(i) If $L < 1$ the series is absolutely convergent.

(ii) If $L > 1$ or is $= \infty$, then the series is divergent.

(iii) If $L = 1$ or DNE, then the test is indeterminate.

4. ☐ (8.2-4) Series estimation methods for convergent $s = \sum_{n=1}^{\infty} a_n$.

(a) A geometric series does not need to be estimated, the exact value is known: $s = \frac{a}{1-r}$ if $|r| < 1$.

(b) Students should understand the need for series estimation methods. Any *partial sum* s_N can be used as an approximation, but one might want to know how accurate each s_N is – estimation techniques give information as to how close a given partial sum s_N is to s.

(c) Know the definition of a remainder $R_N = s - s_N = \sum_{n=N+1}^{\infty} a_n$. The remainder can be considered the error between an approximation and the exact value.

(d) The remainder estimate for integral tests $a_n = f(n) > 0$:

$$\int_{N+1}^{\infty} f(x)\mathrm{d}x \leq R_N \leq \int_{N}^{\infty} f(x)\mathrm{d}x.$$

(22)

(e) Alternating series estimation theorem: if $s = \sum_{n=1}^{\infty}(-1)^n b_n$, where b_n satisfies the alternating series test hypotheses, then $|R_N| \le b_{N+1}$. (23)

5. ☐ (8.5-6) Power series

   (a) Know the definition of a function defined by a power series in x: $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$.

   (b) Be able to use convergence tests, principally the ratio test, to determine for which values x does the series converge.

   (c) Know the definition of the radius and interval of convergence and be able to determine them for a given series. (24)

   (d) Using the known representation of a geometric power series by $a/(1-r)$, be able to construct other power series by algebraic manipulation, integration, and differentiation. (25, 26, 27)

   (e) Know under which conditions a function $f(x)$ represented by a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable ($f'(x)$ exists) and computable by differentiation of the series term by term $\displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

   (f) Know under which conditions a function $f(x)$ represented by a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ is integrable ($\int f(x)dx$ exists) and computable by integration of the series term by term $C+\displaystyle\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$.

6. ☐ (8.7-8) Taylor and Maclaurin series

   (a) Be able to determine/compute the special power series representation of a function near a point $x = a$ – termed a Taylor series – as

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}, \quad \text{for } |x-a| < R.$$

   This is known as a *Taylor expansion of f at $x = a$*. When $a = 0$, the Taylor series is termed a Maclaurin series. (28, b)

   (b) Know the definition of the Nth partial sum as the order-N Taylor expansion

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

   (c) Know how to use the Taylor's Inequality to determine the bound on the error $R_N(x)$ in a neighborhood of $x = a$: $|x-a| \le d$: If $|f^{(N+1)}(x)| < M$ – the bound on the $(N+1)$th derivative for points $|x-a| \le d$, then the remainder/error between $T_N(x)$ and $f(x)$ is within the range (d, 30, 31)

$$|f(x) - T_N(x)| = |R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}, \quad \text{for} \quad |x-a| \le d.$$

   (d) If given $f(x)$ and a required error tolerance $\epsilon \ge |R_N(x)|$ on $|x-a| \le d$, students should be able to find the correct Taylor polynomial of order N that ensures this error tolerance is met; conversely, given an N, students should be able to determine the maximal error bound of $|R_N(x)|$ on $|x-a| \le d$. (32)

Chapter 9

7. ☐ (9.1) 3D coordinate systems

   (a) Know how to compute distances between points in the Cartesian 3D coordinate system.

   (b) Know how to represent a set of points forming a sphere as an equation of equal distance r from a center point (x_0, y_0, z_0) to the sphere surface:

   $$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

   (c) Know how to represent a set of points forming a cylinder as an equation of equal distance r from the vertical axis located at (x_0, y_0):

   $$(x - x_0)^2 + (y - y_0)^2 = r^2, \text{ where} z \in \mathbb{R}.$$

   (d) If given an equation of a sphere or a cylinder, be able to identify the center point or axis, respectively, and the radius. (33)

   (e) Be able to write out in set notation the points lying in a given surface. (34)

8. ☐ (9.2) Vectors in 2- and 3D space

   (a) Know how to compute the algebraic vector computations of vector addition, subtraction, and scalar multiplication using the algebraic properties on page 643. (35)

   (b) If given graphical depictions of two or more vectors, be able how to graphically represent vector addition, subtraction, and scalar multiplication. (36)

   (c) Given a vector $\vec{u}$, be able to produce a unit-length vector $\vec{U}$ in the same direction $\vec{U} = \frac{\vec{u}}{|\vec{u}|}$; also be familiar with the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. (37)

   (d) Be able to recognize and use various vector notations of angle brackets $\vec{u} = \langle u_1, u_2, u_2 \rangle$, and using standard unit vectors $\vec{u} = u_1\vec{i} + u_2\vec{j}, +u_2\vec{k}$.

   (e) Be able to compute the magnitude of a vector using the formula $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

9. ☐ (9.3) Dot product

   (a) Know how to compute the dot product between two vectors

   $$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

   (38)

   (b) Be able to use the dot product angle-formula:

   $$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta).$$

   (39)

   (c) Know how to identify the dot product number in relation to the angle θ between the vectors when depicted graphically.

   (d) Given a force vector $\vec{F}$ on an object moving on a path given by $\vec{d}$, students should be able to compute the work W done on the object as

   $$W = \vec{F} \cdot \vec{d}.$$

   (41)

(e) Be able to compute the angle between vectors using the dot product-based formula

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right).$$

(39)

(f) Know that two vectors are orthogonal when $\vec{u}\cdot\vec{v} = 0$. (41)

(g) Be able to perform vector calculations using the algebraic properties of the dot product shown on page 651. (42)

(h) Be able to find the scalar component of vector $\vec{u}$ in the direction of vector $\vec{v}$ using the *comp* formula

$$comp_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}.$$

(i) Be able to graphically depict the quantity $comp_{\vec{v}}(\vec{u})$ and explain how it relates to the vectors. (43)

(j) Be able to find the vector projection of vector $\vec{u}$ in the direction of vector $\vec{v}$ using the *proj* formula

$$proj_{\vec{v}}(\vec{u}) = comp_{\vec{v}}(\vec{u})\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}.$$

(44)

(k) Be able to graphically depict the vector projection as the vector of length $comp_{\vec{v}}(\vec{u})$ in the direction of $\vec{v}$.

10. ☐ (9.4) Cross product

(a) Students should be able to compute cross product between two vectors to be

$$\vec{u}\times\vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle.$$

(45)

(b) Students should be able to utilize all the algebraic properties of the cross product (page 656) to compute various vector calculations. (46)

(c) The magnitude of the cross product relates to the angle between vectors according to the torque formula in (a)

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\theta),$$

and is equal to the area of the parallelogram formed by the $\vec{u}$ and $\vec{v}$ sides. (47)

(d) Students should be able to infer the properties of the cross product for two special $\vec{u}$ and $\vec{v}$ relationships.

(i) $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

(ii) If $\vec{u}$ and $\vec{v}$ point along the same direction—either $\theta = 0$ or $\theta = \pi$–then $\vec{u}\times\vec{v} = \vec{0}$.

(e) Students should be able to compute torque: If a vector force $\vec{F}$ is applied to a lever arm $\vec{r}$ to effect a twisting force on an axis passing through the origin (the base of $\vec{r}$), the torque τ is the force vector that points along the axis of twisting, and has magnitude

$$|\vec{\tau}| = |\vec{F}||\vec{r}|\sin(\theta),$$

where θ is the angle between $\vec{F}$ and $\vec{r}$. The direction of $\vec{\tau}$ is chosen with the right hand rule. (48)

(f) Students should be able to compute the volume V of a parallelepiped formed by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ using the triple product formula

$$V = \vec{w}\cdot(\vec{u}\times\vec{v}).$$

(49)

11. ☐ (9.5) Equations for lines and planes

   (a) Students should commit to memory the two equation forms for lines and be able to be able to produce one from the other:
      (i) Symmetric: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
      (ii) Parametric: $\vec{r}(t) = \vec{v}t + \vec{r}_0$, $t \in \mathbb{R}$.

      (50)
   (b) Students should commit to memory the three equation forms for planes and be able to be able to produce any one from the others:
      (i) Normal form: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$
      (ii) Scalar: $ax + bx + cy = d$.
      (iii) Parametric: $\vec{r}(s,t) = \vec{u}t + \vec{v}s + \vec{r}_0$, $s, t \in \mathbb{R}$.

      (51)
   (c) Students should be able to produce the correct equation for a line or plane if given two, or three points respectively, that reside in the line/plane. (51)
   (d) Given two objects, either points, lines, or plane objects, students should be able to the object defined by their intersection (if it exists). (52)
   (e) Students should be able to find the angle of intersection between two line or plane objects. (53)
   (f) Students should be able to find the distance between a plane or line and a point in space by finding the closest-distance point. (55)
   (g) Given a line or a plane, students should be able to find orthogonal or parallel lines or planes, possibly subject to additional conditions. (56,57)

12. ☐ (9.6) Multivariate functions and surfaces

   (a) Be able to determine the domain and range of a multivariable function $z = f(x, y)$ and express the domain as a 2D set.
   (b) Be able to draw or graphically depict the rudiments of the surface generated from a multivariable function $z = f(x, y)$. This can be done several strategies, when applicable.
      (i) Holding one variable fixed (e.g., y) and drawing cross sections in the other variable (e.g., x).
      (ii) Using special geometric principles if applicable (e.g., distance formulas, etc).
      (iii) Finding level sets: fix z and draw a contour lines in the x-y plane.
   (c) Be familiar with certain prototypical surfaces and their corresponding functions: Paraboloids, hyperbolic paraboloids, ellipsoids, cones, ... (see page 679).

   (58)

13. ☐ (9.7) Cylindrical and spherical coordinates

   (a) Students should be able to convert between cartesian coordinates and cylindrical coordinates:

$$x = r\cos(\theta), \quad y = r\sin(\theta),$$
$$\tan(\theta) = \frac{y}{x}, \quad r^2 = x^2 + y^2, \quad z = z.$$

      (59)
   (b) Students should be able to convert between cartesian coordinates and spherical coordinates:

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi)$$
$$\tan(\theta) = \frac{y}{x}, \quad \rho^2 = x^2 + y^2 + z^2, r = \rho\sin(\phi).$$

      (59)
   (c) Be able to identify basic surfaces defined by equations expressed in these coordinate systems, like cones, spheres, etc. (60)

Chapter 10

14. ☐ (10.1) Vector functions

    (a) Students should be able to graphically represent space curves given by vector functions of the form

    $$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in [a, b].$$

    (61)

    (b) Given a description of a basic space curve, students should be able to formulate a vector function $\vec{r}(t)$ that represents the curve.(62)

    (c) Be able to formulate a space curve as a vector function $\vec{r}(t)$ if the space curve is defined by a set of equations. (63)

15. ☐ (10.2) Derivatives and integrals of vector functions.

    (a) Students should be able to compute derivatives of vector functions

    $$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle, \quad t \in [a, b].$$

    (64)

    (b) Students should be able to graphically depict the direction and magnitude of $\vec{r}'(t)$ as tangent to the space curve with magnitude equal to the speed $v(t) = |\vec{r}'(t)|$.

    (c) Students should be able to compute derivatives of vector expressions using the differentiation rules on page 704. (65)

    (d) Students should be able to compute integrals of vector functions

    $$\int_a^b \vec{r}(t)dt = \langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \rangle.$$

16. ☐ (10.3) Arc length and curvature

    (a) Students should be able to produce and compute the arc length L formula of a space curve, defined as

    $$L = \int ds = \int_a^b |\vec{r}'(t)|dt,$$

    where $ds = |\vec{r}'(t)|dt$. (66,67)

    (b) Know how the graphical meaning and how to compute the tangent vector: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

    (c) Students should be able to explain and compute the curvature $\kappa(t)$ of a space curve as the magnitude of rate of change of $\vec{T}$ with respect to arc length ds: $\kappa = \left|\frac{d\vec{T}}{ds}\right|$. The constructive formulas for computing curvature are

    $$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3},$$

    where $ds = |\vec{r}'(t)|dt$. (68)

    (d) Students should know how to compute the, tangent, normal and binormal vectors forming a local coordinate system $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ at every point $\vec{r}(t)$ of a space curve (see page 712-713. Students should be able to graphically depict the directions of these vectors where appropriate. (69)

    (e) Know the definitions of the normal and osculating planes of a space curve, know their normal vectors and how to find equations of the planes. Be able to describe the behavior of these planes as a function of time t.

17. ☐ (10.4) Motion in space: velocity and acceleration

   (a) Students should be able to compute the velocity and acceleration vectors given a position vector function $\vec{r}(t)$"
   $$\vec{v}(t) = \vec{r}'(t), \quad \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

   (b) Students should be able to decompose the velocity vector into its direction unit vector $\vec{T}(t)$, and magnitude given by speed $v(t) = |\vec{v}(t)|$:
   $$\vec{v}(t) = v(t)\vec{T}(t)$$

   (c) Students should be able to decompose the acceleration vector into its tangential component $\vec{T}$ (fore/aft direction), and its normal component $\vec{N}$ (side-to-side, or steering direction) as expressed as:
   $$\vec{a}(t) = v'(t)\vec{T}(t) + \kappa(t)v^2(t)\vec{N}(t),$$
   where $v'(t)$ is the tangential component of acceleration
   $$v'(t) = a_T(t) = \vec{T}(t) \cdot \vec{a}(t),$$
   and $\kappa(t)v^2(t)$ is the normal component of acceleration:
   $$\kappa(t)v^2(t) = a_N(t) = \vec{N}(t) \cdot \vec{a}(t).$$
   (70)

   (d) Given a vector $\vec{x} = \langle x_1, x_2, x_3 \rangle$ in the standard coordinate system $(\vec{i}, \vec{j}, \vec{k})$, be able to represent $\vec{x}$ in an alternative coordinate system defined by orthogonal vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ as
   $$\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$$
   where the triplet (a, b, c) are the coordinates for $\vec{x}$ in the new system given by
   $$a = \frac{\vec{x} \cdots \vec{u}}{|\vec{u}|^2}, \quad b = \frac{\vec{x} \cdots \vec{v}}{|\vec{v}|^2}, \quad a = \frac{\vec{x} \cdots \vec{w}}{|\vec{w}|^2}.$$
   (71)

Chapter 11

18. ☐ (11.1) Functions of several variables $f(x, y) = z$.

   (a) Students should be able to determine the domain and range of multivariate functions.

   (b) Students should be able to determine the nature of the relationship between points (x, y) and z using a methodical approach involving the synthesis of multiple approaches:

      (i) Forming a table of values for a relevant range of example points (x, y, z).

      (ii) Studying the algebraic relationship between (x, y) and z.

      (iii) Form graphical depictions of the surface by setting a variable equal to constants and drawing the resulting traces or level curves of the remaining variables. Note that for each trace formed at each (x, y) point, the traces must intersect for the surface to be defined by a function. (72,73,74)

   (c) The aforementioned items must also be performed on any 3D coordinate system: Euclidian, cylindrical, spherical.

19. ☐ (11.2) Limits and continuity of $f(x, y) = z$.

(a) Students should understand that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that $f(x,y)$ takes on z-values that are exceedingly close to L whenever (x,y) is near (a,b). This means that any trajectory $(x,y) \to (a,b)$ yields $z \to L$.

(b) Students should be able to craft a clear argument that a given limit *does not exist* by taking two limits $(x,y) \to (a,b)$ along two distinct paths C_1 and C_2 and establishing that the limits (if they exists) are different. If the limits on the two paths are the same, then the no conclusions can be made about the existence of the limit. (75,76)

(c) Continuity definition: Students should know the definition of continuity and use it to assert existence of a limit. Definition: If a limit exists at a point $\lim_{(x,y)\to(a,b)} f(x,y) = L$, and $f(a,b) = L$, then the function is continuous at (a,b). Limit existence: If a function is known to be continuous, students should be able to use this fact to assert what the limit is (L).

(d) Determining continuity: Just as with single-variable functions, certain categories of functions like polynomials, trigonometrics, logs, exponentials,, and function compositions, sums, products, and divisions of these functions, are continuous, if not at singular points (e.g., division by zero points or other singularities). That is, if a function is in such a category and non singular at the point in question, the function is continuous at that point. Students should be able to identify these functions and be able to assert continuity when needed. (77)

20. ☐ (11.3) Partial derivatives

(a) Students should be able to compute partial derivatives using the limit-based definition.

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}, \quad \frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

(b) Students should be able to compute partial derivatives using standard differentiation rules. (78,79)

(c) Students should be familiar with the various partial derivative notations

$$\frac{\partial f}{\partial x} = f_x = \partial_x f = D_x f, \quad \frac{\partial f}{\partial y} = f_y = \partial_y f = D_y f,$$

and second partial derivative notations, for instance

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}\partial_{xx} f = D_{xx} f, \quad \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx} = \partial_{yx} f = D_{yx} f.$$

(d) Given a differentiable function (see below), students should be able to determine the slope of a function's tangent lines in the x or y directions using partial derivatives and be able to use this information to understand the local properties of the surface of f.

(e) Students should be able to use Clairaut's theorem to assert the equality of mixed partials: If f is defined in a region around (a,b) and the mixed partials f_{xy} and f_{yx} are both continuous there (i.e. the mixed partials smooth), then $f_{xy} = f_{yx}$ at (a,b).

21. ☐ (11.4) Tangent planes and linear approximations

(a) Students should be able to compute/find tangent plane of a surface defined by $f(x,y) = z$ at a point (x_0, y_0, z_0) is defined by

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0),$$

where the normal vector to the tangent plane is $\vec{n} = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle$.(80)

(b) Students should be able to use the tangent plane of function at a point (x_0, y_0, z_0) as a way to approximate $f(x,y)$ at points (x,y) near (x_0, y_0):

$$f(x,y) = z \approx z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

(81)

(c) Differentiability definition: f is differentiable at a point (a, b) if the error $R(x, y)$ between a function and its linear approximation goes to zero as $(x, y) \to (a, b)$ in such a manner that $R(x, y)/(x-a) \to 0$ and $R(x, y)/(y - b) \to 0$. That is, a tangent plane well approximates the surface in a way that R goes to zero faster than linearly (e.g., quadratically). This is a difficult definition put here for reference only.

(d) Students should be a able to use the differentiability theorem—If the partials of f exists and are continuous at (a, b), then f is differentiable—to determine differentiability and the usability of a tangent plane to make good approximations.

22. ☐ (11.5) Chain rule

(a) Given a differentiable trajectory in 2D $\vec{r}(t) = \langle x(t), y(t)\rangle$, students should be able to compute the derivative of $f(x, y) = z$ with respect to time on the trajectory using the chain rule in 2D:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

(82)

(b) Student should be able to compute derivatives similarly defined for functions of three variables $u = f(x, y, z)$ defined on 3D trajectories $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$:

$$\frac{du}{dt} = f_x x' + f_y y' + f_z z'.$$

(83)

(c) Students should be able to graphically depict and/or describe $z(t)$ and its surrounding local surface based on information in $z'(t)$ computed as above in (a). (84)

(d) Implicit functions: Given an equation $F(x, y) = 0$ that defines implicitly a function $f(x) = y(x) = y$, students should be able to compute the derivative of $f(x) = y$ using the chain rule:

$$\frac{d}{dx}\Big(F(x, y(x))\Big) = 0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} \implies \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

23. ☐ (11.6) Directional derivatives and gradient vector.

(a) Students should be able to compute the directional derivative of a differentiable function $f(x, y) = z$ at a point (x, y) in any direction $\vec{u} = \langle a, b\rangle = \langle\cos(\theta), \sin(\theta)\rangle$ defined as a unit vector $|\vec{u}| = 1$ given by the formula

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

(85)

(b) Students should be able to graphically identify $D_{\vec{u}} f(x, y)$ as the slope of the tangent line on the surface of f in the direction $\vec{u}$.

(c) Students should know how to compute the gradient vector $\nabla f = \langle f_x, f_y\rangle$

(d) Students should be able identify direction of travel with the steepest ascent of the surface is defined by the maximal $D_u f$, which will always be in the direction of ∇f. Moreover, ∇f will be orthogonal to any level curve tangent vector. (86)

(e) Students should be able to express directional derivative by the gradient vector as $D_{\vec{u}} f(x, y) = \nabla f \cdot \vec{u}$.

(f) Given a surface defined by the equation $F(x, y, z) = k$, students should be able to find the normal vector of the tangent plane of the surface at a point (x_0, y_0, z_0) as $\vec{n}(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$, with tangent plane equation given by

$$0 = \vec{n}(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle.$$

(87)

24. ☐ (11.7) Minimum and maximum values of a surface

   (a) Students should utilize Fermat's theorem (not his last theorem) to aid in finding local min and max values: if f has a local min/max, at (x, y), then $\nabla f(x, y) = \vec{0}$—the point (x, y) is a critical point if $\nabla f(x, y) = \vec{0}$. (a)

   (b) Students should be able to use the second derivative test of a differentiable f to determine if a critical point (x, y) is a local minimum, maximum, saddle, or indeterminate point: Define

$$D_{(x,y)} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

   If $D > 0$ and $f_{xx} > 0 \implies$ local min; if $D > 0$ and $f_{xx} < 0 \implies$ local max; $D < 0 \implies$ saddle; if $D = 0 \implies$ indeterminate.(b,89)

   (c) Students should be able to determine absolute min and max values on a closed set U by checking all critical points within U and its boundary points. (90)

25. ☐ (11.8) Lagrange multipliers—optimization under constraints.

   (a) Students should recognize that many optimization problems in science and engineering involve maximizing/minimizing a given value $z = f(x, y)$ subject to some given constraint on (x, y) to satisfy a given equation $g(x, y) = 0$—this could also be expressed in three or more independent variables.

   (b) Solving the problem in (a) involves solving

$$\nabla f = \lambda \nabla g,$$

   for an unknown scalar λ to find critical point (x, y). (91, 92)

   (c) Students should be able to graphically and mathematically explain the rationale behind (c): $g(x, y) = 0$ defines a level curve $\vec{r}(t)$. The min/max $z(t)$ critical points will be attained traveling along on $\vec{r}(t)$: $z(t) = f(\vec{r}(t))$. $z(t)$ will be a critical point when $z'(t) = 0$. By the chain rule $z'(t) = f_x x' + f_y y' = 0 = \nabla f \cdot \vec{r}'(t)$, meaning that the gradient of f is orthogonal to the level curve only at critical points. Note also that $0 = \frac{dg}{dt} = \nabla g \cdot \vec{r}'(t)$, meaning the gradient of g is orthogonal to the level curve everywhere. Therefore ∇g and ∇f must pointing along the same direction at critical points; or rather, they are scalar multiples of each other, thus verifying the equation in (c) defining the critical points.

Chapter 12

26. ☐ (12.1) Double integrals over rectangular regions.

   (a) Be able to visualize functions $f(x, y) = z$ over rectangular regions $[a, b] \times [c, d]$.

   (b) Understand that the volume under the surface of a function $f(x, y) = z$ over rectangular region $R = [a, b] \times [c, d]$ can be computed by a double Riemann sum

$$V \approx \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_{ij}^*, y_{ij}^*) \Delta A \,,$$

   where R is broken up into many rectangular patches $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ with area $\Delta A = \Delta x \Delta y$, where

$$x_i = a + i\Delta x \,, \quad y_j = c + j\Delta y \,, \quad \Delta x = \frac{b-a}{N} \,, \quad \Delta y = \frac{d-c}{M}$$

   with sample points (x_{ij}^*, y_{ij}^*) in R_{ij}.

(c) Be able to numerically approximate the volume using the double Riemann sum and choosing the sample points by one of the following methods (93)

   i. Basic midpoint rule: The sample points are in the middle of the rectangle R_{ij}, that is $x_{ij}^* = \bar{x}_i = \frac{x_i + x_{i-1}}{2}$ and $y_{ij}^* = \bar{y}_j = \frac{y_j + y_{j-1}}{2}$.

   ii. Left-endpoint rule: The sample points are in the lower left corner of the rectangle R_{ij}, that is $x_{ij}^* = x_{i-1}$ and $y_{ij}^* = y_{j-1}$.

   iii. Right-endpoint rule: The sample points are in the upper right corner of the rectangle R_{ij}, that is $x_{ij}^* = x_i$ and $y_{ij}^* = y_j$.

(d) Be able specify the a Riemann sum as a limit of finite sums from above as the definition of a double integral

$$V = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_{ij}^*, y_{ij}^*)\Delta A \equiv \iint\limits_R f(x,y)\; dA\,.$$

27. ☐ (12.2) Iterated integrals.

(a) Be able to apply Fubini's theorem: If $f(x,y) = z$ is continuous on the rectangle $R = [a,b] \times [c,d]$, then the double integral can be computed by either of the following iterated integrals:

$$\iint\limits_R f(x,y)\; dA = \int_c^d \int_a^b f(x,y)\; dx\; dy = \int_a^b \int_c^d f(x,y)\; dy\; dx\,,$$

where either iterated integral is computed by first computing the inner integral in one variable, keeping the other variable a fixed parameter, then subsequently computing the outer integral afterwards.

(b) Be able to identify, where relevant, when one iterated integral is more advantageous or computationally simpler than the other. (94)

(c) When the integrand is a product of two functions in one variable, that is $f(x,y) = g(x)h(y)$, be able to exploit the identity: (105)

$$\int_c^d \int_a^b g(x)h(y)\; dx\; dy = \left(\int_a^b g(x)\; dx\right)\left(\int_c^d h(y)\; dy\right)\,.$$

28. ☐ (12.3) Integration over general regions.

(a) Be able to specify type 1 and type 2 regions D in set notation:

Type I: $D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$

Type II: $D = \{(x,y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$

(b) Given a set of equations and/or lines defining the boundaries of a region D, be able find the intersection points for the boundary curves and formulate the region D as a type I or II region when possible. If the full region is not type I or II, be able to split the region into subsets that are when possible. (98)

(c) Given a type I or II region, be able to express the double integral $\iint_D f(x,y)\; dA$ as one of the iterated integrals containing functions of the outer integral variable in limits of integration of the inner integral:

Type I: $$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\; dy\; dx$$

Type II: $$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\; dx\; dy$$

Be able to compute these integrals. (95, 96)

(d) If a region D is both type I and II, be able to assess which iterated integral is most efficient to compute. (97)

(e) Be able to compute the area of a region D, as (101)

$$A = \iint_D 1 \; dA \,.$$

(f) If a region D cannot be written as a region of type I or II, be able to break it up into regions D_1 and D_2 of type I and/or II, that do not overlap, and know

$$\iint_D f(x,y)\; dA = \iint_{D_1} f(x,y)\; dA + \iint_{D_2} f(x,y)\; dA \,.$$

29. ☐ (12.4) Integration with polar coordinates.

(a) Be able to convert 2D regions expressed in Cartesian coordinates (x, y) into polar coordinate (r, θ) through the relations: (99)

$$x = r\cos(\theta)\,, \quad y = r\sin(\theta)\,, \quad r^2 = x^2 + y^2 \,.$$

(b) Commit to memory the area element in polar coordinates: $dA = r\; dr\; d\theta$.

(c) Be acquainted with the derivation of the area element formula, and understand the essential principle that the area ΔA covered due to an angle change $\Delta\theta$ and radial change Δr is in direct proportion to the distance r of the area from the origin. That is, if the area in question is further from the origin, the area in question will be larger than if it were closer to the origin: $\Delta A \propto r$.

(d) Be able to compute a double integral over a region $D = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\}$ by transforming it into an iterated integral in polar coordinates:

$$\iint_D f(x,y)\; dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta)) r\; dr\; d\theta \,.$$

Be able to do the same when the bounds of r, respectively θ, depend on the other variable. (100, 101)

30. ☐ (12.5) Applications of double integrals.

(a) Mass from density: If $\rho(x, y)$ is the density (mass/area) of a laminar object with spatial extent given by a 2D region D, know the mass m is (102)

$$m = \iint_D \rho(x,y)\; dA \,.$$

(b) Center of mass: Given the laminar object defined in (a), be able to compute the center of mass, or balance point, $(\bar{x}, \bar{y})$ of the object from the formulas

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x,y)\; dA \quad \text{and} \quad \bar{y} = \frac{1}{m} \iint_D y\rho(x,y)\; dA \,.$$

(c) Probability: Let (X, Y) be a pair of random events that could be observed to take on values in a given region R, where the likelihood of the events is given by $p(x, y)$ is a probability density function. Know the properties of the probability density: (a)

$$p(x,y) \ge 0 \quad \text{and} \quad \iint_R p(x,y)\; dA = 1 \,.$$

(d) Let D be a subset of events of R. Know how to interpret a verbal/written description of D and formulate it in set notation: $D = \{(x, y) | \ldots\}$.

(e) Be able to compute the probability $P((x, y) \in D)$ of the events D as computed by the double integral: (b, 105)

$$P((x, y) \in D) = \iint_D p(x, y)\ dA\,.$$

Example Questions

1. Write out the describe sequence a_n indexed by n in set notation: $\{??\}_{n=?}^{?}$. Indicate start and ending indexes (if there is no end, indicate ∞).
   (a) a_n counts up by one, starting from 1 to infinity.
   (b) a_n counts up by two, starting at 6 to infinity.
   (c) a_n is all positive odd numbers starting at one to infinity.
   (d) a_n has a positive one at the start $n = 1$, and alternates sign for all n increasing.

Solution:
(a) $\{n\}_{n=1}^{\infty}$
(b) $\{2n\}_{n=3}^{\infty}$
(c) $\{2n-1\}_{n=1}^{\infty}$
(d) $\{(-1)^{n+1}\}_{n=1}^{\infty}$

2. Match the equivalent sequences for $n = 0, 1, 2, 3, 4, 5, 6\ldots$, or state that there is no match. Note that $(-1)^{\frac{1}{2}} = i$.
   (a) $a_n = \sin(n\pi/2)$
   (b) $b_n = \cos(n\pi/2)$
   (c) $c_n = \begin{cases} (-1)^{n/2}, & n = 0,\ evens \\ 0, & else \end{cases}$
   (d) $d_n = \begin{cases} (-1)^{(2n+1)/2}, & n =\ odds \\ 0, & else \end{cases}$
   (e) $e_n = \begin{cases} (-1)^{[1+(n+1)/2]}, & n =\ odds \\ 0, & else \end{cases}$

Solution: $b_n = c_n$, $a_n = e_n$, and d_n is unmatched.

3. (8.2) The given series are telescoping series. Find the partial sums and decide whether the series is convergent or divergent. If it is convergent, find its sum.
   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$

Solution: Write the summand as a difference

$$\frac{2}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2} \qquad 2 = Ak + 2A + Bk \qquad A = 1 \qquad B = -1$$

So the nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k+2} = \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=3}^{n+2}\frac{1}{k}$$
$$= \begin{cases} 0 & n=0 \\ 1-\frac{1}{3} & n=1 \\ 1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4} & n=2 \\ 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2} & n>2 \end{cases}$$

Now the limit of the partial sums exists and it is

$$\sum_{k=1}^{\infty}\frac{2}{k(k+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = 1+\frac{1}{2} = \frac{3}{2}$$

(b) $\displaystyle\sum_{n=2}^{\infty}\ln\left(\frac{n}{n+1}\right)$

Solution: Write each log ratio term as a difference of logs:

$$\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$$

So, the partial sum to N can be expressed as

$$s_N = \sum_{n=2}^{N}\ln\left(\frac{n}{n+1}\right) = [\ln(2)-\ln(3)] + [\ln(3)-\ln(4)] + \ldots + [\ln(N)-\ln(N+1)].$$
$$= \ln(2) + [-\ln(3)] + \ln(3)] + [-\ln(4) + \ln(4)] + \ldots - \ln(N+1)$$
$$= \ln(2) - \ln(N+1).$$

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty}[\ln(2) - \ln(N+1)] = -\infty,$$

and so the series is divergent.

(c) $\displaystyle\sum_{k=2}^{\infty}\ln\left(\frac{k^2-1}{k^2+2k}\right)$

Solution: Write the summand as a difference

$$\ln\left(\frac{k^2-1}{k^2+2k}\right) = \ln(k^2-1) - \ln(k^2+2k) = \ln(k^2-1) - \ln((k+1)^2-1)$$

So the nth partial sum for $n \geq 2$ is

$$s_n = \sum_{k=2}^{n}\ln\left(\frac{k^2-1}{k^2+2k}\right) = \sum_{k=2}^{n}\ln(k^2-1) + \sum_{k=2}^{n}\ln((k+1)^2-1)$$
$$= \sum_{k=2}^{n}\ln(k^2-1) - \sum_{k=3}^{n+1}\ln(k^2-1) = \ln(2) - \ln((n+1)^2-1)$$

Now
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\ln(2) - \ln((n+1)^2 - 1)\right) = \infty$$
and so the series is divergent.

4. (8.2) Decide whether the given geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $\displaystyle\sum_{k=0}^{\infty} 3^{2k+1}2^{-k}$

Solution:
$$\sum_{k=0}^{\infty} 3^{2k+1}2^{-k} = \sum_{k=0}^{\infty} 3\left(\frac{9}{2}\right)^k$$
So the common ratio $r = \frac{9}{2} > 1$ and the series is divergent.

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{(-2)^k}{3^{3k-1}}$

Solution:
$$\sum_{k=2}^{\infty} \frac{(-2)^k}{3^{3k-1}} = \sum_{k=2}^{\infty} 3\left(\frac{-2}{27}\right) = \frac{3\left(\frac{-2}{27}\right)^2}{1 - \frac{-2}{27}} = \frac{\frac{4}{9\cdot 27}}{\frac{29}{27}} = \frac{4}{9\cdot 29} = \frac{4}{261}$$
since the common ratio is $r = \frac{-2}{27}$, $|r| < 1$ and the series is convergent.

5. (8.2) Decide whether the following series are divergent or convergent.

(a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k} + \frac{1}{2^k}\right)$

Solution: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent, since it is a harmonic series, and $\sum_{k=1}^{\infty}\frac{1}{2^k}$ is convergent, since it is a geometric series with common ration $\frac{1}{2}$. So $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k} + \frac{1}{2^k}\right)$ has to be divergent.

(b) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k^2} + \frac{1}{2^k}\right)$

Solution: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent, since it is a p-series with $p > 1$, and $\sum_{k=1}^{\infty}\frac{1}{2^k}$ is convergent, since it is a geometric series with common ration $\frac{1}{2}$. So
$$\sum_{k=1}^{\infty}\left(\frac{1}{k^2} + \frac{1}{2^k}\right) = \sum_{k=1}^{\infty}\frac{1}{k^2} + \sum_{k=1}^{\infty}\frac{1}{2^k}$$
is convergent.

6. (8.2) Use the Test for Divergence on the following series. Decide whether it shows the series is divergent or it is inconclusive.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^2+1}$

   Solution:
   $$\lim_{k\to\infty} \frac{k}{k^2+1} = \lim_{k\to\infty} \frac{1}{k+\frac{1}{k}} = 0$$
   So the test is inconclusive.

   (b) $\displaystyle\sum_{k=1}^{\infty} k\cos(k)$

   Solution: The limit $\lim_{k\to\infty} k\cos(k)$ does not exist, so the series is divergent.

   (c) $\displaystyle\sum_{k=1}^{\infty} \cos\left(\frac{1}{k}\right)$

   Solution:
   $$\lim_{k\to\infty} \cos(\left(\frac{1}{k}\right) = 1$$
   So the series is divergent.

7. (8.3) Use the integral test to decide whether the following series are divergent or convergent. Make sure to check, that it is possible to use the integral test.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^2+1}$

   Solution: The function $\frac{x}{x^2+1}$ is positive and continuous. It is decreasing, since
   $$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{x^2+1} = \frac{(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \qquad \text{for } x > 1$$
   $$\int_0^{\infty} \frac{x}{x^2+1}\mathrm{d}x = \int_1^{\infty} \frac{1}{u}\frac{1}{2}\mathrm{d}u = \frac{1}{2}\left[\ln|u|\right]_1^{\infty} = \infty$$
   using substitution $u = x^2+1$, $\mathrm{d}u = 2x\mathrm{d}x$. Since the integral diverges, the series is divergent.

   (b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$

   Solution: The function $\frac{1}{x\ln(x)^2}$ is positive and continuous. It is decreasing, since the denominator is increasing.
   $$\int_2^{\infty} \frac{1}{x\ln(x)^2}\mathrm{d}x = \int_{\ln(2)}^{\infty} \frac{1}{u^2}\mathrm{d}u = \left[-\frac{1}{u}\right]_{\ln(2)}^{\infty} = \frac{1}{\ln(2)}$$
   using substitution $u = \ln(x)$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$. Since the integral converges, the series is convergent.

(c) $\sum_{k=1}^{\infty} k^2$

Solution: It is not possible to use the integral test, since the sequence $\{k^2\}_{k=1}^{\infty}$ is increasing and thus any function, that takes these values at the integers, can't be decreasing.

8. (8.3) Use the comparison test to compare the given series to the indicated series. Decide

   (i) whether the indicated series is convergent or divergent

   (ii) whether the comparison test shows the given series is convergent, divergent or if the comparison test is inconclusive

   (a) Compare $\sum_{k=1}^{\infty} \frac{1}{k+1}$ to $\sum_{k=1}^{\infty} \frac{1}{k}$.

   Solution:

   (i) $\sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series, so it is divergent.

   (ii) Now
   $$\frac{1}{k+1} \leq \frac{1}{k}$$
   So the comparison test is inconclusive.

   (b) Compare $\sum_{k=0}^{\infty} \frac{3^k}{5^k+3}$ to $\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k$

   Solution:

   (i) $\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k$ is a geometric series with common ration $r = \frac{3}{5} < 1$, so the series is convergent.

   (ii) Now
   $$\frac{3^k}{5^k+3} \leq \frac{3^k}{5^k} = \left(\frac{3}{5}\right)^k$$
   So by the comparison test, $\sum_{k=0}^{\infty} \frac{3^k}{5^k+3}$ is convergent.

   (c) Compare $\sum_{k=0}^{\infty} \frac{3^{k+1}}{2^k-4}$ to $\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k$

   Solution:

   (i) $\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k$ is a geometric series with common ration $r = \frac{3}{2} > 1$, so the series is divergent.

   (ii) Now
   $$\frac{3^{k+1}}{2^k-4} \geq 3\frac{3^k}{2^k} \geq \frac{3^k}{2^k} = \left(\frac{3}{2}\right)^k$$

So by the comparison test, $\displaystyle\sum_{k=0}^{\infty} \frac{3^{k+1}}{2^k - 4}$ is divergent.

9. (8.3) Use the limit comparison test to decide whether the given series are convergent or divergent. Use either a geometric series or a p-series to compare the series to.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k^3 - \sqrt{2k}}$

   Solution: To find a series to compare this series to, take the highest powers of numerator and denominator, that is $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^2}$. This is a p-series with $p = 2 > 1$, so it is convergent. Now using the limit comparison test

   $$\lim_{k\to\infty} \frac{\frac{\sqrt{k^2+1}}{k^3-\sqrt{2k}}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2\sqrt{k^2+1}}{k^3 - \sqrt{2k}} = \lim_{k\to\infty} \frac{\frac{1}{k^3}}{\frac{1}{k^3}} \frac{k^2\sqrt{k^2+1}}{k^3 - \sqrt{2k}} = \lim_{k\to\infty} \frac{\sqrt{1+\frac{1}{k^2}}}{1 - \sqrt{2\frac{1}{k^5}}} = 1$$

   So $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k^3 - \sqrt{2k}}$ is also convergent.

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{3}{2^k - k}$

   Solution: Since 2^k growths faster than k, compare this series to $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k}$. This is a geometric series with $r = \frac{1}{2} < 1$, so it is convergent. Now using the limit comparison test

   $$\lim_{k\to\infty} \frac{\frac{3}{2^k-k}}{\frac{1}{2^k}} = \lim_{k\to\infty} 3\frac{2^k}{2^k - k} = \lim_{k\to\infty} \frac{3}{1 - \frac{k}{2^k}} = 3$$

   since $\displaystyle\lim_{k\to\infty} \frac{k}{2^k} = 0$. So $\displaystyle\sum_{k=0}^{\infty} \frac{3}{2^k - k}$ is convergent.

10. (8.4) Use the alternating series test on the following series. Decide whether is shows the series is convergent or if the test is inconclusive.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

   Solution: To see the sequence $b_k = \frac{k}{k^2+1}$ is decreasing, consider the function $f(x) = \frac{x}{x^2+1}$. If this function is decreasing, then the sequence is decreasing. Now

   $$f'(x) = \frac{x^2 + 1 - x(2x)}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} < 0 \qquad \text{for } x > 1$$

   So the function and especially the sequence is decreasing. Also

   $$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{k}{k^2+1} = 0$$

So by the alternating series test the given series is convergent.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{(-1)^k(k+2)}$

Solution: Consider the sequence $b_k = \frac{k}{k+2}$. It is

$$\lim_{k\to\infty} \frac{k}{k+2} = 1$$

So this series does not satisfy the conditions of the alternating series test and the test is inconclusive.

11. (8.4) Use the ratio test on the following series. Decide whether it shows the series is convergent, divergent or if the test is inconclusive.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{3^k}$

Solution:

$$\lim_{k\to\infty} \left| \frac{\frac{(k+1)!}{3^{k+1}}}{\frac{k!}{3^k}} \right| = \lim_{k\to\infty} \left(\frac{(k+1)!}{k!} \frac{3^k}{3^{k+1}} \right) = \lim_{k\to\infty} \frac{k+1}{3} = \infty$$

So the series is divergent.

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{k^2}{k^4 - \sqrt{k}}$

Solution:

$$\lim_{k\to\infty} \left| \frac{\frac{(k+1)^2}{(k+1)^4 - \sqrt{k+1}}}{\frac{k^2}{k^4 - \sqrt{k}}} \right| = \lim_{k\to\infty} \left(\frac{(k+1)^2}{k^2} \frac{k^4 - \sqrt{k}}{(k+1)^4 - \sqrt{k+1}} \right) = 1$$

since in each fraction the highest power in numerator and denominator are the same. So the root test is inconclusive.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{2^k + k^2}$

Solution:

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}(k+1)}{2^{k+1}+(k+1)^2}}{\frac{(-1)^k k}{2^k + k^2}} \right| = \lim_{k\to\infty} \left(\frac{k+1}{k} \frac{2^k + k^2}{2^{k+1} + (k+1)^2} \right) = \lim_{k\to\infty} \left(\frac{k+1}{k} \frac{1 + \frac{k^2}{2^k}}{2 + \frac{(k+1)^2}{2^k}} \right) = \frac{1}{2}$$

Since $\displaystyle\lim_{k\to\infty} \frac{k^2}{2^k} = 0 = \lim_{k\to\infty} \frac{(k+1)^2}{2^k}$. So the series is convergent.

12. Consider the converging p-series $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

(a) Find the remainder of the s_9 partial sum. Express using summation notation.

Solution:

$$R_9 = s - s_9 = \sum_{n=10}^{\infty} \frac{1}{n^2}.$$

(b) Using the integral remainder estimate, estimate how close s_9 is to s.

Solution:

$$\int_9^{\infty} \frac{1}{x^2} dx = 1/9, \quad \int_{10}^{\infty} \frac{1}{x^2} dx = 1/10.$$

So

$$\frac{1}{10} \leq |R_9| \leq \frac{1}{9}$$

(c) Is s_9 within 0.1 of s?

Solution: no, s_9 is more than a tenth from s because the error/remainder is larger than 1/10.

13. Consider the converging alternating series $s = \sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{n}}$.

(a) Find the remainder of the s_{15} partial sum. Express using summation notation.

Solution:

$$R_{15} = s - s_{15} = \sum_{n=16}^{\infty} (-1)^n e^{-\sqrt{n}}.$$

(b) Using the alternating series remainder estimate, estimate how close s_{15} is to s.

Solution:

$$|R_{15}| = |a_{16}| = e^{-\sqrt{16}} = e^{-4} \approx 0.0183.$$

(c) Is s_{15} within 1/16th of s?

Solution: Yes, $1/16 = 2^{-4} = 0.0625 > e^{-4} = 0.0183$.

14. (8.5) Find the interval of convergence and the radius of convergence for the following power series.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{(x+3)^k}{2^k}$

Solution: This is a geometric series with common ration $r = \frac{x+3}{2}$. This converges for $\left|\frac{x+3}{2}\right| < 1$ and diverges for $\left|\frac{x+3}{2}\right| \geq 1$. So it converges if and only if $|x+3| < 2$, that is

$$-2 < x+3 < 2 \qquad -5 < x < -1$$

So the radius of convergence is 2 and the interval of convergence is $(-5, -1)$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(2x+1)^k}{3^{2k-1}k}$

Solution: Use the ratio test

$$\lim_{k\to\infty}\left|\frac{\frac{(2x+1)^{k+1}}{3^{2(k+1)-1}(k+1)}}{\frac{(2x+1)^k}{3^{2k-1}k}}\right| = \lim_{k\to\infty}\left(|2x+1|\frac{1}{3^2}\frac{k}{k+1}\right) = \frac{|2x+1|}{9}$$

So the test gives, that the series converges for $|2x+1| < 9$, diverges for $|2x+1| > 9$ and it is inconclusive for $|2x+1| = 9$. So now test $x = 4$ and $x = -5$ seperately.

$$x = 4 : \sum_{k=1}^{\infty}\frac{9^k}{3^{2k-1}k} = \sum_{k=1}^{\infty}\frac{3}{k}$$

This series is a multiple of the harmonic series, so it diverges.

$$x = -5 : \sum_{k=1}^{\infty}\frac{(-9)^k}{3^{2x-1}k} = \sum_{k=1}^{\infty}\frac{3(-1)^k}{k}$$

This series is a multiple of the alternating harmonic series, so it converges. So the interval of convergence for the power serie is $[-5, 4)$ and the radius of convergence is $\frac{9}{2}$.

(c) $\displaystyle\sum_{k=0}^{\infty}\frac{k(x-2)^k}{k!}$

Solution: Use the ration test

$$\lim_{k\to\infty}\left|\frac{\frac{(k+1)(x-2)^{k+1}}{(k+1)!}}{\frac{k(x-2)^k}{k!}}\right| = \lim_{k\to\infty}\left(\frac{k+1}{k}\frac{k!}{(k+1)!}|x-2|\right) = \lim_{k\to\infty}\left(\frac{k+1}{k}\frac{1}{k+1}|x-2|\right) = 0$$

So the series converges for all x. That is the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

15. (8.6) Use $\displaystyle\frac{1}{1-x} = \sum_{k=0}^{\infty}x^k$ to find a power series for the following functions.

(a) $\displaystyle\frac{3}{3-x}$

Solution:

$$\frac{3}{3-x} = \frac{1}{1-\frac{x}{3}} = \sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k = \sum_{k=0}^{\infty}\frac{1}{3^k}x^k$$

(b) $\displaystyle\frac{x^2}{2+x}$

Solution:

$$\frac{x^2}{2+x} = \frac{x^2}{2}\frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{x^2}{2}\sum_{k=0}^{\infty}\left(-\frac{x}{2}\right)^k = \sum_{k=2}^{\infty}\frac{(-1)^k}{2^{k-1}}x^k$$

(c) $\displaystyle\frac{1+x}{1-x}$

Solution:

$$\frac{1+x}{1-x} = (1+x)\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} x^{k+1}$$

Note that $\sum_{k=0}^{\infty} x^{k+1} = \sum_{k=1}^{\infty} x^k$, and $\sum_{k=0}^{\infty} x^k = 1 + \sum_{k=1}^{\infty} x^k$, so

$$\frac{1+x}{1-x} = 1 + 2\sum_{k=1}^{\infty} x^k.$$

16. (8.6) Use $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$ to find a power series in $(x-a)$ for the following functions.

(a) $\dfrac{1}{2+x}$ with $a=-1$

Solution:

$$\frac{1}{2+x} = \frac{1}{1-(-(x+1))} = \sum_{k=0}^{\infty}(-(x+1))^k = \sum_{k=0}^{\infty}(-1)^k(x+1)^k$$

(b) $\dfrac{1}{1+x}$ with $a=1$

Solution:

$$\frac{1}{1+x} = \frac{1}{2-(-(x-1))} = \frac{1}{2}\frac{1}{1-\frac{-(x-1)}{2}} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{-(x-1)}{2}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+1}}(x-1)^k$$

(c) $\dfrac{4}{x^2+4x}$ with $a=-2$

Solution:

$$\frac{4}{x^2+4x} = \frac{4}{(x+2)^2-4} = -\frac{1}{1-\frac{(x+2)^2}{4}} = -\sum_{k=0}^{\infty}\left(\frac{(x+2)^2}{4}\right)^k = \sum_{k=0}^{\infty}\frac{-1}{4^{2k}}(x+2)^{2k}$$

17. (8.6) Consider the power series $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$ with radius of convergence 1. Use differentiation and integration to find power series for the following functions. What is the radius of convergence?

(a) $\dfrac{1}{(1-x)^3}$

Solution: It is

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{1}{(1-x)^2} \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{1}{1-x} = \frac{2}{(1-x)^3}$$

This gives

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{k=0}^{\infty} x^k = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-2} = \sum_{k=0}^{\infty}\frac{(k+2)(k+1)}{2}x^k$$

Differentiated series have the same radius of convergence as the original series, so the radius of convergence is $R = 1$.

(b) $\dfrac{1}{(1+x)^2}$

Solution: It is

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1+x} = -\frac{1}{(1+x)^2}$$

This gives

$$\begin{aligned}\frac{1}{(1+x)^2} &= -\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1+x} = -\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-(-x)} = -\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{\infty}(-x)^k \\ &= -\sum_{k=1}^{\infty}(-1)^k k x^{k-1} = \sum_{k=0}^{\infty}(-1)^k(k-1)x^k\end{aligned}$$

The radius of convergence is $R = 1$.

(c) $\ln(1+x^2)$

Solution: It is

$$\ln(1+x) = \int \frac{1}{1+x}\mathrm{d}x = \int \sum_{k=0}^{\infty} x^k \mathrm{d}x = C + \sum_{k=0}^{\infty}\frac{1}{k+1}x^{k+1}$$

Since $\ln(1+0) = 0$, $C = 0$ and it is

$$\ln(1+x^2) = \sum_{k=0}^{\infty}\frac{1}{k+1}x^{2k+2} = \sum_{k=1}^{\infty}\frac{1}{k}x^{2k}$$

The radius of convergence is $R = 1$.

18. (8.7) Find the Taylor series of the given functions about the indicated points. Use Sigma notation to express it, if possible.

(a) $f(x) = e^{2x}$ about $x = 0$

Solution:

$$f'(x) = 2e^{2x} \qquad f''(x) = 2^2e^{2x} \qquad f^{(k)}(x) = 2^k e^{2x}$$

This gives $f^{(k)}(0) = 2^k$ and the Taylor series is

$$T(x) = \sum_{k=0}^{\infty}\frac{2^k}{k!}x^k$$

(b) $f(x) = \sin(x)$ about $x = \frac{\pi}{4}$ (Hint: Write the Taylor series using two or more sums.)

Solution:

$$f'(x) = \cos(x) \qquad f''(x) = -\sin(x) \qquad f^{(3)}(x) = -\cos(x) \qquad f^{(4)}(x) = \sin(x)$$

This gives

$$f^{(k)}\left(\frac{\pi}{4}\right) = \begin{cases} \frac{\sqrt{2}}{2} & k = 4l \\ \frac{\sqrt{2}}{2} & k = 4l+1 \\ -\frac{\sqrt{2}}{2} & k = 4l+2 \\ -\frac{\sqrt{2}}{2} & k = 4l+3 \end{cases}$$

and the Taylor series is

$$\begin{aligned} T(x) =& \frac{\sqrt{2}}{2}\left(\sum_{l=0}^{\infty} \frac{x^{4l}}{(4l)!} + \sum_{l=0}^{\infty} \frac{x^{4l+1}}{(4l+1)!} - \sum_{l=0}^{\infty} \frac{x^{4l+2}}{(4l+2)!} - \sum_{l=0}^{\infty} \frac{x^{4l+3}}{(4l+3)!}\right) \\ =& \frac{\sqrt{2}}{2}\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right) \end{aligned}$$

(c) $f(x) = x^5 + 1$ about $x = -1$

Solution:

$$f'(x) = 5x^4 \qquad f''(x) = 20x^3 \qquad f^{(3)}(x) = 60x^2 \qquad \ldots$$

This gives

$$\begin{aligned} T(x) =& 0 \cdot (x+1)^0 + \frac{5}{1}(x+1)^1 + \frac{-20}{2}(x+1)^2 + \frac{60}{6}(x+1)^3 \\ &+ \frac{-120}{24}(x+1)^4 + \frac{120}{120}(x+1)^5 \\ =& 5(x+1) - 10(x+1)^2 + 10(x+1)^3 - 5(x+1)^4 + (x+1)^5 \end{aligned}$$

19. (8.7) Remember the hyperbolic functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

(a) Find the first three derivatives of $\sinh(x)$, that is $\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x)$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sinh(x)$ and $\frac{\mathrm{d}^3}{\mathrm{d}x^3}\sinh(x)$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \cosh(x) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sinh(x) = \sinh(x) \qquad \frac{\mathrm{d}^3}{\mathrm{d}x^3}\sinh(x) = \cosh(x)$$

(b) What is the Maclaurin series of $\sinh(x)$? Use the sum notation to write it down.

Solution: It is $\sinh(0) = 0$ and $\cosh(0) = 1$, so the Maclaurin series of $\sinh(x)$ is

$$0 + 1x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

(c) What is the radius of convergence of the Maclaurin series?

Solution: Use the ratio test

$$\lim_{k\to\infty}\left|\frac{\frac{x^{2(k+1)+1}}{(2(k+1)+1)!}}{\frac{x^{2k+1}}{(2k+1)!}}\right| = \lim_{k\to\infty}\frac{x^2}{(2k+3)(2k+2)} = 0$$

So the radius of convergence is $R=\infty$ and the Maclaurin series converges everywhere.

(d) Show, that the Maclaurin series is equal to $\sinh(x)$.

Solution: It is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sinh(x) = \begin{cases}\sinh(x) & n \text{ even}\\ \cosh(x) & n \text{ odd}\end{cases}$$

Now it is for $|x|\leq d$

$$|\sinh(x)| \leq \frac{e^x}{2} \leq e^x \leq e^d \qquad |\cosh(x)| \leq e^x \leq e^d$$

and so

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sinh(x)\right| \leq e^d$$

for all x with $|x|\leq d$. Set $M=e^d$. So Taylor's Inequality gives

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$$

and

$$\lim_{n\to\infty}|R_n(x)| = e^d \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$$

Since the limit of the remainder converges to 0, the Taylor polynomials converge to $\sinh(x)$. That is the Maclaurin series is equal to $\sinh(x)$.

20. (8.7) Use Taylor's inequality to show that the given function and their Taylor series are equal on the open interval of convergence. That is, show that $|R_n(x)|$ goes to zero as $n\to\infty$ by bounding $|R_n(x)|$ by an expression involving n.

(a) $f(x)=\cos(x)$ with Taylor series $\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$, which converges for all x

Solution: The derivatives of $\cos(x)$ are up to a sign either $\sin(x)$ or $\cos(x)$. Both are bounded above by 1, so $|f^{(n+1)}(x)|\leq 1$ for all x. So set $M=1$ and Taylor's inequality gives

$$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1} \xrightarrow{n\to\infty} 0$$

So $\cos(x)$ is equal to its Taylor series.

(b) $f(x)=(x-1)^5$ with Taylor series $x^5-5x^4+10x^3-10x^2+5x-1$ which converges for all x

Solution: The higher derivatives are zero, that is $f^{(n+1)}(x)=0$ for $n\geq 5$. So 0 is an upper bound and Taylor's inequality gives

$$|R_n(x)| \leq 0 \xrightarrow{n\to\infty} 0$$

So $(x-1)^5$ equals its Taylor series.

21. Consider $f(x) = \sqrt{x}$. Use the linear $T_1(x)$ and quadratic $T_2(x)$ truncated Taylor series, expanded about $a = 1$ to approximate $\sqrt{2}$. Which approximation does better?

Solution: The, zeroth, first, and second derivatives are

$$f(1) = 1, \quad f'(1) = \frac{1}{2}, \quad f''(1) = -\frac{1}{4}$$

So, the linear approximation is

$$T_1(x) = 1 + \frac{1}{2}(x-1),$$

and the quadratic is

$$T_1(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2.$$

The approximations are

$$T_1(2) = 1 + 1/2 = 3/2 = 1.5 \approx \sqrt{2} = 1.4142\ldots,$$

and

$$T_2(2) = 1 + 1/2 - 1/8 = 11/8 = 1.3758 \approx \sqrt{2} = 1.4142\ldots$$

The errors: $|T_1(2) - \sqrt(2)| = 0.0858$, and $|T_2(2) - \sqrt(2)| = 0.0392$, so the quadratic approximation does better.

22. (8.7) Let $f(x) = e^{-x}$ and $a = 1$. Use Taylor's inequality to find the smallest N so that the Taylor polynomial $T_N(x)$ is within a 0.1 error bound of $f(x)$ on the interval $(0, 2)$.

Solution: The derivatives are $f^{(n)}(x) = (-1)^n e^{-x}$. For each n, the bound for the absolute value of the derivatives on the interval $(0, 2)$ is found on its left edge: $x = 0$. That is, $M = e^0 = 1 > |f^{(n)}(x)|$ for $x \in (0, 2)$. The Taylor inequality states

$$|f(x) - T_N(x)| \leq \frac{M}{(N+1)!}|x-1|^{N+1},$$

for $|x - 1| < 1$ (i.e., $x \in (0, 2)$). Therefore, the bound ϵ is

$$\epsilon = \frac{1}{(N+1)!}.$$

The goal is to make the bound less than $0.1 = 1/10$, so $(N+1)! > 10$, the smallest N for which this is satisfied is $N = 3$: $\epsilon = 1/4! = 1/24$.

23. (9.1) Decide whether the given equation describes a sphere. If yes, find its radius and center.

(a) $x^2 + y^2 + z^2 + 2z - 6x + 3 = 0$

> **Solution:** It is
>
> $$0 = x^2 + y^2 + z^2 + 2z - 6x + 3 = (x-3)^2 + y^2 + (z+1)^2 - 9 - 1 + 3 = 0$$
>
> $$(x-3)^2 + y^2 + (z+1)^2 = 7$$
>
> So the equation describes a sphere with radius $\sqrt{7}$ and center $(3, 0, -1)$.

(b) $x^2 + 2y^2 = 4x - 2 + 4y - z^2$

> **Solution:** It is
>
> $$0 = x^2 + 2y^2 + z^2 - 4x - 4y + 2 = (x-2)^2 + 2(y-1)^2 + z^2 - 4 - 2 + 2$$
>
> $$(x-2)^2 + 2(y-1)^2 + z^2 = 4$$
>
> This does not describe a sphere, since $(y-2)^2$ is weight more than $(x-2)^2$ and z^2. This describes an ellipsoid.

(c) $x^2 + y^2 + z^2 + 4z + 2y + 12 = 0$

> **Solution:** It is
>
> $$0 = x^2 + (y+1)^2 + (z+2)^2 - 1 - 4 + 12$$
>
> $$x^2 + (y+1)^2 + (z+2)^2 = -7$$
>
> This does not describe a sphere, since not tuple (x, y, z) satisfies this equation.

24. (9.1) Write the set of all the points, that lie in the described object.

(a) A cylinder about the x-axis with radius 2 and height from $x = 0$ to $x = 3$.

> **Solution:**
>
> $$\{(x, y, z) | y^2 + z^2 = 4 \text{ and } 0 \leq x \leq 3\}$$

(b) A sphere of radius 3 with center $(0, 3, 1)$.

> **Solution:**
>
> $$\{(x, y, z) | x^2 + (y-3)^2 + (z-1)^2 = 9\}$$

(c) A disk with center $(3, 1, -5)$ of radius 2 parallel to the yz-plane.

> **Solution:**
>
> $$\{x, y, z | x = 3 \text{ and } (y-1)^2 + (x+5)^2 \leq 4\}$$

25. (9.2) Consider

$$\vec{u} =< 3, -2, 12 > \qquad \vec{v} =< -2, 5, 0 > \qquad \vec{w} = 3\vec{i} - 3\vec{k}$$

Evaluate the following expressions.

(a) $\vec{u} + \vec{v}$

Solution:

$$\vec{u}+\vec{v} = <1, 3, 12>$$

(b) $3\vec{w}$

Solution:

$$3\vec{w} = 9\vec{i} - 9\vec{k}$$

(c) $2\vec{v} - \vec{w}$

Solution:

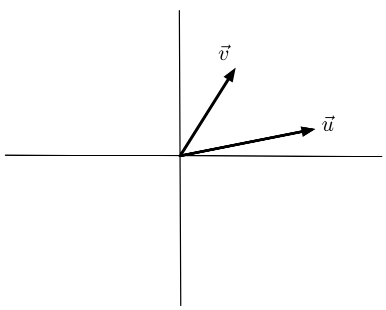
$$2\vec{v} - \vec{w} = < -7, 10, 3 > = -7\vec{i} + 10\vec{j} + 3\vec{k}$$

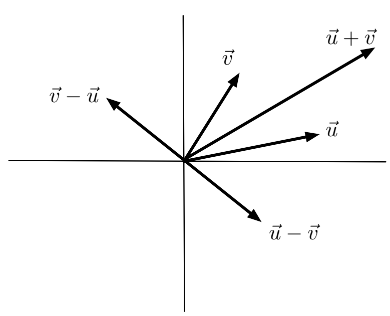
(d) $|\vec{w}|$

Solution:

$$|\vec{w}| = \sqrt{9+0+9} = 3\sqrt{2}$$

26. (9.2) Consider $\vec{u}$ and $\vec{v}$ in the figure. Draw on the figure the following vector calculations
    (a) $\vec{u}+\vec{v}$
    (b) $\vec{u}-\vec{v}$
    (c) $\vec{v}-\vec{u}$





Solution:

27. (9.2) For the given vectors, find the unit vector with the same direction as the given vector. Express it in component form *and* using the standard basis.

(a) $\vec{u} =< 2, -3, 5 >$

Solution:

$$|\vec{u}| = \sqrt{4 + 9 + 25} = \sqrt{38}$$

So the unit vector is

$$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{38}} < 2, -3, 5 >=< \frac{2}{\sqrt{38}}, -\frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} >= \frac{2}{\sqrt{38}}\vec{i} - \frac{3}{\sqrt{38}}\vec{j} + \frac{5}{\sqrt{38}}\vec{k}$$

(b) $\vec{v} = -\vec{j} + 4\vec{k}$

Solution:

$$|\vec{v}| = \sqrt{1 + 16} = \sqrt{17}$$

So the unit vector is

$$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{17}}\left(-\vec{j} + 4\vec{k}\right) = \langle 0, -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle$$

(c) $\vec{w}$ the vector from $P(0, 12, -3)$ to $Q(3, 8, 1)$

Solution:

$$\vec{w} =< 3, -4, 4 > \qquad |\vec{w}| = \sqrt{9 + 16 + 16} = \sqrt{41}$$

So the unit vector is

$$\frac{\vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{41}} < 3, -4, 4 >=< \frac{3}{\sqrt{41}}, -\frac{4}{\sqrt{41}}, \frac{4}{\sqrt{41}} >= \frac{3}{\sqrt{41}}\vec{i} - \frac{4}{\sqrt{41}}\vec{j} + \frac{4}{\sqrt{41}}\vec{k}$$

28. (9.3) Compute the dot product $\vec{u} \cdot \vec{v}$ for the given vectors $\vec{u}$ and $\vec{v}$.

(a) $\vec{u} =< 3, 7, -1 >$ and $\vec{v} =< 2, 0, 3 >$

Solution:

$$\vec{u} \cdot \vec{v} = 6 + 0 - 3 = 3$$

(b) $\vec{u} = -2\vec{i} + 4\vec{k} + \vec{j}$ and $\vec{v} =< -1, 2, 1 >$

Solution:

$$\vec{u} \cdot \vec{v} = (-2)(-1) + 1 \cdot 2 + 4 \cdot 1 = 8$$

(c) $\vec{u}$ the vector from $P(1, 3, 2)$ to $Q(2, 3, -2)$, $\vec{v} = 5\vec{i} - 3\vec{k}$

Solution:

$$\vec{u} =< 1, 0, -4 > \qquad \vec{u} \cdot \vec{v} = 5 + 12 = 17$$

29. (9.3) Compute the angle between the given vectors $\vec{u}$ and $\vec{v}$.

(a) $\vec{u} =< 3, -2, 1 >$ and $\vec{v} =< 2, 0, -1 >$

Solution:
$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{6-1}{\sqrt{9+4+1}\sqrt{4+1}} = \frac{5}{\sqrt{14}\sqrt{5}}$$
$$\theta = \arccos\left(\frac{5}{\sqrt{14}\sqrt{5}}\right) \approx 0.93 = 53.3^\circ$$

(b) $\vec{u}$ the vector from $P(3,-1,4)$ to $Q(-1,3,0)$ and $\vec{v}$ the vector from P to $R(-2,9,-1)$

Solution:
$$\vec{PQ} =< -4,4,-4 >= 4 < -1,1,-1 > \qquad \vec{PR} =< -5,10,-5 >= 5 < -1,2,-1 >$$
Since the length does not have an effect on the angle, one can use multiples to calculate the angle
$$\cos(\theta) = \frac{< -1,1,-1 > \cdot < -1,2,-1 >}{| < -1,1,-1 > || < -1,2,-1 > |} = \frac{1+2+1}{\sqrt{3}\sqrt{6}} = \frac{4}{3\sqrt{2}}$$
$$\theta = \arccos\left(\frac{4}{3\sqrt{2}}\right) \approx 0.34 = 19.47^\circ$$

30. (9.3) A sled is pulled along a horizontal path by a rope. A force of 30 N is applied to the rope at an angle of 30° from the horizontal. How much work is done, when the sled moved 20 m?

Solution: The force $\vec{F}$ has a magnitude of $|\vec{F}| = 30\,\mathrm{N}$ and has an angle of 30° from the horizontal, so
$$\vec{F} = 30\,\mathrm{N}\left(\cos(30^\circ)\vec{i} + \sin(30^\circ)\vec{j}\right) = 30\,\mathrm{N}\left(\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}\right)$$
The sled moves along the horizontal, so the distance vector is $\vec{d} = 20\,\mathrm{m}\vec{i}$. So the work is
$$W = \vec{F}\cdot\vec{d} = 30\,\mathrm{N}\frac{\sqrt{3}}{2}20\,\mathrm{m} = \sqrt{3}300\,\mathrm{J} \approx 519.6\,\mathrm{J}$$

31. (9.3) Consider vectors $\langle 3,2,-1\rangle$ and $\langle 1,2,c\rangle$, where c is a constant. Find a value for c that makes the two vectors orthogonal.

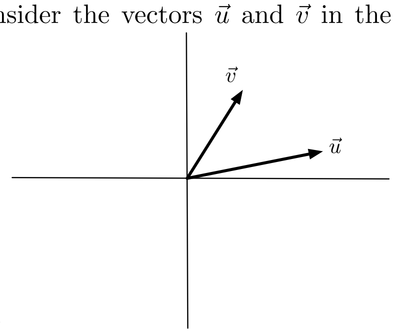
Solution: $\langle 3,2,-1\rangle \cdot \langle 1,2,c\rangle = 3+4-c = 0 \implies c = 7.$

32. (9.3) Consider the vectors $\vec{u} = \langle 3,1\rangle$, $\vec{v} = \langle 1,2\rangle$ and $\vec{w} = \langle -1,2\rangle$. Find numbers a and b in order represent the vector $\vec{w}$ in terms of $\vec{u}$ and $\vec{v}$:
$$\vec{w} = a\vec{u} + b\vec{v}.$$
Hint: take dot products of the above expression with $\vec{u}$ and $\vec{v}$ to solve for a and b. Verify your answer using direct computation.

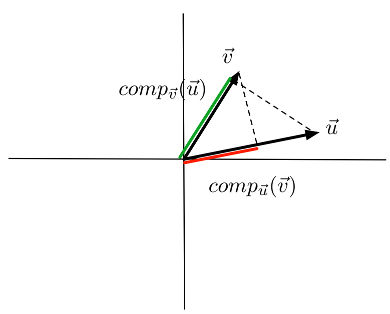
Solution:

$$\begin{aligned}\vec{w} = a\vec{u} + b\vec{v}\vec{w}\cdot\vec{u} &= -1\\ \vec{w}\cdot\vec{v} &= 3\\ \vec{u}\cdot\vec{v} &= 5\\ |\vec{u}|^2 &= 10\\ |\vec{v}|^2 &= 5\\ \vec{w}\cdot\vec{u} = (a\vec{u}+b\vec{v})\cdot\vec{u} = -1 &= a10 + b5\\ \vec{w}\cdot\vec{v} = (a\vec{u}+b\vec{v})\cdot\vec{v} = 3 &= a5 + b5\\ \implies -4 = 5a \implies a &= -4/5\\ \implies 3 = -4+5b \implies b &= 7/5.\\ -4/5\vec{u} + 7/5\vec{v} = -4/5\langle 3,1\rangle + 7/5\langle 1,2\rangle\\ \langle -12/5+7/5, -4/5+14/5\rangle = \langle -5/5, 10/5\rangle = \langle -1,2\rangle = \vec{w}\end{aligned}$$

33. (9.3) Consider the vectors $\vec{u}$ and $\vec{v}$ in the figure. Draw on the figure the components $comp_{\vec{v}}(\vec{u})$ and



$comp_{\vec{u}}(\vec{v})$



Solution:

34. (9.3) Consider the orthogonal vectors $\vec{u} = \langle 1,1\rangle$, $\vec{v} = \langle -1,1\rangle$. Verify by direct computation that the vector $\vec{w} = \langle -1,2\rangle$ can be expressed as

$$\vec{w} = proj_{\vec{u}}(\vec{w}) + proj_{\vec{v}}(\vec{w}).$$

Note this representation of $\vec{w}$ is only possible when $\vec{u}$ and $\vec{v}$.

Solution:

$$\begin{aligned}\vec{w}\cdot\vec{u} &= 1\\ \vec{w}\cdot\vec{v} &= 3\\ |\vec{u}|^2 &= 2\\ |\vec{v}|^2 &= 2\\ proj_{\vec{u}}(\vec{w}) &= \frac{1}{2}\langle 1,1\rangle\\ proj_{\vec{v}}(\vec{w}) &= \frac{3}{2}\langle -1,1\rangle\\ proj_{\vec{u}}(\vec{w})+proj_{\vec{v}}(\vec{w}) &= \langle -1,2\rangle = \vec{w}.\end{aligned}$$

35. (9.4) Calculate the cross product of the given vector $\vec{u}$ and $\vec{v}$.

(a) $\vec{u} =< 3,-1,4 >$ and $\vec{v} =< 2,-1,6 >$

Solution:

$$< 3,-1,4 > \times < 2,-1,6 >=< -2,-10,-1 >$$

(b) $\vec{u} = 3\vec{i} + 2\vec{k}$ and $\vec{v} = 2\vec{j} - \vec{k}$

Solution:

$$\vec{u}\times\vec{v} = -4\vec{i} + 3\vec{j} + 7\vec{k}$$

36. (9.4) Compute the following

(a) $\vec{i}\times\vec{j}$

(b) $\vec{i}\times\vec{i}$

(c) $\vec{j}\times\vec{k}$

(d) $\vec{i}\times\vec{k}$

(e) $\vec{i}\times(\vec{j}\times\vec{i})$

(f) $\vec{k}\times(\vec{i}\times\vec{j})$

Solution:

(a) $\vec{k}$

(b) $\vec{0}$

(c) $\vec{i}$

(d) $-\vec{j}$

(e) $\vec{j}$

(f) $\vec{0}$

37. (9.4) Consider the triangle formed by the points $P(1,0,-3)$, $Q(2,-1,5)$ and $R(0,3,1)$.

(a) Find the length of the sides of the triangle.

Solution:

$$|PQ| = \sqrt{1+1+64} = \sqrt{66} \qquad |QR| = \sqrt{4+16+16} = \sqrt{36}$$
$$|PR| = \sqrt{1+9+16} = \sqrt{26}$$

(b) Find the angle at Q.

Solution: If θ is the angle at Q, then

$$\cos(\theta) = \frac{\vec{QP}\cdot\vec{QR}}{|\vec{QP}||\vec{QR}|} = \frac{< -1, 1, -8 > \cdot < -2, 4, -4 >}{\sqrt{66}\sqrt{36}} = \frac{40}{6\sqrt{66}} = \frac{20}{3\sqrt{66}}$$
$$\theta = \arccos\left(\frac{20}{3\sqrt{66}}\right) \approx 0.608 \approx 34.85^\circ$$

(c) Find the area of the triangle.

Solution: The area of the triangle is half of the area of the parallelogram spanned by $\vec{QP}$ and $\vec{QR}$. So

$$A = \frac{1}{2}|\vec{QP}\times\vec{QR}| = \frac{1}{2}| < 1\cdot 4-(-8)(-4), (-8)(-2)-(-4)(-1), (-1)4-1(-2) > |$$
$$= \frac{1}{2}| < -28, 12.2 > | = | < -14, 6, 1 > | = \sqrt{196+36+1} = \sqrt{233}$$

38. (9.4) To tighten a bolt a force of 12 N is applied to a wrench at a distance of 15 cm. The angle between the wrench and the applied force is 30°. What is the magnitude of the torque?

Solution:

$$|\vec{\tau}| = 15\,\mathrm{cm}\cdot 12\,\mathrm{N}\cdot\sin(30^\circ) = 90\,\mathrm{cN\,m} = 0.9\,\mathrm{N\,m}$$

39. (9.4) Find the volume of the parallelepiped spanned by $\vec{a} =< 3, -1, 3 >$, $\vec{b} =< 1, 0, 4 >$ and $\vec{c} =< 0, -3, 1 >$.

Solution:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})| = | < 3, -1, 3 > \cdot < 12, -1, -3 > | = |36+1-9| = 28$$

The volume of the parallelopiped is 28.

40. (9.5) Find the vector equation and the parametric equations for the given lines.

(a) The line through $P(3, -1, 12)$ parallel to the vector $< 1, 0, 3 >$.

Solution: The vector equation is

$$\vec{r} =< 3, -1, 12 > +t < 1, 0, 3 >$$

and the parametric equations are

$$x = 3 + t \qquad y = -1 \qquad z = 12 + 3t$$

(b) The line through $P(4, -1, 0)$ and $Q(3, 2, 1)$.

Solution: A possible direction vector is $< 3 - 4, 2 - (-1), 1 - 0 >=< -1, 3, 1 >$, so the vector equation is

$$\vec{r} =< 4, -1, 0 > +t < -1, 3, 1 > \qquad \text{alternatively: } \vec{r} =< 3, 2, 1 > +t < -1, 3, 1 >$$

and the parametric equations are

$$x = 4 - t \qquad y = -1 + 3t \qquad z = t$$

41. (9.5) Find the vector equation and the scalar equation for the given planes.

(a) The plane containing $P(3, 1, 6)$, $Q(-2, 1, 4)$ and $R(3, 1, -1)$.

Solution: The vectors $\vec{PQ}$ and $\vec{PR}$ lie on the plane, so the normal vector is

$$\vec{n} = \vec{PQ} \times \vec{PR} =< -5, 0, -2 > \times < 0, 0, -7 >=< 0, -35, 0 >= (-35) < 0, 1, 0 >$$

So the vector equation is

$$< 0, 1, 0 > \cdot(\vec{r}- < 3, 1, 6 >) = 0$$

and the scalar equation is

$$y - 1 = 0$$

(b) The plane containing $P(6, 2, -2)$ and $Q(1, -2, 0)$, that is parallel to the vector $< -1, 4, 2 >$.

Solution: The vector $\vec{PQ}$ lies in the plane, so the normal vector is perpendicular to it. The normal vector also has to be perpendicular to $< -1, 4, 2 >$, so

$$\begin{aligned} \vec{n} &= \vec{PQ}\times < -1, 4, 2 >=< -5, -4, 2 > \times < -1, 4, 2 >=< -8 - 8, -2 + 10, -20 - 4 > \\ &= < 0, 8, -24 >= 8 < 0, 1, -3 > \end{aligned}$$

So the vector equation is

$$< 0, 1, -3 > \cdot(\vec{r}- < 6, 2, -2 >) = 0$$

and the scalar equation is

$$y - 2 + (-3)(z + 2) = 0 \qquad y - 3z = 8$$

(c) The plane perpendicular to $< 2, -2, 5 >$ containing $P(0, 3, -2)$.

Solution: The vector equation is

$$< 2, -2, 5 > \cdot (\vec{r} - < 0, 3, -2 >) = 0$$

and the scalar equation is

$$2x + (-2)(y-3) + 5(z+2) = 0 \qquad 2x - 2y + 5z = -4$$

42. (9.5) Find the intersection of the given objects.

(a) The line $\vec{r} = \langle 3, 5, -4 \rangle + t\langle 2, -1, 0 \rangle$ and the line $x = 3 + s$, $y = -2 - 4s$, $z = 2 + 3s$.

Solution: To find the intersection, set the lines equal, that gives three equations

$$3 + 2t = 3 + s \quad 5 - t = -2 - 4s \quad -4 = 2 + 3s$$

The third equation gives $s = -2$, plugged into the first two

$$3 + 2t = 3 - 2 \quad 5 - t = -2 + 8 \qquad 2t = -2 \quad -t = 1$$

Both are solved by $t = -1$. So the intersection point is $(1, 6, -4)$.

(b) The line $\vec{r} = \langle 0, 4, -1 \rangle + t\langle 3, -7, 7 \rangle$ and the plane $3x - 5y + 2z = 10$

Solution: To parametric equations of the line are

$$x = 3t \qquad y = 4 - 7t \qquad z = -1 + 7t$$

Plugging these into the scalar equation of the plane gives

$$3(3t) - 5(4 - 7t) + 2(-1 + 7t) = 10 \qquad 9t - 20 + 25t - 2 + 14t = 10$$

$$48t = 32 \qquad t = \frac{32}{48} = \frac{2}{3}$$

So the point of intersection is $(2, 4 - \frac{14}{3}, -1 + \frac{14}{3}) = (2, -\frac{2}{3}, \frac{11}{3})$.

(c) the plane $< 4, 0, -2 > \cdot (\vec{r} - < 1, 0, 5 >) = 0$ and the plane $-2x + 6y - z = 9$

Solution: The planes are not parallel, since the normal vectors $< 4, 0, -2 >$ and $< -2, 6, -1 >$ are not parallel. So the intersection is a line. The direction of the line is orthogonal to both normal vectors, so

$$< 4, 0, -2 > \times < -2, 6, -1 > = < 12, 8, 2 > = 2 < 6, 4, 1 >$$

is the direction vector of the line of intersection. It remains to find one point, that lies on both planes. Consider the scalar equation of each plane

$$4x - 2z = -6 \qquad -2x + 6y - z = 9$$

Setting $x = 0$, gives $z = 3$ and $y = 2$. So $(0, 2, 3)$ lies on both planes and the line of intersection is

$$\vec{r} = < 0, 2, 3 > + t < 6, 4, 1 >$$

43. (9.5) Find the angle in which the given planes intersect.

(a) $< 1, -3, 0 > \cdot (\vec{r} - < 1, 0, 4 >) = 0$ and $< 0, -2, 4 > \cdot (\vec{r} - < 5, -2, 0 >) = 0$

Solution: The planes are clearly not parallel, so the intersect in a line. The angle in which they intersect is the same as the angle between the normal vectors. So

$$\cos(\theta) = \frac{< 1, -3, 0 > \cdot < 0, -2, 4 >}{| < 1, -3, 0 > || < 0, -2, 4 > |} = \frac{6}{\sqrt{10} 2\sqrt{5}} = \frac{3}{5\sqrt{2}}$$

$$\theta = \arccos\left(\frac{3}{5\sqrt{2}}\right) \approx 1.13 = 64.9^\circ$$

So the planes intersect at an angle of 64.9°.

(b) $2x - 7y + 3z = 12$ and $< 1, 5, 2 > \cdot (\vec{r} - < 5, -6, 2 >) = 0$

Solution: The planes are clearly not parallel, so the intersect in a line. The angle in which they intersect is the same as the angle between the normal vectors. So

$$\cos(\theta) = \frac{< 2, -7, 3 > \cdot < 1, 5, 2 >}{| < 2, -7, 3 > || < 1, 5, 2 > |} = \frac{2 - 35 + 6}{\sqrt{4 + 49 + 9}\sqrt{1 + 25 + 4}} = \frac{-27}{\sqrt{62}\sqrt{30}}$$

$$\theta = \arccos\left(\frac{-27}{\sqrt{62}\sqrt{30}}\right) \approx 2.25 = 128.8^\circ$$

So the planes intersect at an angle of 51.2° (the angle of intersection between two planes can always be choosen $\leq 90^\circ$).

44. (9.5) Derive the equation to find the point $\vec{x}$ in a plane $\vec{n} \cdot (\vec{r} - \vec{p}) = 0$ that is closest to a given point in space $\vec{y}$ is given by the formula in terms of the plane normal vector $\vec{n}$ and plane reference point $\vec{p}$. Also find the distance formula between the points.

Solution: The minimum-distance point $\vec{x}$ is found when $\vec{x} - \vec{y} = A\vec{n}$ for some A scalar. So, we need to find A. That is, the vectors are collinear, but may differ by a constant. Solving for $\vec{x}$:

$$\vec{x} = \vec{y} + A\vec{n}$$

$\vec{x}$ needs to also be in the plane, so

$$0 = \vec{n} \cdot (\vec{x} - \vec{p}) = \vec{n} \cdot (\vec{y} + A\vec{n} - \vec{p}),$$

so

$$\vec{n} \cdot \vec{y} + A||\vec{n}||^2 - \vec{n} \cdot \vec{p} = 0$$

Solving for A:

$$A = \frac{\vec{n} \cdot \vec{p} - \vec{n} \cdot \vec{y}}{||\vec{n}||^2},$$

which yields the formula

$$\vec{x} = \vec{y} - \vec{n}\Big(\frac{(\vec{y} - \vec{p}) \cdot \vec{n}}{||\vec{n}||^2}\Big)$$

and the distance equation is

$$||\vec{x} - \vec{y}|| = \frac{|(\vec{y} - \vec{p}) \cdot \vec{n}|}{||\vec{n}||}$$

45. (9.5) Find the distance of the given objects.

(a) The plane $\langle 3,0,-4\rangle \cdot (\vec{r} - \langle 1,3,7\rangle) = 0$ and the point $(2,6,3)$

Solution: A point $\vec{x}$ in the plane that is closest to a given point in space $\vec{y}$ is given by the formula in terms of the plane normal vector $\vec{n}$ and plane reference point $\vec{p}$:

$$\vec{x} = \vec{y} - \vec{n}\Big(\frac{(\vec{y}-\vec{p})\cdot\vec{n}}{||\vec{n}||^2}\Big)$$

and the distance equation is

$$||\vec{x} - \vec{y}|| = \frac{|(\vec{y}-\vec{p})\cdot\vec{n}|}{||\vec{n}||}$$

To use the formula for the distance between a point and a plane, one needs the scalar equation of the plane. For this plane the scalar equation is

$$3x - 4z + 25 = 0$$

So the distance is

$$d = \frac{|3\cdot 2 - 4\cdot 3 + 25|}{\sqrt{3^2+4^2}} = \frac{19}{5}$$

(b) The plane $5x - 8y + 2z = 2$ and the line $\vec{r} = \langle 2,-4,3\rangle + t\langle 4,2,-2\rangle$

Solution: To see plane and the line are parallel, check if the normal vector of the plane and the direction vector of the line are orthogonal

$$< 5,-8,2 > \cdot < 4,2,-2 >= 20 - 16 - 4 = 0$$

So they are orthogonal. Now the distance of the line and the plane is the same as the distance from any point on the line and the plane. A point on the line is $(2,-4,3)$. The distance of the plane to this point is

$$d = \frac{|5\cdot 2 - 8\cdot(-4) + 2\cdot 3 - 2|}{\sqrt{25+64+4}} = \frac{46}{\sqrt{91}}$$

This is also the distance of the line and the plane.

(c) the plane $2x - 6y + 2z = -11$ and the plane $< -1,3,-1 > \cdot(\vec{r} - < -2,3,5 >) = 0$

Solution: First check, if the planes are parallel. The normal vectors $< 2,-6,2 >$ and $< -1,3,-1 >$ are clearly multiples of each other. So the planes are parallel. Now the distance between the two planes is the same as the distance of any point on one plane to the other plane. So it is enough to find the distance of $(-2,3,5)$, a point on the second plane, to the first plane. The distance between these is

$$d = \frac{|2(-2) - 6\cdot 3 + 2\cdot 5 + 11|}{\sqrt{4+36+4}} = \frac{1}{2\sqrt{11}}$$

This is the distance between the two planes.

(d) the line $\vec{r} =< -3,0,7 > +t < 3,-2,1 >$ and the line $x = 3t - 2$, $y = -t + 2$, $z = 11$

Solution: First construct a plane, that contains the first line and is parallel to the second line. The normal vector $\vec{n}$ of this plane has to be orthogonal to the direction vector of both planes, which are $< 3, -2, 1 >$ and $< 3, -1, 0 >$, so take

$$\vec{n} = < 3, -2, 1 > \times < 3, -1, 0 > = < 1, 3, 3 >$$

Then the plane is

$$< 1, 3, 3 > \cdot (\vec{r} - < -3, 0, 7 >) = 0 \qquad x + 3y + 3z + 18 = 0$$

Now the distance between the two lines is the distance of this plane to a point on the second line. Take $(-2, 2, 11)$ as a point on the second line, then the distance is

$$d = \frac{|1(-2) + 3(2) + 3(11) + 18|}{\sqrt{1 + 9 + 9}} = \frac{55}{\sqrt{19}}$$

This is the distance between the two lines.

46. (9.5) Decide whether the given lines are the same, parallel, intersect or neither.

(a) $L_1 : \vec{r} = < 2, 4, -1 > +t < 3, 4, 1 >$ and $L_2 : y = 2, \frac{x-2}{3} = \frac{z-1}{-1}$

Solution: The direction vector of L_2, which is given by symmetric equations, is $< 3, 0, -1 >$. So the lines are neither the same nor parallel. To check, if they intersect, look at the parametric equations of L_1

$$x = 2 + 3t \qquad y = 4 + 4t \qquad z = -1 + t$$

Now for L_2, it is $y = 2$, so $t = \frac{1}{2}$. This gives the point $(\frac{7}{2}, 2, -\frac{1}{2})$. Now check, if this point is on L_2:

$$\frac{\frac{7}{2} - 2}{3} = \frac{3}{2} = \frac{-\frac{1}{2} - 1}{-1}$$

So the lines intersect.

(b) $L_1 : \vec{r} = < 3, -1, 2 > +t < 3, -6, 9 >$ and $L_2 : \vec{r} = < 0, 5, -7 > +t < -1, 2, -3 >$

Solution: The lines are the same or parallel, since the direction vectors are multiples of each other $< 3, -6, 9 > = (-3) < -1, 2, -3 >$. To check, if they are equal, just check, if a point, that lies on L_1, also lies in L_2. Take $P(3, -1, 2)$. Now solve the equations

$$3 = -t \qquad -1 = 5 + 2t \qquad 2 = -7 - 3t$$

$t = -3$ solves all these equations simultaneously, so the lines are the same.

(c) $L_1 : x = 2 + t, y = -3 + 2t, z = -5t$ and $L_2 : \vec{r} = < 1, 0, 2 > +t < -2, -4, 10 >$

Solution: A direction vector of the first line is $< 1, 2, -5 >$. Since $< -2, -4, 10 > = (-2) < 1, 2, -5 >$, the lines are parallel or the same. Now check, if the point $P(2, -3, 0)$, which lies on L_1, also lies on L_2. That is solve the equations

$$2 = 1 - 2t \qquad -3 = -4t \qquad 0 = 2 + 10t$$

The point does not lie on L_2, since the first equation is solved by $t = \frac{1}{2}$ and the second by $t = \frac{3}{4}$. So the lines are parallel.

47. (9.5) Decide whether the given planes are the same, parallel or intersect.

(a) Plane 1: $< 1, 0, -3 > \cdot (\vec{r} - < 2, 0, 3 >) = 0$ and plane 2: $2x - 3y + 2z + 4 = 0$

> **Solution:** The normal vector of the first plane is $< 1, 0, -3 >$ and the normal vector of the second plane is $< 2, -3, 2 >$. These are not parallel, so the planes are neither parallel nor the same. So they intersect.

(b) Plane 1: $-4x + 3y - z = 12$ and plane 2: $< 12, -9, 3 > \cdot (\vec{r} - < -2, 1, -1 >) = 0$

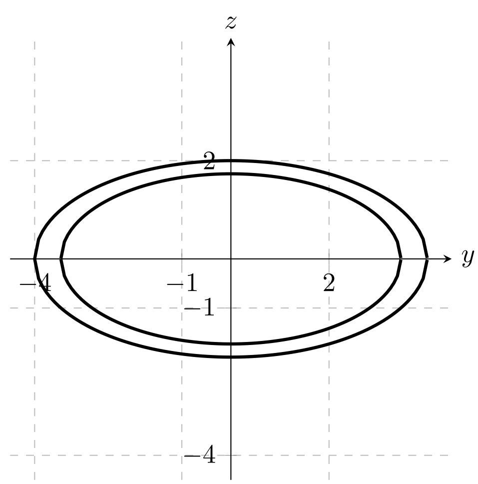
> **Solution:** The normal vector of the first plane is $< -4, 3, -1 >$ and the normal vector of the second plane is $< 12, -9, 3 >$. These vectors are multiplies of each other: $< 12, -9, 3 >= (-3) < -4, 3, -1 >$. So the planes are parallel or the same. $< -2, 1, -1 >$ is a point in the second plane, plugging it into the equation of the first plane
>
> $$-4(-2) + 3 \cdot 1 - (-1) = 8 + 3 + 1 = 12$$
>
> So the point also lies in the first plane. So the planes are the same.

(c) Plane 1: $8x - 4y = 2z - 2$ and plane 2: $-4x + 2y + z + 14 = 0$

> **Solution:** The normal vector of the first plane is $< 8, -4, -2 >$ and the normal vector of the second plane is $< -4, 2, 1 >$. These are multiples of each other: $< 8, -4, -2 >= (-2) < -4, 2, 1 >$. So the planes are parallel or the same. A point on the first plane is $(0, 0, 1)$. Plugging this into the second equation
>
> $$-4 \cdot 0 + 2 \cdot 0 + 1 + 14 = 15 \neq 0$$
>
> so the point does not lie in the plane. So the planes are parallel.

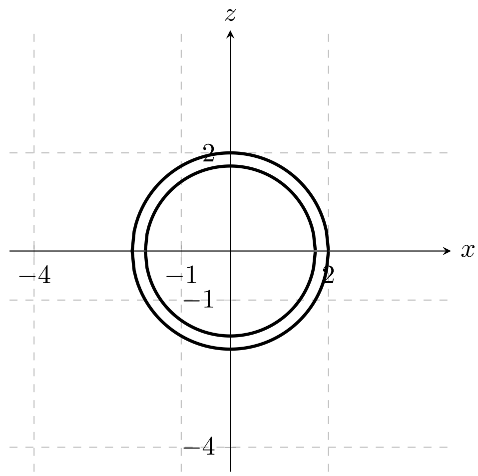
48. (9.6) Consider the surface $4x^2 + y^2 + 4z^2 = 16$.

(a) Draw the traces in $x = k$ for three different values for k.



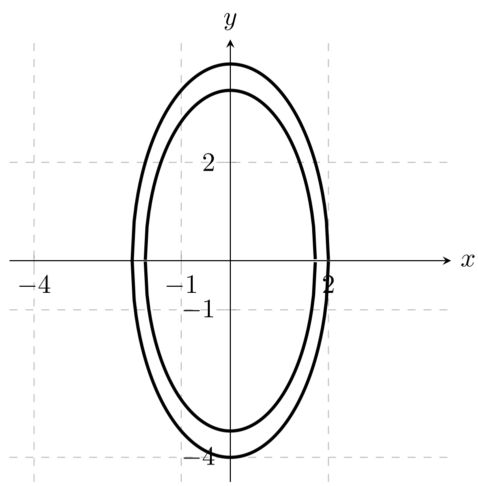
> **Solution:** The outer trace is for $k = 0$, the inner for $k = \pm 1$. If $-2 > k$ or $k > 2$, the equation has not solution.

(b) Draw the traces in $y = k$ for three different values for k.



Solution: The outer trace is for $k = 0$, the inner for $k = \pm 1$. If $-4 > k$ or $k > 4$, the equation has not solution.

(c) Draw the traces in $z = k$ for three different values for k.



Solution: If $-2 > k$ or $k > 2$, the equation has not solution.

(d) Describe and or sketch the shape of the surface in three dimensions.

Solution: The shape is an ellipsoid (an egg), with the long axis on the y-axis.

49. (9.7) Each point is given in either rectangular, cylindrical or spherical coordinate. Fill in the missing coordinates.

Rectengular (x, y, z)	**Cylindrical** (r, θ, z)	**Spherical** (ρ, θ, ϕ)
$(2, 2, 0)$	$(2\sqrt{2}, \frac{\pi}{4}, 0)$	$(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2})$
$(0, 0, 3)$	$(0, 0, 3)$	$(3, 0, 0)$
$(1, -1, 1)$	$(\sqrt{2}, \frac{7\pi}{4}, 1)$	$(\sqrt{3}, \frac{7\pi}{4}, \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 0.96 \approx 54.7°)$

50. (9.7) Consider surface the cone with the tip at the origin and who's axis goes along the negative z-axis down to a base with radius radius 1 and height 2 from the tip.

(a) Find the equation describing the cone in rectangular coordinates.

> **Solution:**
> $$x^2 + y^2 = (-\frac{z}{2})^2, \quad 0 \le z \le 2$$

(b) Find the equation describing the cone in cylindrical coordinates.

> **Solution:**
> $$r = -\frac{z}{2}$$

(c) Find the equation describing the cone in spherical coordinates.

> **Solution:**
> $$0 \le \rho \le \sqrt{1+4} \qquad \phi = \frac{5\pi}{6}$$
> θ is free to vary.

51. (10.1) Sketch the curve of the given vector equation.

(a) $\vec{r}(t) = \langle \sin(t), 3\cos(t), 2\rangle$.

(b) $\vec{r}(t) = \langle t, -1, t^2\rangle$.

(c) $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(t)\rangle$.

> **Solution:**
>
> (a) An ellipse in the plane $z = 2$, with y-diameter 3 and x-diameter 1.
>
> (b) An upward u-shaped parabola lying in the $y = -1$ plane where $z = x^2$.
>
> (c) An ellipse lying on a slanted plane with y-diameter 1 and x-diameter 1. The normal vector of the plane can be computed as
>
> $$\vec{n} = \vec{r}' \times \vec{r}''$$
> $$det \begin{bmatrix} i & j & k \\ -\sin(t) & \cos(t) & -\sin(t) \\ -\cos(t) & -\sin(t) & -\cos(t) \end{bmatrix}$$
> $$= \langle -1, 0, 1\rangle.$$

52. (10.1) Specify a vector function that draws the following described paths

(a) A circle in the x-y plane ($z = 0$).

(b) Any spiral emanating from the origin and expanding outward that lies in the y-z plane

(c) A helix that begins in the x-y plane at point $(0, -1, 0)$ at $t = 0$ and spirals upward, drawing a circle with radius 1 when observed looking down along the z-axis onto the x-y plane.

Solution: These are examples. Other formulas could work as well.

(a) $\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$

(b) $\vec{r}(t) = \langle 0, t\cos(t), t\sin(t) \rangle$

(c) $\vec{r}(t) = \langle \sin(t), -\cos(t), t \rangle$

53. (10.1) Harder: Find the vector function, that represents the curve of intersection of the given surfaces.

(a) $x^2 + z^2 = y^2$ and $x + 2y = 1$

Solution: Because of the first equation, set

$$x = y\cos(t) \qquad z = y\sin(t)$$

Plugging this into the second one gives

$$y\cos(t) + 2y = 1 \qquad y = \frac{1}{2+\cos(t)}$$

So the vector function for the intersection is

$$\vec{r}(t) =< \frac{\cos(t)}{2+\cos(t)}, \frac{1}{2+\cos(t)}, \frac{\sin(t)}{2+\cos(t)} > \qquad 0 \leq t \leq 2\pi$$

(b) $x^2 - y^2 + z = 1$ with $z \geq 0$ and $2y - z = 1$

Solution: Solving the second for z gives $z = 2y - 1$, plugging this into the first equation

$$x^2 - y^2 + 2y = 2 \qquad x^2 - (y-1)^2 = 1 \qquad y \geq \frac{1}{2}$$

So it is $y = \sqrt{x^2-1} + 1$. Set $t = x$, so the vector function of the intersection is

$$\vec{r}(t) =< t, \sqrt{t^2-1} + 1, 2\sqrt{t^2-1} + 1 >$$

54. (10.2) Find the tangent line for the given vector functions at the given points.

(a) $\vec{r}(t) =< t^2, 2t, t^3 >$ at $(1, 2, 1)$

Solution: The tangent vector is

$$\vec{r}'(t) =< 2t, 2, 3t^2 > \qquad \vec{r}'(1) =< 2, 2, 3 >$$

Since the t-value at $(1, 2, 1)$ is $t = 1$. So the tangent line is

$$x = 1 + 2t \qquad y = 2 + 2t \qquad z = 1 + 3t$$

(b) $\vec{r}(t) = \frac{1}{t}\vec{i} - \ln(t)\vec{j} + \sqrt{t}\vec{k}$ at $(1, 0, 1)$

Solution: The tangent vector is

$$\vec{r}'(t) = -\frac{1}{t^2}\vec{i} - \frac{1}{t}\vec{j} + \frac{1}{2\sqrt{t}}\vec{k} \qquad \vec{r}'(1) = -\vec{i} - \vec{j} + \frac{1}{2}\vec{k}$$

Since the t-value at $(1,0,1)$ is $t = 1$. So the tangent line is

$$x = 1 - t \qquad y = -t \qquad z = 1 + \frac{1}{2}t$$

55. (10.2) Given two vector function $\vec{u}(t)$, $\vec{v}(t)$ and $\vec{w}(t)$ with

$$\vec{u}(1) =< 3,-1,3 > \qquad \vec{u}'(1) =< -2,0,6 > \qquad \vec{v}(1) =< 0,2,1 > \qquad \vec{v}'(1) =< 9,1,3 >$$
$$\vec{w}(1) =< 1,0,3 > \qquad \vec{w}'(1) =< 2,4,1 >$$

Evaluate the following expressions.

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{u}(t)\cdot\vec{v}(t)\right)$ at $t = 1$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{u}(t)\cdot\vec{v}(t)\right)\Big|_{t=1} = \left(\vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)\right)_{t=1} = 6 + (18 - 1 + 9) = 32$$

(b) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{v}(t)\times\vec{u}(t)\right)$ at $t = 1$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{v}(t)\times\vec{u}(t)\right)\Big|_{t=1} = \vec{v}'(1)\times\vec{u}(1) + \vec{v}(1)\times\vec{u}(1) =< 18,-10,-8 >$$

(c) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{w}(t)\cdot(\vec{v}(t)\times\vec{u}(t))\right)$ at $t = 1$

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{w}(t)\cdot(\vec{v}(t)\times\vec{u}(t))\right)\Big|_{t=1} =& \vec{w}'(1)\cdot(\vec{v}(1)\times\vec{u}(1)) + \vec{w}(1)\times\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{u}(t)\cdot\vec{v}(t)\right)\Big|_{t=1}\\ =& < 2,4,1 > \cdot < 7,3,-6 > + < 1,0,3 > \cdot < 18,-10,-8 >\\ =& 14 + 12 - 6 + 18 - 24 = 14\end{aligned}$$

56. (10.3) Find the arc length of the given curves.

(a) $\vec{r}(t) =< 1 - 2t^2, t^3, -t^3 >$ for $0 \le t \le \sqrt{2}$

Solution:

$$\vec{r}'(t) =< -4t, 3t^2, -3t^2 > \qquad |\vec{r}'(t)| = \sqrt{16t^2 + 9t^4 + 9t^4} = t\sqrt{16 + 18t^2}$$

$$L = \int_0^{\sqrt{2}} t\sqrt{16 + 18t^2}\mathrm{d}t = \int_{16}^{36}\frac{1}{36}\sqrt{u}\mathrm{d}u = \frac{1}{36}\left[\frac{2}{3}u^{3/2}\right]_{16}^{36} = \frac{1}{72}\left(6^3 - 4^3\right) = \frac{19}{9}$$

with substitution $u = 16 + 18t^2$, $\mathrm{d}u = 36t\mathrm{d}t$.

(b) $\vec{r}(t) = e^t\vec{i} - \sqrt{2}t\vec{j} + e^{-t}\vec{k}$ for $-1 \leq t \leq 1$

Solution:

$$\vec{r}'(t) = e^t\vec{i} - \sqrt{2}\vec{j} - e^{-t}\vec{k} \qquad |\vec{r}'(t)| = \sqrt{e^2t - 2 + e^{-2t}} = \sqrt{(e^t - e^{-t})^2} = \left|e^t - e^{-t}\right|$$

$$L = \int_{-1}^{1} \left|e^t - e^{-t}\right| \mathrm{d}t = 2\int_0^1 \left(e^t - e^{-t}\right) \mathrm{d}t = 2\left[e^t + e^{-t}\right]_0^1 = 2\left(e + \frac{1}{e} - 2\right)$$

57. (10.3) Find the parametrization with respect to arc length measured from the point where $t = 0$.

(a) $\vec{r}(t) = (1 + 3t)\vec{i} - t\vec{j} + (2 - t)\vec{k}$

Solution: First find the arc length function

$$\vec{r}'(t) = 3\vec{i} - \vec{j} - \vec{k} \qquad |\vec{r}'(t)| = \sqrt{11}$$

$$s(t) = \int_0^t |\vec{r}'(u)|\mathrm{d}u = \int_0^t \sqrt{11}\mathrm{d}u = \sqrt{11}t$$

So the reparametrization is

$$\vec{r}(t(s)) = (1 + \frac{3s}{\sqrt{11}})\vec{i} - \frac{s}{\sqrt{11}} + (2 - \frac{s}{\sqrt{11}})\vec{k}$$

(b) $\vec{r}(t) =< 1, e^t\sin(t), -e^t\cos(t) >$

Solution: First find the arc length function

$$\vec{r}'(t) =< 0, e^t\sin(t) + e^t\cos(t), -e^t\cos(t) + e^t\sin(t) >$$

$$|\vec{r}'(t)| = \sqrt{e^{2t}\left(\sin(t) + \cos(t)\right)^2 + e^{2t}\left(-\cos(t) + \sin(t)\right)^2} = e^t\sqrt{2}$$

$$s(t) = \int_0^t e^u\sqrt{2}\mathrm{d}u = \left[\sqrt{2}e^u\right]_0^t = \sqrt{2}\left(e^t - 1\right)$$

Solving this for t gives

$$\frac{s}{\sqrt{2}} = e^t - 1 \qquad \frac{s}{\sqrt{2}} + 1 = e^t \qquad t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

So the reparametrization is

$$\vec{r}(t(s)) =< 1, \left(\frac{s}{\sqrt{2}} + 1\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right), -\left(\frac{s}{\sqrt{2}} + 1\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right) >$$

58. (10.3) Find the curvature of the given curves and the specified points.

(a) $\vec{r}(t) =< e^{2t}, 3 - t^2, t >$ at $(1, 3, 0)$

Solution:

$$\vec{r}'(t) =< 2e^{2t}, -2t, 1 > \qquad \vec{r}''(t) =< 4e^{2t}, -2, 0 >$$

$$\vec{r}'(t) \times \vec{r}''(t) =< 2, 4e^{2t}, -4e^{2t} + 8te^{2t} >$$

$$|\vec{r}'(t)| = \sqrt{4e^{4t} + 4t^2 + 1} \qquad |\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{4 + 16e^{4t} + 16e^{4t}(1 - 4t + 4t^2)}$$

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{4 + 16e^{4t} + 16e^{4t}(1 - 4t + 4t^2)}}{(\sqrt{4e^{4t} + 4t^2 + 1})^3}$$

The t-value at $(1, 3, 0)$ is $t = 0$. So the curvature is

$$\kappa(0) = \frac{\sqrt{4 + 16 + 16}}{(\sqrt{4 + 1})^3} = \frac{6}{5\sqrt{5}}$$

(b) $\vec{r} = t\sin(t)\vec{i} + 4t\vec{j} + t\cos(t)\vec{k}$ at $\left(\frac{\pi}{2}, 2\pi, 0\right)$

Solution:

$$\vec{r}'(t) = (t\cos(t) + \sin(t))\vec{i} + 4\vec{j} + (-t\sin(t) + \cos(t))\vec{k}$$

$$\vec{r}''(t) = (-t\sin(t) + 2\cos(t))\vec{i} + (-t\cos(t) - 2\sin(t))\vec{k}$$

$$\vec{r}'(t) \times \vec{r}''(t) = 4(-t\cos(t) - 2\sin(t))\vec{i} + (t^2 + 2)\vec{j} + (-4)(-t\sin(t) + 2\cos(t))\vec{k}$$

$$|\vec{r}'(t)| = \sqrt{t^2 + 1 + 16} \qquad |\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{16t^2 + 64 + t^4 + 2t^2 + 4}$$

$$\kappa(t) = \frac{\sqrt{t^4 + 17t^2 + 68}}{(\sqrt{t^2 + 17})^3}$$

The t-value at $\left(\frac{\pi}{2}, 2\pi, 0\right)$ is $t = \frac{\pi}{2}$. So the curvature is

$$\kappa\left(\frac{\pi}{2}\right) = \frac{\sqrt{\left(\frac{\pi}{2}\right)^4 + 17\left(\frac{\pi}{2}\right)^2 + 68}}{\left(\sqrt{\left(\frac{\pi}{2}\right)^2 + 17}\right)^3}$$

59. (10.3) Find the unit tangent vector, the unit normal vector and the binormal vector of the given curves.

(a) $\vec{r}(t) = t\vec{i} - t^2\vec{k}$

Solution:

$$\vec{r}'(t) = \vec{i} - 2t\vec{k} \qquad |\vec{r}'(t)| = \sqrt{1 + 4t^2}$$

$$\vec{T}(t) = \frac{1}{\sqrt{1 + 4t^2}}\left(\vec{i} - 2t\vec{k}\right)$$

$$\vec{T}'(t) = \frac{-8t}{2(1 + 4t^2)^{3/2}}\left(\vec{i} - 2t\vec{k}\right) + \frac{1}{\sqrt{1 + 4t^2}}\left(-2\vec{k}\right) = \frac{-1}{(1 + 4t^2)^{3/2}}\left(4t\vec{i} + 2\vec{k}\right)$$

$$|\vec{T}'(t)| = \frac{1}{(1 + 4t^2)^{3/2}}\sqrt{16t^2 + 4} \qquad \vec{N}(t) = \frac{-1}{\sqrt{1 + 4t^2}}\left(2t\vec{i} + \vec{k}\right)$$

$$\vec{B}(t) = \vec{j}$$

(b) $\vec{r}(t) =< \cos(3t), 4t, \sin(3t) >$

Solution:

$$\vec{r}'(t) =< -3\sin(3t), 4, 3\cos(3t) > \qquad |\vec{r}'(t)| = \sqrt{9+16} = 5$$

$$\vec{T}(t) =< -\frac{3}{5}\sin(3t), \frac{4}{5}, \frac{3}{5}\cos(3t) >$$

$$\vec{T}'(t) =< -\frac{9}{5}\cos(3t), 0, -\frac{9}{5}\sin(3t) > \qquad |\vec{T}'(t)| = \frac{9}{5}$$

$$\vec{N}(t) =< -\cos(3t), 0, -\sin(3t) >$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) =< -\frac{4}{5}\sin(3t), \frac{3}{5}, \frac{4}{5}\cos(3t) >$$

60. (10.4) Consider the vector function $\vec{r}(t) = \langle \sin(t), \cos(t), t\rangle$. Find the decomposition of the acceleration into the osculating plane components:

$$\vec{a} = a_T(t)\vec{T}(t) + a_N(t)\vec{N}(t)$$

Solution:

$$\begin{aligned} \vec{r}'(t) &= \langle \cos(t), -\sin(t), 1\rangle \\ \vec{a}(t) = \vec{r}''(t) &= \langle -\sin(t), -\cos(t), 0\rangle \\ \vec{T}(t) &= \frac{1}{\sqrt{2}}\langle \cos(t), -\sin(t), 1\rangle \\ a_T(t) &= \vec{T}(t)\cdot\vec{a}(t) = 0. \\ \vec{N}(t) &= \langle -\sin(t), -\cos(t), 0\rangle \\ a_N(t) &= \vec{N}(t)\cdot\vec{a}(t) = -1 \\ \vec{a}(t) &= a_N\vec{N}(t). \end{aligned}$$

61. (10.4) Consider $\vec{x} = \langle 1, 2, 3\rangle$ and an orthogonal coordinate system $\vec{u} = \langle 1, 1, -1\rangle$, $\vec{v} = \langle 0, 1, 1\rangle$, $\vec{w} = \langle 2, -1, 1\rangle$. Find the triplet (a, b, c) such that

$$\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}.$$

Solution:

$$\begin{aligned} \vec{x}\cdot\vec{u} &= -1 \\ \vec{x}\cdot\vec{v} &= 5 \\ \vec{x}\cdot\vec{w} &= 3 \\ |\vec{u}|^2 &= 3 \\ |\vec{v}|^2 &= 2 \\ |\vec{w}|^2 &= 6 \\ (a, b, c) &= \left(\frac{-1}{3}, \frac{5}{2}, \frac{1}{2}\right). \end{aligned}$$

62. On an x-y plane, draw precise level curves of the function $f(x, y) = e^{-x^2-y^2} = z$ at the values $f(x, y) = k$, where $k = e^{-1}, e^{-1/4}, e^{-1/16}$. Describe the shape of the surface.

Solution: $e^{-x^2-y^2} = e^{-1} \implies x^2 + y^2 = 1$, which is a circle of radius 1 centered at the origin. The remaining two level curves are circles of radius 2 and 4. The surface looks like a mountain with summit located at the origin.

63. Draw precise traces and level curves of the function $f(x, y) = x^2 - 4y^2 = z$ at the values near the origin

(a) $x = 0$.

(b) $y = 0$

(c) $f(x, y) = 1$

(d) $f(x, y) = -1$

and then describe the shape of the surface near the origin.

Solution: The surface is shown in the figure. The traces and level curves that are embedded in the surface are found as follows:

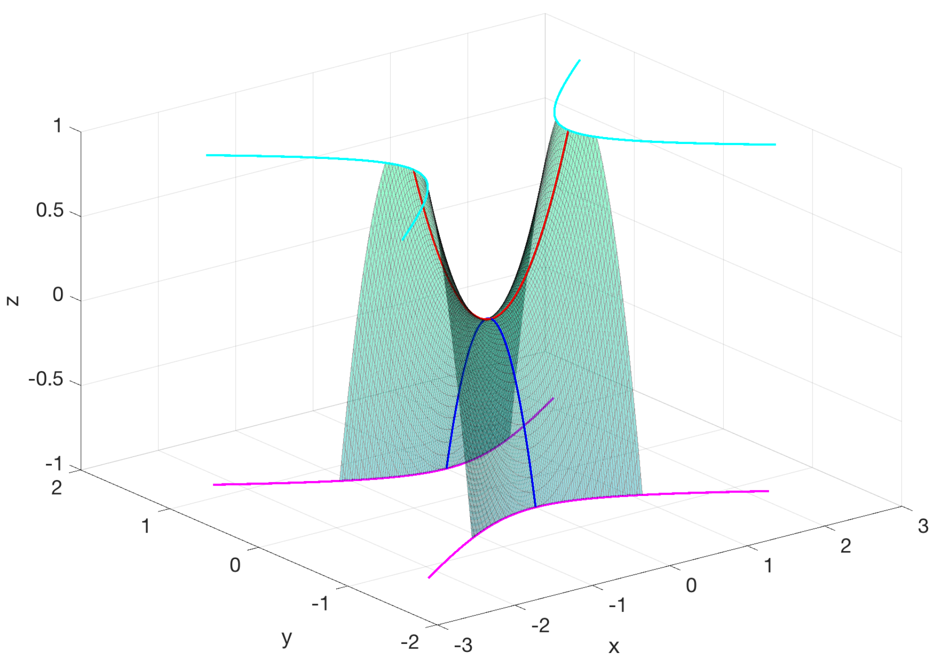
(a) $z = -4y^2$ dark blue downward parabola.

(b) $z = x^2$ dark red upward parabola.

(c) $x = \pm\sqrt{1 + 4y^2}$ light blue hyperbola pairs.

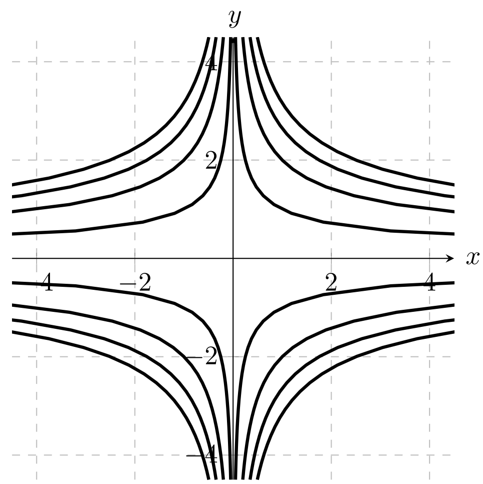
(d) $y = \pm\frac{1}{2}\sqrt{1 + x^2}$ magenta hyperbola pairs.

The surface resembles a mountain pass.



64. (11.1) Draw a contour map (i.e., level curves) for the function

$$f(x, y) = xy^2$$



65. Compute the two limit paths for the function $\frac{|x|-|y|}{|x|+|y|} = z$.

(a) The path $y = 0$ and $x \to 0^+$ to the origin.

(b) The path $x = 0$ and $y \to 0^+$ to the origin.

and determine if $\lim_{(x,y)\to(0,0)} \frac{|x|-|y|}{|x|+|y|}$ exists.

Solution:

(a) $\lim_{y=0, x\to 0^+} \frac{|x|-0}{|x|+0} = 1$.

(b) $\lim_{x=0, y\to 0^+} \frac{0-|y|}{0+|y|} = -1$.

The limits on the two paths do not agree, so the limit as $(x, y) \to (0, 0)$ cannot exist.

66. (11.2) Show that the following limits do not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x)^2 + y}{x^2 + y^2}$

Solution: Set $f(x, y) = \frac{\sin(x)^2+y}{x^2+y^2}$. Along the x-axis it is

$$f(x, 0) = \frac{\sin(x)^2}{x^2} = \left(\frac{\sin(x)}{x}\right)^2 \xrightarrow{x\to 0} 1$$

Along the y-axis it is

$$f(0, y) = \frac{y}{y^2} = \frac{1}{y}$$

which diverges. So the limit does not exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^3}$

Solution: Set $f(x, y) = \frac{xy}{x^2+y^3}$. Along the x-axis it is

$$f(x, 0) = 0 \xrightarrow{x \to 0} 0$$

Along the diagonal $x = y$ it is

$$f(x, x) = \frac{x^2}{x^2 + x^3} = \frac{1}{1 + x} \xrightarrow{x \to 0} 1$$

Since these limits are different, thelmits does not exist.

67. Determine if the following limits exist or not. If so, compute them; if not, explain why.
   (a) $\lim_{(x,y)\to(0,0)} \frac{\ln(x+y)}{2+x+y}$
   (b) $\lim_{(x,y)\to(0,0)} \frac{x^2+2y+3}{2+x+y}$
   (c) $\lim_{(x,y)\to(1,3)} \frac{x+y}{6x-2y}$
   (d) $\lim_{(x,y)\to(1,3)} \frac{\sin(\pi y)x^2-y+2}{x^2-2y+3}$

Solution:
   (a) DNE because the numerator is not defined at the origin.
   (b) The function is rational with no singularity at the origin, so the function is continuous and the limit exists $L = 3/2$.
   (c) DNE because the denominator is zero at the origin while the numerator is not zero.
   (d) The function is continuous because it is comprised of continuous component functions. The limit is $L = -1/(1 - 6 + 3) = 1/2$.

68. Compute the partial derivatives
   (a) $\frac{\partial}{\partial x}[x \cos(xe^y)]$
   (b) $\frac{\partial}{\partial y}[x \cos(xe^y)]$
   (c) $\left(x^2 e^{xy}\right)_{xy}$
   (d) $\left(x^2 e^{xy}\right)_{xx}$

Solution: Wolfram alpha should do it.

69. (11.3) Given the function

$$f(x, y) = xy^3 + \sin(xy) - e^{x^2}$$

Find the following partial derivatives.
   (a) $f_x(x, y)$

Solution:

$$f_x(x, y) = y^3 + y\cos(xy) - 2xe^{x^2}$$

(b) $\dfrac{\partial f}{\partial y}(x, y)$

> **Solution:**
> $$\frac{\partial f}{\partial y}(x, y) = 3xy^2 + x\cos(xy)$$

(c) $f_{xy}(x, y)$

> **Solution:**
> $$f_{xy}(x, y) = 3y^2 + \cos(xy) - xy\sin(xy)$$

(d) $\dfrac{\partial^2 f}{\partial x^2}$

> **Solution:**
> $$\frac{\partial^2 f}{\partial x^2} = 6xy - x^2\sin(xy)$$

70. Find the tangent plane equation to the function $f(x, y) = z = 5 - x^2 - y^2$ at the point $(1, 2)$.

> **Solution:**
> $$f(1, 2) = 1 = z_0, \quad f_x(1, 2) = -2, \quad f_y(1, 2) = -4.$$
> So
> $$z - 1 = -2(x - 1) - 4(y - 2).$$

71. Consider $z = \sqrt{11 + x^2 + y^2}$. Use a linear approximation at $(x, y) = (1, 2)$ to estimate the z-value when $(x, y) = (2, 3)$.

> **Solution:** Note $f(1, 2) = 4$, and $f_x = \frac{x}{\sqrt{11+x^2+y^2}}$, and $f_y = \frac{y}{\sqrt{11+x^2+y^2}}$, therefore $f_x(1, 2) = \frac{1}{4}$, and $f_y(1, 2) = \frac{1}{2}$. The linear approximation
> $$f(2, 3) = z \approx 4 + \frac{1}{4}(x - 1) + \frac{1}{2}(y - 2).$$
> At $(x, y) = (2, 3)$, the approximate z-value is
> $$z = \sqrt{24} \approx 4 + \frac{1}{4} + \frac{1}{2} = 4 + \frac{3}{4} = 4.75$$
> Note that the true value is $\sqrt{24} = 4.8990....$

72. Consider the function $f(x, y) = z = y\sin(x)$, evaluated on the trajectory $\vec{r}(t) = \langle t, t^2 \rangle$ from $t = 0$ to $\pi/2$. Compute $\frac{dz}{dt}$ at $t = 1$.

Solution: $f_x = y\cos(x)$, $f_y = \sin(x)$, $x'(t) = 1$, $y'(t) = 2t$. So,

$$\frac{dz}{dt} = \pi/2\cos(\pi/2)1 + \sin(\pi/2)2\pi/2 = \pi.$$

73. (11.5) Consider

$$z = xy + e^{x^2} \qquad x(r,s,t) = rs + t \qquad y(r,s,t) = r^2 s$$

Find the following expressions.

(a) $\dfrac{\partial z}{\partial r}$

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \left(y + 2xe^{x^2}\right)s + x2rs$$

Plugging in $x(r,s,t)$ and $y(r,s,t)$, we get:

$$\frac{\partial z}{\partial r} = \left(r^2 s + 2(rs+t)e^{(rs+t)^2}\right)s + (rs+t)2rs.$$

(b) $\dfrac{\partial^2 z}{\partial r^2}$ (do this if covered in class)

Solution:

$$\frac{\partial z}{\partial r^2} = \frac{\partial}{\partial r}\left(\left(y + 2xe^{x^2}\right)s + x2rs\right) = \left((2+4x^2)e^{x^2}s + 2rs\right)s + s2rs + x2s.$$

Then substitute $x(s,t)$ and $y(s,t)$ into the above.

74. Consider the function $z = x^2 + 4y^2$, evaluated on the trajectory $\vec{r}(t) = \langle \frac{1}{2}\cos(t), \frac{1}{2}\sin(t)\rangle$.

(a) Describe the trajectory graphically over $t = 0$ to 2π.

(b) Compute $\frac{dz}{dt}$. The double angle formula may be useful: $2\sin(t)\cos(t) = \sin(2t)$.

(c) At what times t does $\frac{dz}{dt} = 0$. Which are max and min values?

(d) Using the above, describe the motion of $z(t)$ in 3D.

Solution:

(a) It's a circular path with radius 1/2 .

(b) Using the chain rule: $dz/dt = -\cos(t)\sin(t) + 4\sin(t)\cos(t) = 3\sin(t)\cos(t) = \frac{3}{2}\sin(2t)$ (use double-angle formula)

(c) $\sin(2t) = 0$ when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$, that alternate from min, max, min, max, min, respectively.

(d) At what times is $\frac{dz}{dt}$ largest in absolute value?

(e) It's like a up-down-up-down roller coaster.

75. (11.6) Consider the function $f(x,y) = x^3y - \sqrt{xy}$.

(a) Find the directional derivative at $(2,2)$ in direction of $\langle -2, 1\rangle$.

Solution:
$$\vec{\nabla} f(x,y) = \langle 3x^2y - \frac{y}{\sqrt{xy}}, x^3 - \frac{x}{\sqrt{xy}}\rangle \qquad \vec{\nabla} f(2,2) = \langle 23, 7\rangle$$
The unit vector in the direction of $\langle -2,1\rangle$ is $\frac{1}{5}\langle -2,1\rangle$. So the directional derivative is
$$\langle 23,7\rangle \cdot \frac{1}{5}\langle -2,1\rangle = \frac{1}{5}(-46+7) = -\frac{39}{5}$$

(b) Find the directional derivative at $(1,4)$ in the direction of the vector given by the angle $\theta = \frac{\pi}{3}$.

Solution:
$$\vec{\nabla} f(1,4) = \langle 10, \frac{1}{2}\rangle$$
So the directional derivative is
$$\langle 10, \frac{1}{2}\rangle \cdot \langle \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\rangle = \langle 10, \frac{1}{2} \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle = 5 + \frac{\sqrt{3}}{4}$$

76. (11.6) Find the direction, in which the tangent line of $f(x,y) = xe^{x+y}$ at $(1,-1)$ is the steepest.

Solution:
$$\vec{\nabla} f(x,y) = \langle (x+1)e^{x+y}, xe^{x+y}\rangle \qquad \vec{\nabla} f(1,-1) = \langle 2,1\rangle$$
So the tangent line is the steepest in the direction of $\langle 2,1\rangle$.

77. (11.6) Find the tangent plane to the surface $2x^2 - y^2 + 4z^2 = 8$ at $(2,2,1)$.

Solution: This surface is a level surface with
$$F(x,y,z) = 2x^2 - y^2 + 4z^2 = 8$$
So the normal vector of the tangent plane is
$$\vec{\nabla} F(x,y,z) = \langle 4x, -2y, 8z\rangle \qquad \vec{\nabla} F(2,2,1) = \langle 8,-4,8\rangle = 4\langle 2,-1,2\rangle$$
So the tangent plane is
$$2(x-2) - (y-2) + 2(z-1) = 0 \qquad 2x - y + 2z = 4$$

78. (11.7) Consider the function $f(x,y) = \frac{x-y}{x^2y^2+3}$.

(a) Find all the critical points of $f(x,y)$.

Solution:

$$f_x(x,y) = \frac{x^2y^2+3-(x-y)2xy^2}{(x^2y^2+3)^2} \qquad f_y(x,y) = \frac{-(x^2y^2+3)-(x-y)x^22y}{(x^2y^2+3)^2}$$

Setting each equal to zero gives the equations

$$x^2y^2+3-2x^2y^2+2xy^3 = 0 \qquad -x^2y^2-3-2x^3y+2x^2y^2 = 0$$
$$3-x^2y^2+2xy^3 = 0 \qquad 3-x^2y^2+2x^3y = 0$$

This gives $xy^3 = x^3y$ and

$$0 = xy^3 - x^3y = xy(y^2-x^2) = xy(y-x)(y+x)$$

Now clearly $x = 0$ or $y = 0$ does not solve the equations above. Now if $y = x$, both equations become

$$3 + x^4 = 0$$

These equations have no solution. If $y = -x$, both equations become

$$0 = 3 - 3x^4 = -3\left(x^4-1\right) = -3\left(x^2+1\right)(x+1)(x-1)$$

So the critical points are $(-1,1)$ and $(1,-1)$.

(b) Decide whether the critical points are local minima, local maxima or saddle points.

Solution: To use the second derivative test, find the second derivatives

$$f_{xx}(x,y) = \frac{\left(-2xy^2+2y^3\right)\left(x^2y^2+3\right)-\left(3-x^2y^2+2xy^3\right)2xy^2}{(x^2y^2+3)^3}$$

$$f_{yy}(x,y) = \frac{\left(x^22y-2x^3\right)\left(x^2y^2+3\right)-\left(-3+x^2y^2-2x^3y\right)x^22y}{(x^2y^2+3)^3}$$

$$f_{xy}(x,y) = \frac{\left(-x^22y+6xy^2\right)\left(x^2y^2+3\right)-\left(3-x^2y^2+2xy^3\right)x^22y}{(x^2y^2+3)^3}$$

$$f_{xx}(1,-1) = \frac{(-4)(4)-0}{4^3} = -\frac{1}{4} \qquad f_{xx}(-1,1) = \frac{(4)(4)-0}{4^3} = \frac{1}{4}$$

$$f_{yy}(1,-1) = \frac{(-4)(4)-0}{4^3} = -\frac{1}{4} \qquad f_{yy}(-1,1) = \frac{(4)(4)-0}{4^3} = \frac{1}{4}$$

$$f_{xy}(1,-1) = \frac{(8)(4)-0}{4^3} = 1 \qquad f_{xy}(-1,1) = \frac{(-8)(4)-0}{4^3} = -1$$

So it is

$$D(1,-1) = \frac{1}{16} - 1 < 0 \qquad D(-1,1) = \frac{1}{16} - 1 < 0$$

Thus both points are saddle points.

79. Find and characterized the critical points as local min, max, or saddles, of $f(x,y) = 3xy - x^3 - y^3$ using the second derivative test.

Solution: First find the critical points

$$f_x(x,y) = 3y - 3x^2 \qquad f_y(x,y) = 3x - 3y^2$$

Setting both zero gives

$$y = x^2 \qquad x = y^2$$

Where these two parabolas intersect at $(0,0)$ and $(1,1)$ defining the critical points. The second derivative test

$$f_{xx} = -6x, \quad f_{yy} = -6y, \quad f_{xy} = f_{yx} = 3,$$

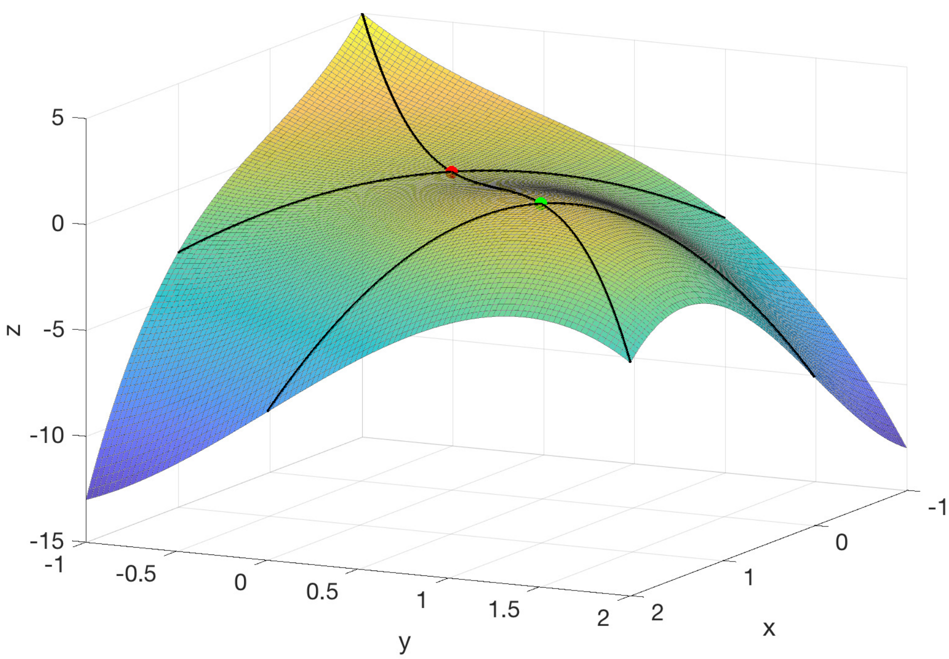
defining the D number

$$D(x,y) = 36xy - 9.$$

$D(0,0) = -9 < 0$, implying the origin critical point is a saddle point.

$D(1,1) = 36 - 9 > 0$, and $f_{xx}(1,1) = -6 < 0$ implying $(1,1)$ critical point is local maxima.

The graph of the surface and the two critical points are shown in the figure: red dot is the saddle, with diagonal traces (black lines) showing one concave up curve and another concave down curve. The green dot is the local max, with both diagonal traces being concave down. Note, the function is unbounded as (x,y) gets increasingly far from the origin, but we are only concerned with local critical points.



80. (11.7) Find the absolute maximum and the absolute minimum of $f(x,y) = xy^2 - x^2y + x$ on the closed rectangle with edges $(0,0)$, $(0,4)$, $(3,4)$ and $(3,0)$.

Solution: First find the critical points

$$f_x(x,y) = y^2 - 2xy + 1 \qquad f_y(x,y) = 2xy - x^2$$

Setting both zero gives

$$0 = y^2 - 2xy + 1 \qquad 0 = 2xy - x^2 = x(2y - x)$$

From the second equation it is $x = 0$ or $x = 2y$. If $x = 0$, then the second equation gives $y^2 + 1 = 0$. This has no solution. If $x = 2y$, the second equation is

$$0 = y^2 - 4y^2 + 1 = -3y^2 + 1 = -3(y - \frac{1}{\sqrt{3}})(y + \frac{1}{\sqrt{3}})$$

So the critical points are $\left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. The function values at these points are

$$f\left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2-4+6}{3\sqrt{3}} = \frac{4}{3\sqrt{3}} \qquad f\left(-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{-2+4-6}{3\sqrt{3}} = \frac{-4}{3\sqrt{3}}$$

Now check the boundary for extrema.

- $(0,0) - (0,4)$: $f(0,y) = 0$
- $(0,4) - (3,4)$: $f(x,4) = 16x - 4x^2 + x = -4x^2 + 17x$, which has critical point for $8x = 17$, that is at $x = \frac{17}{8}$. So the possible extrema are

$$f(0,4) = 0 \qquad f(3,4) = 15 \qquad f(\frac{17}{8}, 4) = \frac{17^2}{16}$$

- $(3,4) - (3,0)$: $f(3,y) = 3y^2 - 9y + 3$, which has critical point for $6y = 9$, that is $y = \frac{3}{2}$. So the possible extrema are

$$f(3,4) = 15 \qquad f(3,0) = 3 \qquad f(3, \frac{3}{2}) = -\frac{15}{4}$$

- $(3,0) - (0,0)$: $f(x,0) = x$, so the possible extrema are

$$f(3,0) = 3 \qquad f(0,0) = 0$$

So the absolute maximum is $\frac{17^2}{16}$ and the absolute minimum $-\frac{15}{4}$.

81. (11.8) Find the maximum of $z = 6x^2 + y$, when $y^2 + x^2 = 4$.

Solution: Using Lagrange multipliers gives the equations

$$12x = \lambda 2x \qquad 1 = \lambda 2y \qquad x^2 + y^2 = 4$$

The first equation gives

$$0 = 6x - \lambda x = x(6 - \lambda)$$

So $x = 0$ or $\lambda = 6$. If $x = 0$, then $y = 2$ from the third equation. If $\lambda = 6$, then $y = \frac{1}{12}$ from the second equation and the third equation gives $x = \frac{\sqrt{575}}{12}$. So the extrema are at $(0,2)$ and $(\frac{1}{12}, \frac{\sqrt{575}}{12})$. The first gives the minimum and the second the maximum.

82. (11.8) Find the minimum of z-value of $z = x^2 - y^2$, subject to the constraint $x + 2y = -1$.

Solution: Using Lagrange multipliers

$$\nabla f =< 2x, -2y >, \quad \nabla g =< 1, 2 >$$

gives the equations

$$2x = \lambda, \quad -2y = 2\lambda$$

The first equation gives

$$x = \lambda/2.$$

and the second gives

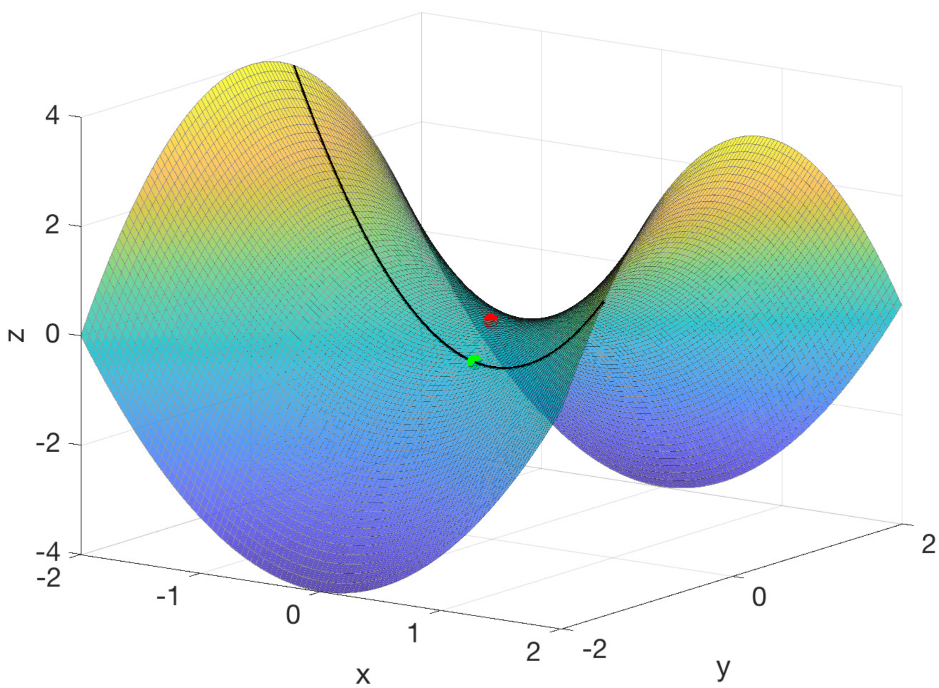
$$-y = \lambda$$

so

$$x = -y/2$$

meaning the line $y = -2x$ for any lambda-value solves the Lagrange equation. The line $y = -2x$ satisfies/intersects the constraint equation $x + 2y = -1$ when $x - 4x = -1$. So, $x = 1/3$ and $y = -2/3$ is the critical point and $\lambda = 2/3$. This point is a minimum $z = 1/9 - 4/9 = -1/3$.

A figure of the saddle surface $z = x^2 - y^2$ overlaid by constraint equation line $x + 2y = -1$ (black line) shows that at $(1/3, -2/3)$ the minimum point $z = -1/3$ is attained (green dot). For reference, the saddle location at the origin (red dot) is also shown.



83. (12.1) Given the function $f(x, y) = x^2 + y$ over the rectangle $R = [1, 3] \times [2, 5]$. Divide the rectangle into smaller rectangle, by dividing it into eight pieces in x-direction and six pieces in y-direction. Set up the Riemann sum using the midpoint rule to estimate the volume of the function over R. Do not evaluate the sum.

Solution:

$$N = 8\,, \quad \Delta x = \frac{3-1}{8} = \frac{1}{4}\,, \quad x_k = 1 + \frac{k}{4}$$
$$M = 6\,, \quad \Delta y = \frac{5-2}{6} = \frac{1}{2}\,, \quad y_l = 2 + \frac{l}{2}$$

The sample points using the midpoint rule are

$$x^*_{kl} = \frac{x_k + x_{k-1}}{2} = 1 + \frac{k}{4} - \frac{1}{8} \quad \text{and} \quad y^*_{kl} = \frac{y_l + y_{l-1}}{2} = 2 + \frac{l}{2} - \frac{1}{4}\,.$$

So the Riemann sum is

$$V \approx \sum_{k=1}^{8}\sum_{l=1}^{6} f(x^*_{kl}, y^*_{kl})\Delta x \Delta y = \sum_{k=1}^{8}\sum_{l=1}^{6}\left(\left(1 + \frac{k}{4} - \frac{1}{8}\right)^2 + 2 + \frac{l}{2} - \frac{1}{4}\right)\cdot\frac{1}{8}\,.$$

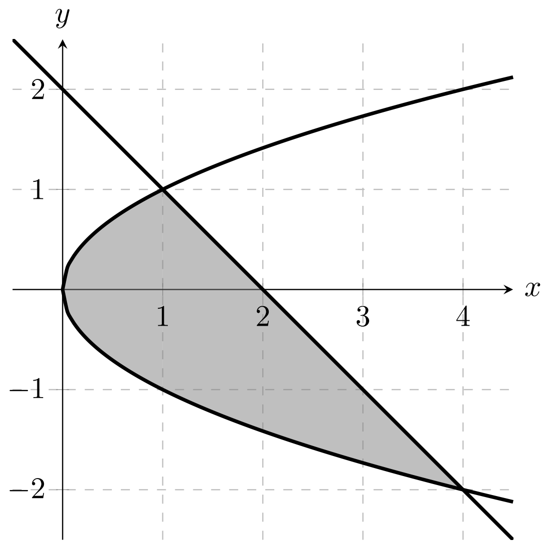
84. (12.2) Find the volume of the solid that lies above the square $R = [0,2] \times [0,4]$ and below the surface defined by $z = ye^{xy}$.

Solution: To integrate this function it is better to first integrate with respect to x. That gives

$$\begin{aligned} V &= \iint\limits_R ye^{xy}\ dA = \int_0^4 \int_0^2 ye^{xy}\ dx\ dy \\ &= \int_0^4 \left[e^{xy}\right]_{x=0}^2\ dy = \int_0^4 \left(e^{2y} - 1\right)\ dy = \left[\frac{1}{2}e^{2y} - y\right]_0^4 \\ &= \frac{1}{2}e^8 - 8 - \frac{1}{2} \end{aligned}$$

85. (12.3) Evaluate $\iint\limits_D xy\ dA$, where D is the region bounded by $x = y^2$ and $y = -x + 2$.

Solution: To describe the region, first sketch it



To find the intersection points, set the equations, after solving both for x, equal

$$y^2 = x = -y + 2 \implies 0 = y^2 + y - 2 = (y-1)(y+2)\,.$$

Then the intersection points are $(1,1)$ and $(4,-2)$, so

$$D = \{(x,y) | -2 \leq y \leq 1, y^2 \leq x \leq -y+2\}\,.$$

Then the integral gives

$$\begin{aligned}
\iint_D xy\ dA = &\int_{-2}^{1}\int_{y^2}^{-y+2} xy\ dx\ dy = \int_{-2}^{1}\left[\frac{1}{2}x^2 y\right]_{x=y^2}^{-y+2}\ dy \\
=&\frac{1}{2}\int_{-2}^{1} y\left((-y+2)^2 - y^4\right)\ dy = \frac{1}{2}\int_{-2}^{1}\left(-y^5 + y^3 - 4y^2 + 4y\right)\ dy \\
=&\frac{1}{2}\left[-\frac{1}{6}y^6 + \frac{1}{4}y^4 - \frac{4}{3}y^3 + 2y^2\right]_{-2}^{1} \\
=&\frac{1}{2}\left(-\frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2 - \left(-\frac{32}{3} + 4 + \frac{32}{3} + 8\right)\right) = -\frac{45}{8}
\end{aligned}$$

86. (12.3) Integrate the function $f(x,y) = xy$ over the set $D = \{(x,y) | 0 \leq x \leq 1, y \geq x, y \leq 4\}$. Choose the integration order so that can you compute $\iint_D f(x,y)\ dA$ in only a single iterated integral.

Solution:

$$\iint_D f(x,y)\ dA = \int_0^1 \int_x^4 xy\ dy\ dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_{y=x}^4\ dx$$
$$= \frac{1}{2}\int_0^1 \left(16x - x^3\right)\ dx = \frac{1}{2}\left[8x^2 - \frac{1}{4}x^4\right]_0^1$$
$$= \frac{1}{2}\left(8 - \frac{1}{4}\right) = \frac{31}{8}$$

87. Integrate the function $f(x,y) = xy$ over the set $D = \{(x,y)|0 \leq x \leq 1, y \geq x, y \leq 4\}$. Choose the integration order so that can you compute the integral $\iint_D f dA$ in only a single iterated integral. Do not compute the integral.

Solution:

$$\int_0^1 \int_x^4 xdydx$$

88. Suppose a laminar object with boundaries $y = x$ and $y = \sqrt{x}$ has a density function $\rho(x,y) = y(1-x^2/2)$. Find the mass of the object.

Solution:

$$\int_0^1 \int_x^{\sqrt{x}} y(1-x^2/2)dydx = \frac{17}{240}$$

89. (12.4) Write the set $D = \{(x,y)|x \geq 0, y \leq 0, x^2 + y^2 \leq 4\}$ in polar coordinates.

Solution: This is a quarter of a circle in the fourth quadrant, so

$$D = \left\{(r,\theta)\middle| r \leq 2, -\frac{\pi}{2} \leq \theta \leq 0\right\}$$

or equivalently

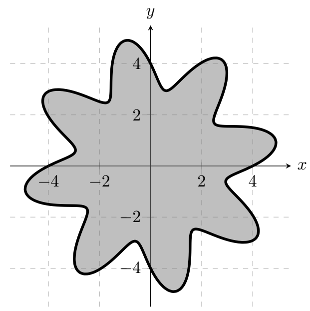
$$D = \left\{(r,\theta)\middle| r \leq 2, \frac{3\pi}{2} \leq \theta \leq 2\pi\right\}.$$

90. (12.4) Using polar coordinates evaluate $\iint\limits_D f(x,y)\ dA$, where D is a disk of radius 2 and $f(r,\theta) = r^2\cos(2\theta)$.

Solution: The region D can be described as $D = \{(r, \theta) | r \leq 2\}$.

$$\iint\limits_D f(x,y)\ dA = \int_0^{2\pi} \int_0^2 r^2 \cos(2\theta) \cdot r\ dr\ d\theta$$
$$= \frac{2^4}{4} \int_0^{2\pi} \cos(2\theta)\ d\theta = \frac{2^4}{4} \left[\frac{1}{2}\sin(2\theta)\right]_0^{2\pi} = 0$$

91. (12.4) Find the area of the flower-shaped region $D = \{(r, \theta) | r \leq \sin(8\theta) + 4\}$.



Solution: Use polar coordinates: $dA = r\ dr\ d\theta$

$$A = \iint\limits_D 1\ dA = \int_0^{2\pi} \int_0^{\sin(8\theta)+4} r\ dr\ d\theta$$
$$= \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_{r=0}^{\sin(8\theta)+4} d\theta = \frac{1}{2}\int_0^{2\pi} 16 + 8\sin(8\theta) + (\sin(8\theta))^2\ d\theta$$
$$= \int_0^{2\pi} 8 + 4\sin(8\theta) + \frac{1}{4} - \frac{1}{4}\cos(16\theta)\ d\theta = \frac{33}{4}\pi$$

since

$$\int_0^{2\pi} \cos(\theta)\ d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \sin(\theta)\ d\theta = 0\,.$$

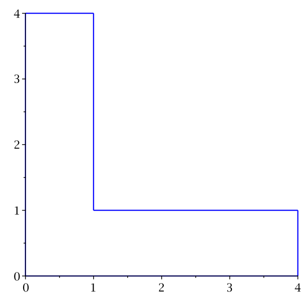
92. (12.5) Suppose a laminar object with boundaries $y = x$ and $y = \sqrt{x}$ has a density function $\rho(x, y) = y\left(1 - \frac{x^2}{2}\right)$. Find the mass of the object.

Solution: The intersections of the boundaries are at $(0, 0)$ and $(1, 1)$. So the region on which the object lies can be described as

$$D = \left\{(x, y) \middle| 0 \leq x \leq 1, x \leq y \leq \sqrt{x}\right\}\,.$$

$$
\begin{aligned}
m =& \int_0^1 \int_x^{\sqrt{x}} y\left(1-\frac{x^2}{2}\right)\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2\left(1-\frac{x^2}{2}\right)\right]_{y=x}^{\sqrt{x}}\,dx \\
=& \frac{1}{2}\int_0^1 \left(x-\frac{1}{2}x^3-\left(x^2-\frac{1}{2}x^4\right)\right)\,dx = \frac{1}{2}\left[\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{3}x^3+\frac{1}{10}x^5\right]_0^1 \\
=& \frac{1}{2}\left(\frac{1}{2}-\frac{1}{8}-\frac{1}{3}+\frac{1}{10}\right) = \frac{17}{240}
\end{aligned}
$$

93. Some objects can have the center of mass outside itself, e.g., a boomerang. Let's consider an idealized boomerang with constant density and the shape shown in the figure. Find its center of mass.



94. (12.5) Let $p(x,y)$ be the probability density function defined for the random variables X and Y in the unit square $D=[0,1]\times[0,1]$, given by

$$p(x,y) = 4y(1-x)\,.$$

(a) Check that this function is a probability density function.

Solution:

$$
\begin{aligned}
\iint\limits_D 4y(1-x)\,dA =& 4\int_0^1\int_0^1 y(1-x)\,dy\,dx \\
=& 4\int_0^1 \left[\frac{1}{2}y^2(1-x)\right]_{y=0}^1\,dx = 2\int_0^1 (1-x)\,dx \\
=& 2\left[x-\frac{1}{2}x^2\right]_0^1 = 1
\end{aligned}
$$

So this function is probability density function.

(b) Set up the iterated integral for the probability that $(x,y)\in A$, where A contains all points (x,y) in D for which x is greater than y. Do not compute the integral.

Solution: The region A can be described as

$$A = \{(x,y)|0\le x\le 1, 0\le y\le x\}\,.$$

Then the probability is

$$P((x,y)\in A) = \iint\limits_A p(x,y)\,dA = \int_0^1\int_0^x 4y(1-x)\,dy\,dx\,.$$

95. (12.5) Suppose a factory must perform two operations on items they are producing. The time (in hours) it takes for operation 1 is X and for operation 2 it is Y. The joint probability distribution is

$$p(x,y) = \begin{cases} 4xye^{-x^2-y^2} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

What is the probability that operation 1 takes more than 2 hours, AND operation 2 takes more than 2 hours?

Solution: The region that describes this occurrence is

$$D = \{(x,y)|x \geq 2, y \geq 2\}.$$

Then the region D cannot be integrated in only one iterated integral. Therefore it is easier to integrate the region outside of D, call it $R = D = \{(x,y)|x \leq 2, y \leq 2\}$ and subtract the result from 1, to get the probability

$$P((x,y) \in D) = 1 - P((x,y) \in D)$$

$$1 - \iint_R p(x,y)\, dA = \int_0^2 \int_0^2 4xye^{-x^2-y^2}\, dy\, dx$$

$$1 - \int_0^2 2ye^{-y^2}\, dy \int_0^2 2xe^{-x^2}\, dx$$

$$= \int_2^\infty 2xe^{-x^2}\, dx \int_2^\infty 2ye^{-y^2}\, dy$$

$$= \left[-e^{-x^2}\right]_2^\infty \cdot \left[-e^{-y^2}\right]_2^\infty = e^{-4} \cdot e^{-4} = e^{-8}.$$

The integral $\int_0^2 2ye^{-y^2}\, dy = 1 - e^{-4}$, so

$$P((x,y) \in D) = 1 - (1 - e^{-4})^2$$