Chapter 2: Geometry of $\mathbb{R}^2, \mathbb{R}^3$

Section 2.1: Vectors and Points

Def: $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R} \quad 1 \le i \le n\}$

A point in $\mathbb{R}^n$ is any tuple $(a_1, \dots, a_n)$.

A vector in $\mathbb{R}^n$ is any tuple $\langle a_1, \dots, a_n \rangle$

where $v = (\langle a_1, \dots, a_n \rangle, \|v\|)$

and $\|v\| = \sqrt{a_1^2 + \dots + a_n^2}$

We get a correspondence

$$\left\{\begin{array}{c}\text{Points in}\\ \mathbb{R}^n\end{array}\right\} \underset{1\cdot v \,\longleftarrow\!\shortmid\, v}{\overset{p \,\longmapsto\, (p, d(p,0))}{\longleftrightarrow}} \left\{\begin{array}{c}\text{Vectors in}\\ \mathbb{R}^n\end{array}\right\}$$

Further

$$\left\{\begin{array}{c}\text{Vectors in}\\ \mathbb{R}^n\end{array}\right\} \underset{v' \,\longleftarrow\!\shortmid\, tv' + r_0}{\overset{v \,\longmapsto\, t\cdot v}{\longleftrightarrow}} \left\{\begin{array}{c}\text{Lines in}\\ \mathbb{R}^n\end{array}\right\}$$

Given two points $p, q \in \mathbb{R}^n$, we can construct a vector

$$\overrightarrow{pq} = \langle q_1 - p_1, \ldots, q_n - p_n \rangle$$

$$\|\overrightarrow{pq}\| = \sqrt{\sum_{i=1}^{n} (q_i - p_i)^2}$$

<u>Def:</u> A vector $v = (p, \|v\|)$ is said to be a <u>unit vector</u> if $\|v\| = 1$.

$\ast$

<u>Lemma:</u> There is a map

$$\left\{ \begin{matrix} \text{Vectors in} \\ \mathbb{R}^n \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Unit Vectors} \\ \text{in } \mathbb{R}^n \end{matrix} \right\}$$

given by $v \longmapsto \dfrac{v}{\|v\|}$.

$\ast$ Operations on Vectors:

If $v, w \in \mathbb{R}^n$, then

i) $v + w = \langle v_1 + w_1, \ldots, v_n + w_n \rangle \in \mathbb{R}^n$

ii) For $\lambda \in \mathbb{R}$, $\lambda v = \langle \lambda v_1, \ldots, \lambda v_n \rangle \in \mathbb{R}^n$

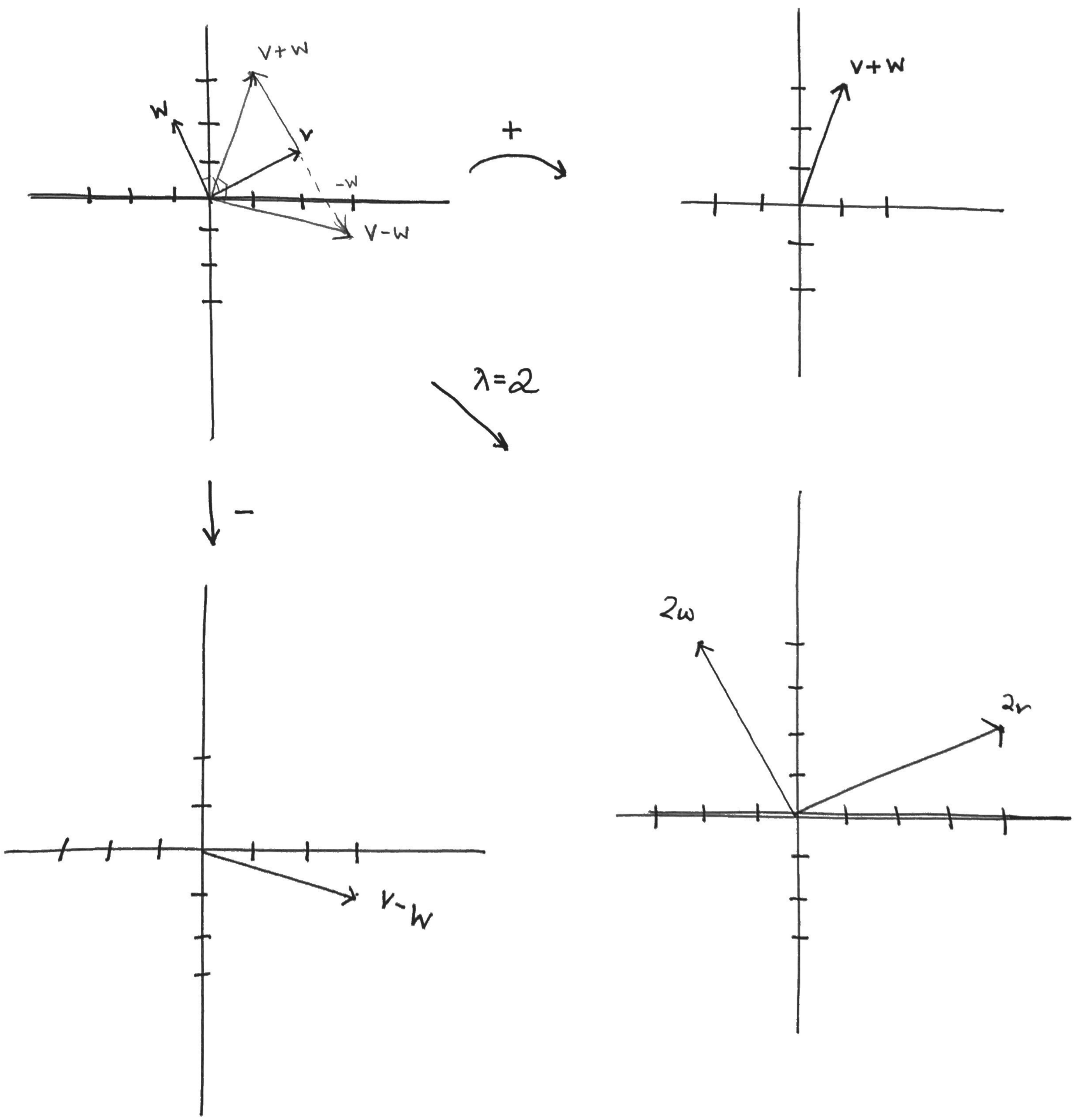
iii) $\lambda(v + w) = \lambda v + \lambda w$

iv) $\exists\, 0 \in \mathbb{R}^n$ such that $0 + v = v$.

Let's draw a few pictures of these interactions in $\mathbb{R}^2$.

$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad w = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad v + w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$v - w = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$



Now that we can add and subtract Vectors, one may ask if we can multiply them. The answer is complicated! As you will see, there are many definitions of products.

Def: Let $v, w \in \mathbb{R}^n$. The dot product or inner product of v and w is denoted

$$v \cdot w = \sum_{i=1}^{n} v_i w_i = \langle v, w \rangle$$

} this can be interpreted as Work (from Physics)

Prop: (Properties of the dot product)

i) $v \cdot w = w \cdot v$

ii) $v \cdot \vec{0} = 0$

iii) $\| v \cdot w \| \leq \|v\| \cdot \|w\|$

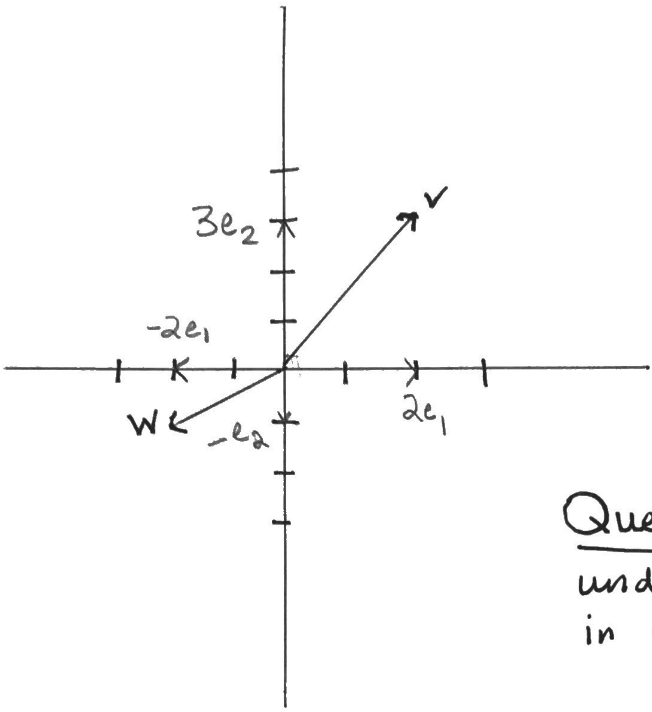
iv) $v \cdot w = \|v\| \cdot \|w\| \cos \theta$ where θ is the angle between v and w.

v) $v \cdot w = 0$ if and only if $v \perp w$ $\left(\theta = 90^\circ = \frac{\pi}{2}\right)$

vi) $v \cdot v = \|v\|^2$, $\|v\| = \sqrt{v \cdot v}$

The Geometric Picture:

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad w = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$



$v \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2$

$v \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3$

$w \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -2$

$w \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$

This shows that taking dot products gives the components of a vector.

Question: What if we want to understand how much v moves in the w-direction?

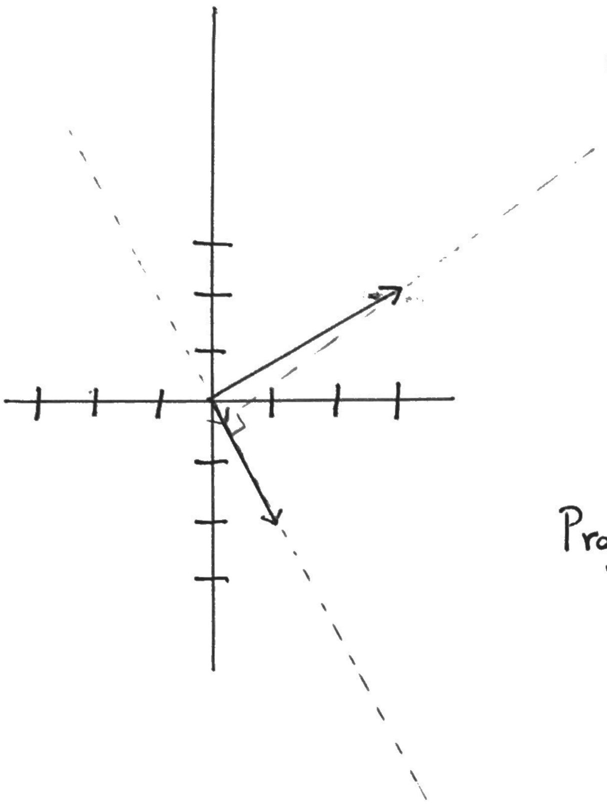
Def: Let $v, w \in \mathbb{R}^n$. Then the projection of v on w is

$$\text{proj}_w(v) = \underbrace{\left[\frac{v \cdot w}{\|w\|^2}\right]}_{\in \mathbb{R}} w$$

and the coefficient of the projection is

$$\text{comp}_w(v) = \frac{v \cdot w}{\|w\|}$$

Example: $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



$$\text{Comp}_w(v) = \frac{v \cdot w}{\|w\|} = \frac{3 \cdot 1 + 2 \cdot (-2)}{\sqrt{1 + (-2)^2}}$$

$$= \frac{-1}{\sqrt{5}}$$

$$= \frac{-\sqrt{5}}{5}$$

$$\text{Proj}_w(v) = \text{Comp}_w(v) \cdot \frac{w}{\|w\|}$$

$$= \frac{-1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \frac{-1}{5} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$$

Example: Computing Angles between vectors

$v = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ $\quad w = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

$$\cos\theta = \frac{v \cdot w}{\|v\| \cdot \|w\|} = \frac{4 \cdot 1 + 2 \cdot 1 - 2 \cdot 3}{\sqrt{11} \cdot \sqrt{24}}$$

$$= 0 \qquad \Rightarrow \theta = \frac{\pi}{2}$$

$$\Rightarrow v \perp w$$

Some other properties of the dot product and vectors:

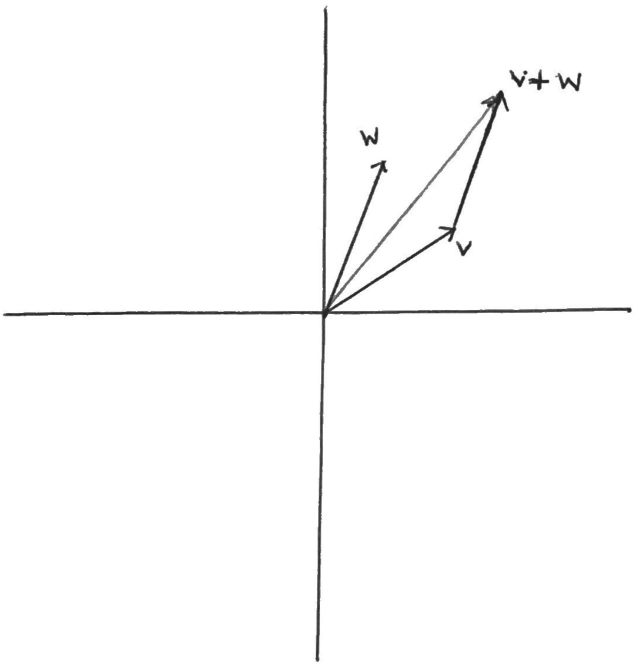
<u>Proposition</u>: Let $v, w \in \mathbb{R}^n$. Then

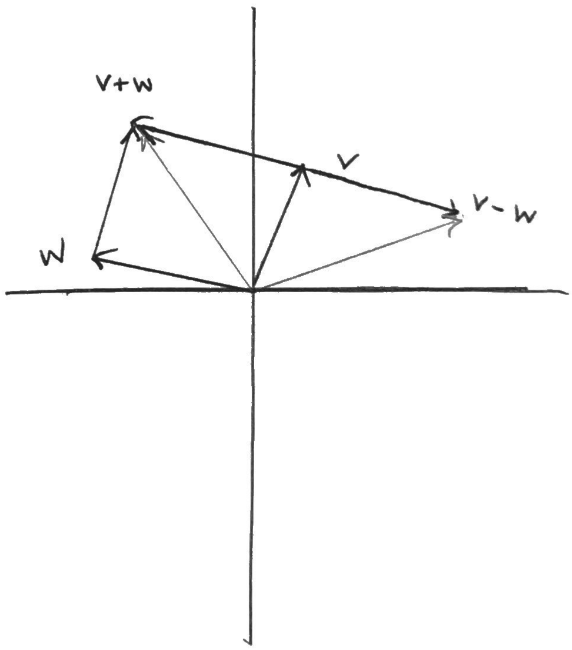
i) $\|v+w\| \leq \|v\| + \|w\|$ (Triangle Inequality)

ii) $\|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right)$

<u>Proof</u>: Use dot product for $v+w$. $\blacksquare$

Geometric Interpretation of the above proposition:





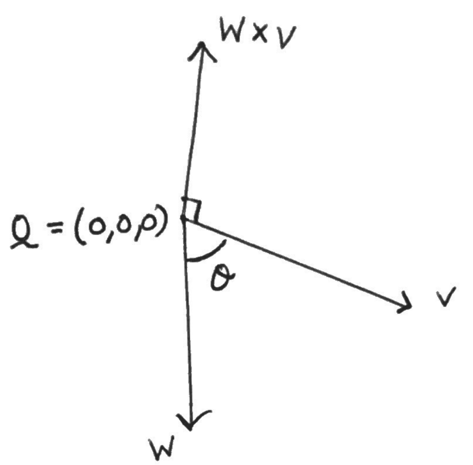
$i \Rightarrow v, w, v+w$ form a valid triangle.

$ii \Rightarrow$ the parallelogram with sides $v+w$, $v-w$, $2v$, 0 forms a parallelogram

The cross product (Lie bracket):

In $\mathbb{R}^3$ we have another product of vectors. This, in contrast to the dot product, produces another vector! Before giving a definition in terms of algebra, we will look at a geometric picture.

$\mathbb{R}^3$



Moral Definition: If v, w are two non-zero vectors in $\mathbb{R}^3$, then $v \times w$ is

$$v \times w = (\|v\| \cdot \|w\| \sin\theta)\, \underset{\sim}{n}$$

where $\underset{\sim}{n}$ is a unit vector perpendicular to both v and w, θ is the angle between v and w, and the direction of $\underset{\sim}{n}$ is given by the right hand rule.

first vector is index finger
Second vector is middle finger
Thumb is direction of cross product.

Properties of the cross product:

i) $v, w \in \mathbb{R}^3$ are parallel if and only if $v \times w = \underset{\sim}{0}$

ii) $v \times w = -w \times v$ (Anti-Symmetry)

iii) $c(v \times w) = (cv) \times w = v \times (cw) \quad \forall c \in \mathbb{R}$

iv) (bi-linearity) For $v, w, u \in \mathbb{R}^3$ we have that

$$(v+w) \times u = v \times u + w \times u$$

$$v \times (w+u) = v \times w + v \times u$$

Example: Let $\hat{i}, \hat{j}, \hat{k}$ denote the unit vectors corresponding to the coordinate axes in $\mathbb{R}^3$. (These are basis vectors). By applying the definition, we see that

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

WARNING: The cross product is not associative. That is, for $v, w, u \in \mathbb{R}^3$

$$(v \times w) \times u \neq v \times (w \times u)$$

Example: $v = w = \hat{i} \quad u = \hat{j}$

We could compute the cross product of the standard basis because we already understood the geometric relationship between $\hat{i}, \hat{j}, \hat{k}$.
What do we do if v, w are arbitrary?

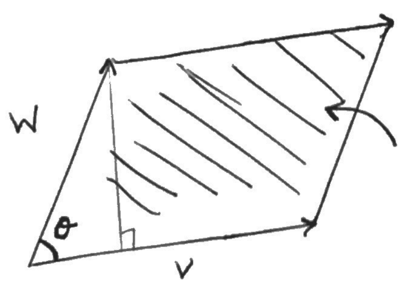
<u>Def</u>: (Algebraic Definition of the Cross product)
Let $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$.
Then

$$v \times w = \left\langle v_2 w_3 - w_2 v_3 \,,\, v_3 w_1 - w_3 v_1 \,,\, v_1 w_2 - w_1 v_2 \right\rangle$$

<u>Exercise</u>: Verify the above example with $\hat{i}, \hat{j}, \hat{k}$ in the coordinate definition.

<u>Interpretations of the cross product</u>:

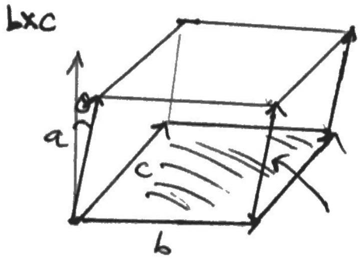
1 Area of a parallelogram:



$$A = \|v \times w\|$$

$$= \|v\| \left(\|w\| \sin\theta \right)$$

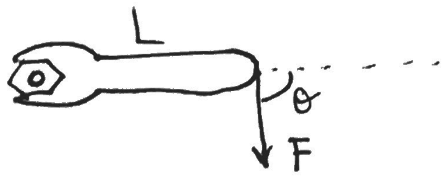
② Volume of Parallelepiped (The triple product)



$$V = Ah = \|b \times c\| \cdot \|a\| \cos\theta$$

$$= |a \cdot (b \times c)|$$

③ Torque



The magnitude of the torque around the bolt is given by:

$$\|\tau\| = \|L \times F\| = \|L\| \cdot \|F\| \sin\theta \; \|\hat{n}\|$$

Exercise: Show that

$$v \times (w \times u) + u \times (v \times w) + w \times (u \times v) = 0$$

for all $u, v, w \in \mathbb{R}^3$.

Exercise: A plane in $\mathbb{R}^3$ is given by the equation

$ax + by + cz + d = 0$ or $a\hat{i} + b\hat{j} + c\hat{k} = d$.

To find a, b, c, d for given points; $P = (1,2,3)$, $Q = (3,4,4)$ and $R = (1,1,2)$ we construct a vector $\underset{\sim}{n} = \langle a, b, c \rangle$ normal to the plane these points define and pick d such that P is in the plane.

Exercise: The unit cube is the set

$$C^3 = \left\{ x \in \mathbb{R}^3 : 0 \leq x_i \leq 1 \ , i \in \{1,2,3\} \right\}$$

ⓐ Find the angle between the main diagonal of the unit cube and one of the face diagonals. (Assume both diagonals pass through a common vertex)

ⓑ Find the projection of the main diagonal

$$v = \langle 1, 1, 1 \rangle \quad \text{onto} \quad w = \langle 1, 1, 0 \rangle$$

Exercise: Verify the Lagrange formula

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$$

for general vectors $a, b, c \in \mathbb{R}^3$.

Section 2.2: Subspaces of $\mathbb{R}^2, \mathbb{R}^3$.

There are two types of subspaces:

i) Linear

ii) Non-Linear

Intuitive distinction:

Linear	Non-Linear
• generated by lines	• Curvy
• "flat"	• "non-constant"
• determined by vectors	• Relatively difficult
• Relatively simple	

Examples:

Lines in $\mathbb{R}^n = tv + p$

Planes in $\mathbb{R}^3 = tv + sw + r$

v, w vectors

$s, t \in \mathbb{R}$.

$r \in \mathbb{R}^3$.

"Linear Algebra"

Examples:

$\left.\begin{array}{l}\text{Curves in } \mathbb{R}^n \\ \text{Surfaces in } \mathbb{R}^3\end{array}\right\}$ given by functions of many variables.

"Differential Geometry"

We will start with linear subspaces.

Def: A line is any geometric object given by a single vector $\hat{v}$ in the following way

$$L = \{tv + p_0 : t \in \mathbb{R}\} \qquad p_0 \in \mathbb{R}^n$$

If $p_0 = \underset{\sim}{0}$, then L is not only a geometric object. For any two points $t_0 v$ and $t_1 v$ we can add them to get

$$(t_0 + t_1) v$$

another point on L. For this reason, the case of $p_0 = 0$ is a genuinely different situation. (*) See page 19 for an additional form.

Lemma: The "dimension" of any Line is 1. That is, a line has at most 1 degree of freedom.

Example: Suppose we fix $v_0 \in \mathbb{R}^n$. Then the function defined by

$$f(t) = tv_0$$

has as its graph, a line.

This Lemma is our first introduction to dimension. We want to formalize this idea.

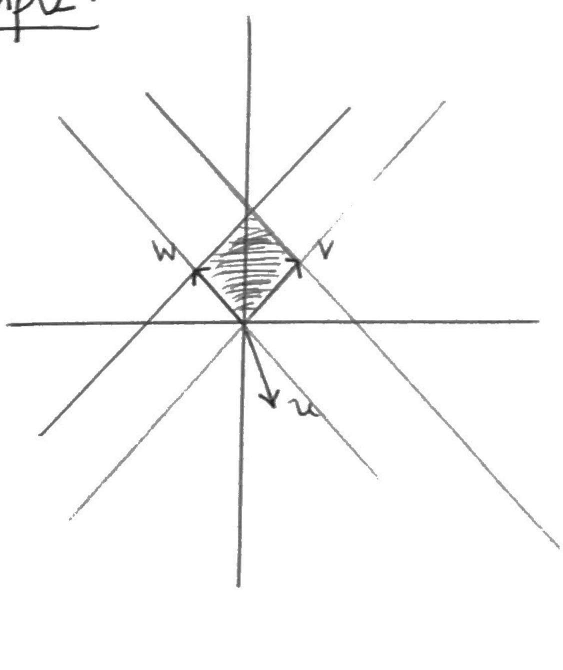
Def: The dimension of a linear subspace is the largest number of vectors in the space which cannot be written in terms of the other vectors.

Said another way, the dimension of a linear space is the size of a maximal linearly independent set.

$\{v_i\}_{i \in I}$ is linearly independent if whenever $\sum_{k=1}^{n} v_{i_k} \cdot a_k = 0$

$\Rightarrow a_k = 0$ for all k.

Example:



$\{v,w\}$ is linearly ind.

$\{v,w,u\}$ is not because $u = \alpha v + \beta w$ for some $\alpha, \beta \in \mathbb{R}$.

Using this definition, we can now define planes.

Def: A 2-dimensional linear subspace is a plane. For this reason, we can define planes by two linearly independent vectors contained in it. That is

$$P = \{ tv + sw + r_0 : t, s \in \mathbb{R} \}$$

To complete the analogy with lines, we need to define functions with more than one input.

A multivariable function is any function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

That is,

$$f(x_1, x_2, \dots, x_n) \in \mathbb{R}$$

Example: Let $n = 2$. Then by defining

$$f(x,y) = x + y$$

we have a multivariable function. In fact, if we put $z = f(x,y)$. Then the graph of f is a plane defined by

$$z - x - y = 0.$$

This last example motives another way of writing (hyper) planes:

General form: $a_1x_1 + \dots + a_nx_n = b$.

For $n=2$, this gives us a line (or at least the general form) and for $n=3$ this gives us

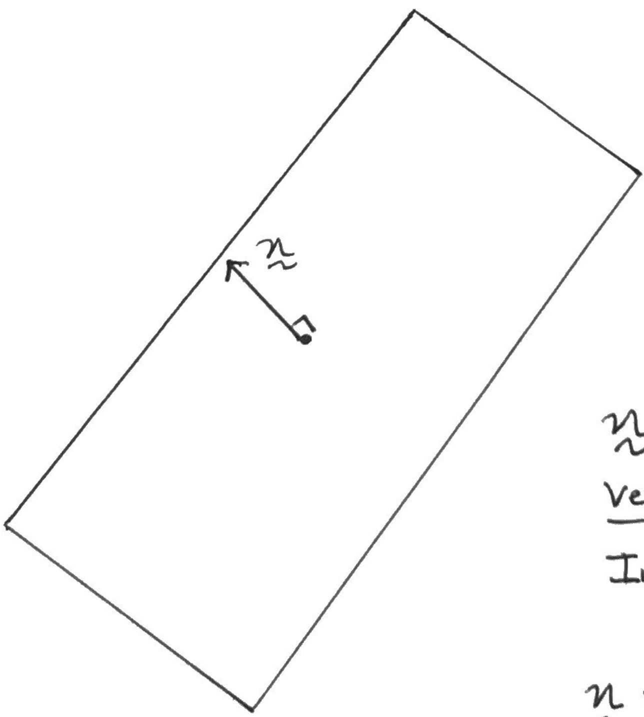
$$ax + by + cz = d$$

as the general form. Notice that if $d=0$, we have a similar story as for lines: we can add points!.

<u>Question</u>: What do the values (a,b,c) determine/tell us about the plane?

<u>Answer</u>:

$\mathbb{R}^3$



$\underset{\sim}{n}$ is the <u>normal vector</u> to the plane. In fact,

$$\underset{\sim}{n} = \langle a, b, c \rangle$$

Example: Computing $\underset{\sim}{n}$.

Idea: cross products determine orthogonal vectors, thus $\underset{\sim}{n}$ should arise as a cross product.

Suppose we [redacted] are given 3 points in $\mathbb{R}^3$.

$P=(p_1,p_2,p_3)$ $\quad$ $Q=(q_1,q_2,q_3)$ $\quad$ $R=(r_1,r_2,r_3)$

The points define a plane assuming they are not colinear. How do we get $\underset{\sim}{n}$ from these?

Step 1: Compute $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Step 2: Take $\overrightarrow{PQ} \times \overrightarrow{PR}$ to get $\underset{\sim}{n}$.

Step 3: Write $\underset{\sim}{n} = \langle a,b,c \rangle$. Then the plane is given by

$$ax+by+cz=d$$

where $d = ap_1 + bp_2 + cp_3$.

For $P=\hat{i}$, $Q=\hat{j}$, $R=0$:

$$\overrightarrow{PQ} = \langle -1,1,0 \rangle \qquad \overrightarrow{PR} = \langle -1,0,0 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1\cdot 0 - 0\cdot 0,\ 0\cdot(-1) - 0\cdot(-1),\ (-1)\cdot 0 - (-1)\cdot 1 \rangle$$

$$= \langle 0,0,1 \rangle = \hat{k}.$$

Thus the plane containing P, Q, R is given by

$$0x + 0y + z = 0$$

$\rightarrow$ $z = 0$ is the plane. :)

(*) For any line $L = tv + r_0$ we can write this in terms of the coordinates:

$$x = tv_1 + r_1 \qquad y = tv_2 + r_2 \qquad z = tv_3 + r_3$$

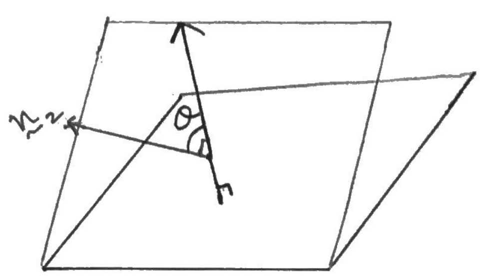
Solving for t in all of the equations we get

$$\frac{x - r_1}{v_1} = \frac{y - r_2}{v_2} = \frac{z - r_3}{v_3}$$

Operations with Planes:

Def: Let $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ be two planes. Then the angle between the planes is

$$\theta = \cos^{-1} \frac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2}{\|\underset{\sim}{n}_1\| \, \|\underset{\sim}{n}_2\|}$$



Example: Consider the two planes

$$x + y = 3 \qquad\qquad y = 25$$

The two normal vectors are

$$\underset{\sim}{n}_1 = \langle 1,1,0 \rangle \quad \text{and} \quad \underset{\sim}{n}_2 = \langle 0,1,0 \rangle$$

then

$$\theta = \cos^{-1} \frac{|\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2|}{\|\underset{\sim}{n}_1\| \; \|\underset{\sim}{n}_2\|} = \cos^{-1} \frac{|1\cdot 0 + 1\cdot 1 + 0\cdot 0|}{\sqrt{2} \cdot 1} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Another operation is finding the distance from a point to a plane.

Let $P = (p_1, p_2, p_3)$ be a point in $\mathbb{R}^3$ and $ax + by + cz = d$ a plane.

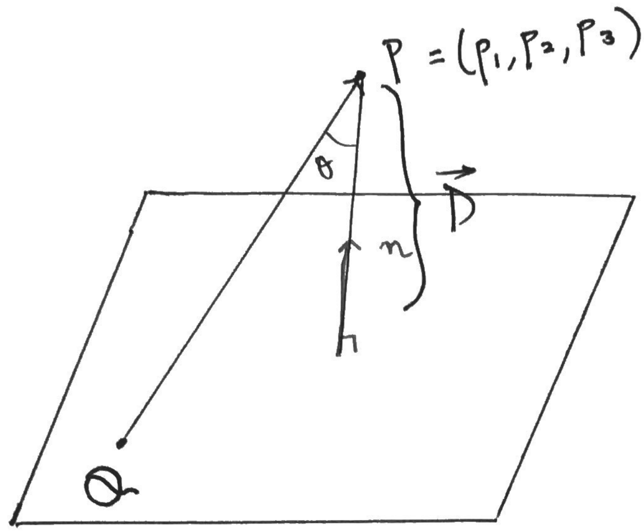
Def: The distance from P to the plane is given by

$$D = \frac{|ap_1 + bp_2 + cp_3 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Let us derive this result from a vector computation:

Let $Q = (q_1, q_2, q_3)$ be an element of the plane.
Let $\overrightarrow{QP}$ be the vector from $Q \to P$. Then,

$$D = \frac{|n \cdot \overrightarrow{QP}|}{\|n\|}$$



$$\vec{D} = \|\overrightarrow{PQ}\| \cos\theta \cdot \frac{n}{\|n\|}$$

$$\Rightarrow \|\vec{D}\| = \frac{\|n\| \, \|\overrightarrow{PQ}\| \cos\theta}{\|n\|} = \frac{|n \cdot \overrightarrow{QP}|}{\|n\|}$$

<u>Example</u>: Let $X = \left\{ t \underset{v}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + s \underset{w}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} : s, t \in \mathbb{R} \right\}$
be a plane and $P = (3, 0, 0)$.
Find $d(P, X)$.

First we need to construct the normal vector.

$$\underset{\sim}{n} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \langle 1\cdot 1 - 0\cdot 0,\ 0\cdot 1 - 1\cdot 1,\ 1\cdot 1 - 0\cdot 1 \rangle$$

$$= \langle 1, -1, 1 \rangle$$

Now, lets pick a point on X. Namely the point given by $t=-1$, $s=-1$ so that

$$Q = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Using the formula from above, we have

$$D = \frac{|\underset{\sim}{n} \cdot \overrightarrow{QP}|}{\|\underset{\sim}{n}\|} = \frac{\left| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right|}{\sqrt{1^2 + (-1)^2 + 1^2}}$$

$$= \frac{3\cdot 1 + (-1)\cdot 0 + (-2)\cdot 1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

☺.

Moving on to non-linear subspaces!

Some examples from highschool:

Circles Ellipses Hyperbolas Cones

These are all quadric objects that is they come from equations involving x^2, y^2.

In general, non-linear (smooth) subspaces are given by functions of many variables. We saw before that planes and lines can be given as the graph of certain multivariable functions. This process is called <u>Parametrizing</u> the subspace and will be studied in detail a few lectures from now. For the moment, we will give intrinsic definitions of certain surfaces.

Examples to Remember: The Quadric Surfaces

1 Spheres (and Circles):

A <u>sphere</u> in $\mathbb{R}^n$, $n=2,3$ is the collection of all points equidistant from a common center. The equation for a sphere is

$$x^2+y^2+z^2=r^2 \quad \text{(centered at origin)}$$

or

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2 \quad \text{(centered at } r=(x_0,y_0,z_0))$$

Slogan: Cross sections are circles!

(2) Ellipsoids (and Ellipses)

Any surface given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (\text{centered at } 0)$$

or

$$\frac{(x-p_1)^2}{a^2} + \frac{(y-p_2)^2}{b^2} + \frac{(z-p_3)^2}{c^2} = 1$$

Lemma: If $a=b=c$, then an ellipsoid is a sphere!

Slogan: Cross sections are ellipses!

(3a) Hyperboloids (One sheet)

Any surface given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (\text{centered at } 0)$$

or

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - \frac{(z-z_0)^2}{c^2} = 1$$

Slogan: Horizontal cross sections are ellipses
Vertical cross sections are hyperbolas

3b Cones

A <u>cone</u> is any surface given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad \text{(centered at } O\text{)}$$

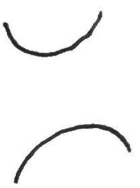
or

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - \frac{(z-z_0)^2}{c^2} = 0$$

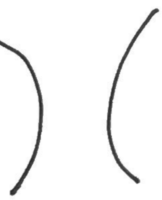
<u>Slogan</u>: Cross sections:

if the slice passes through the Vertex.

if the slice is horizontal



or



if the slice is vertical

3c Hyperboloids (Two Sheeted)

Any surface given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

or

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - \frac{(z-z_0)^2}{c^2} = -1$$

<u>Slogan</u>:
Vertical slices are hyperbolas
Horizonal slices are ellipses.

4a Elliptic Paraboloids

Any surface given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0$$

(centered at 0)

or

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - \frac{z-z_0}{c} = 0$$

Slogan: Vertical Cross sections are parabolas
Horizontal Cross sections are ellipses

4b Hyperbolic Paraboloid

Any surface given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0$$

(centered at 0)

or

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} - \frac{z-z_0}{c} = 0.$$

Slogan: Horizontal Cross sections are hyperbolas
Vertical cross sections are parabolas.

Determining a surface given as the graph of a function:

<u>Def</u>: Let $f:\mathbb{R}^n \to \mathbb{R}$ be a multivariable function. The <u>pre-image</u> of a point $r \in \mathbb{R}$ is the set

$$f^{-1}(r) = \{ v \in \mathbb{R}^n : f(v) = r \} = \{ (v,r) \in \Gamma(f) : f(v) = r \}$$

We also call these <u>level sets</u>.

Notice that

$$\Gamma(f) = \bigcup_{r \in \mathbb{R}} f^{-1}(r)$$

Therefore, we can understand surfaces by understanding their level sets.

<u>Example</u>: What do the level sets look like for the functions:

i) e^{x+y}

As every positive real number can be given as e^c for some c, we find that the level set of e^{x+y} for any positive real number is the set $\{x+y=c\}$ <u>so a line</u>!

ii) $|xy|$

When $r=0$, we see that one of x or y is 0. Thus, we get that $f^{-1}(0) = \{x\text{-axis}\} \cup \{y\text{-axis}\}$

But now we have a problem. How do we determine the rest of the surface?

Additional methods for determining surfaces:

① **Def:** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function and $k \in \mathbb{R}$ any number. Then the

<u>x-slice</u> of f at k is the set $f(k, y)$

<u>y-slice</u> of f at k is the set $f(x, k)$.

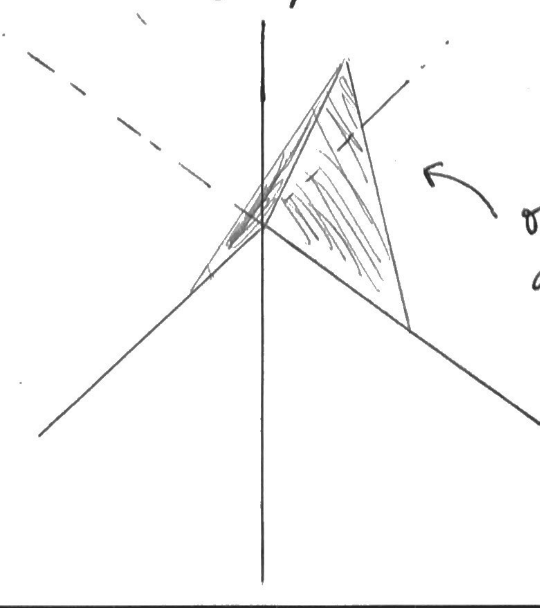
This determines the behavior of f if one of the coordinates is fixed.

---

② <u>Using other functions:</u>

Some functions intersect certain lines in ways that make it easier to graph.

<u>Ex:</u> For $f(x,y) = |xy|$ from before, we can look at when $x = y$. This reduces the complexity of f and makes its graph more clear.



one quarter of $\Gamma(|xy|)$.

Change of Coordinates :

Consider the cylinder

$$x^2+y^2=1$$

how is this a cylinder? It looks like a circle!

in $\mathbb{R}^3$. If we look at the level set / z-slice at $z=0$, we get a circle. Recall from High-School algebra that every point on a circle of radius 1 can be written as

$$(\cos\theta, \sin\theta)$$

If instead of having radius 1, we can replace $\cos\theta$ with $r\cos\theta$ where r is the new radius.

<u>Def</u>: The coordinate system on $\mathbb{R}^2$ given by $\underset{\sim}{x}=(r,\theta)$ where $\underset{\sim}{x}$ is a vector in $\mathbb{R}^2$ is called the <u>Polar Coordinate system</u>.

In $\mathbb{R}^3$ we simply have

$$\underset{\sim}{v} = (r, \theta, z)$$ as the polar coordinate system.

To understand how we translate from the standard (Cartesian) coordinate system to the polar coordinate system, we inspect the cylinder from above.

$$x^2+y^2=r^2$$

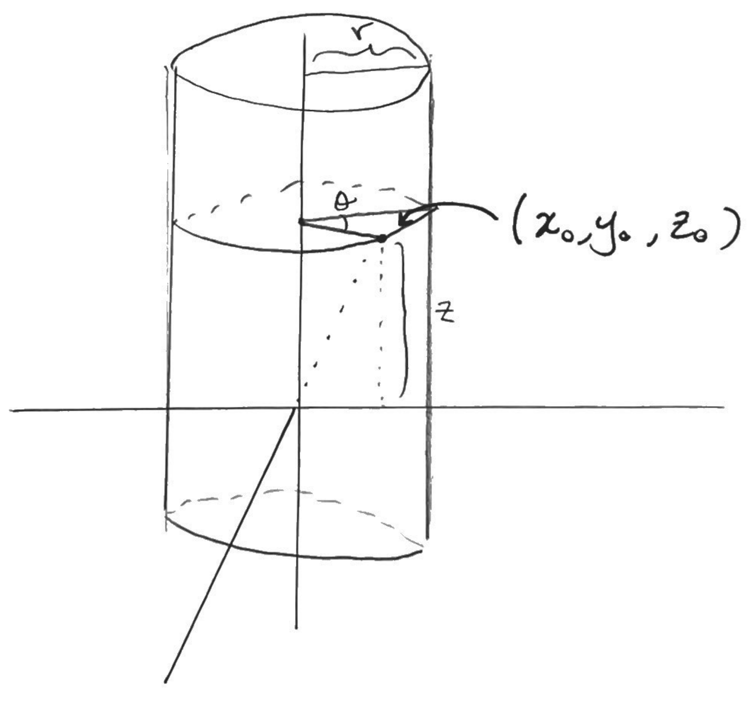
For any triple of points $(x_0,y_0,z_0)\in\mathbb{R}^3$ we get that

$$x_0 = r\cos\theta$$

$$y_0 = r\sin\theta$$

$$z_0 = z$$

where these relations come from the following picture:



For this reason, we also call 3D polar coordinates <u>Cylindrical Coordinates</u>

"Polar" $\equiv$ "Cylindrical"

<u>Example</u>: Convert the point $(1,1,1)$ into Cylindrical coordinates

$1 = r\cos\theta$

$1 = r\sin\theta$

$z = 1$

As $\cos\theta = \sin\theta$ for this point we see that $\theta = \frac{\pi}{4}$

Thus $r = \sqrt{2}$.

Is there another way to compute r?

Let's investigate

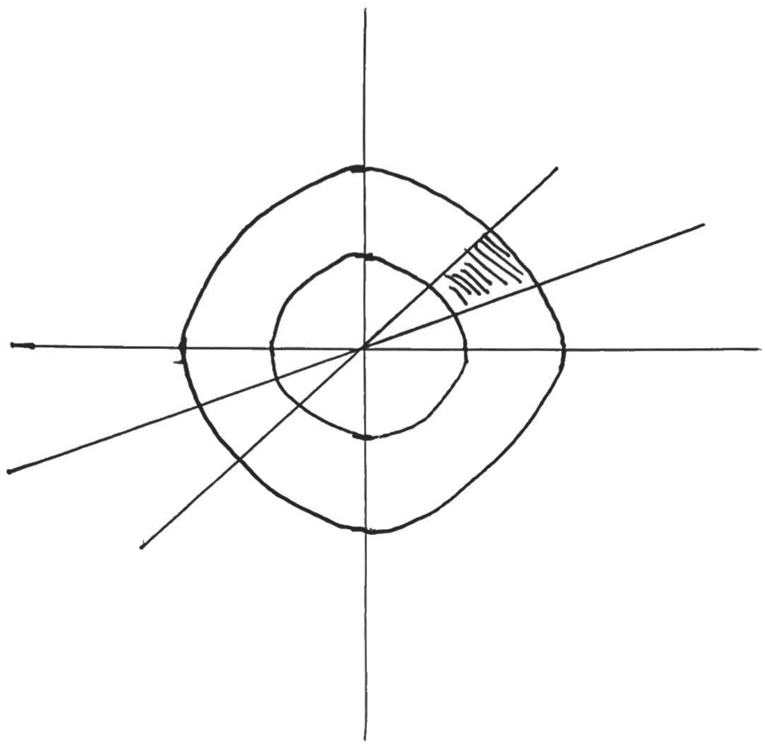
$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$

therefore $r = \sqrt{x^2+y^2}$

Furthermore

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

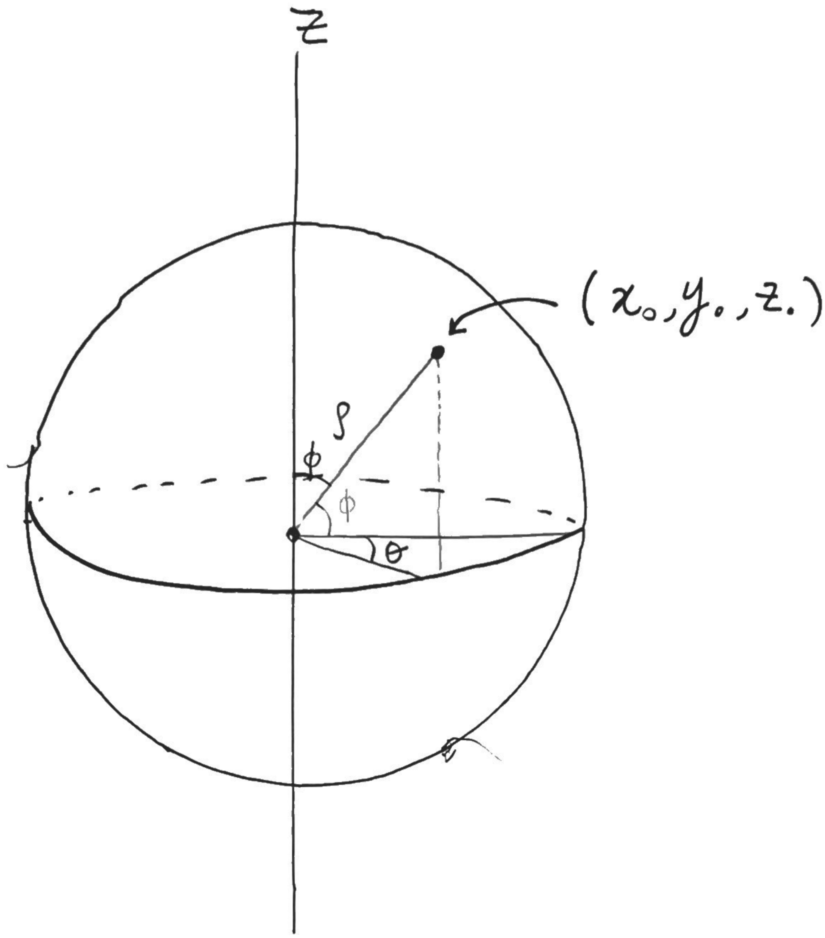
Now, the use of the term "Polar" seems a bit misleading as in $\mathbb{R}^2$ the polar coordinate system is given by concentric rings about the origin:



<u>Q:</u> What is the "correct" analog for this in $\mathbb{R}^3$?

<u>Answer</u>: Spherical Coordinates!

Given a point $(x_0, y_0, z_0) \in S^2$ how can we write x_0, y_0, and z_0 in terms of ρ the radius, θ the horizontal angle, and ϕ the vertical angle.



From the picture, we see that

$$x_0 = \rho \cos\theta \sin\phi$$

$$y_0 = \rho \sin\theta \sin\phi$$

$$z_0 = \rho \cos\phi$$

Claim: $x_0^2 + y_0^2 + z_0^2 = \rho^2$

Proof: $\rho^2 \cos^2\theta \sin^2\phi + \rho^2 \sin^2\theta \sin^2\phi + \rho^2 \cos^2\phi$

$= \rho^2(\cos^2\theta + \sin^2\theta)\sin^2\phi + \rho^2\cos^2\phi$

$= \rho^2(\sin^2\phi + \cos^2\phi) = \rho^2$. ■

Examples:

Convert the point $(1,1,1)$ to spherical coordinates:

$$\rho = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\phi = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx .955$$

So $\left(\sqrt{3}, \frac{\pi}{4}, .955\right)$

Convert the following equation of a hyperboloid into spherical coordinates:

$$x^2 - y^2 - z^2 = 1$$

$$\rho^2 \cos^2\theta \sin^2\phi - \rho^2 \sin^2\theta \sin^2\phi - \rho^2 \cos^2\phi = 1$$

$$\Rightarrow \rho^2\left[\sin^2\phi\left(\cos^2\theta - \sin^2\theta\right) - \cos^2\phi\right] = 1$$

$$\Rightarrow \rho^2\left[\sin^2\phi \cdot \cos 2\theta - \cos^2\phi\right] = 1$$

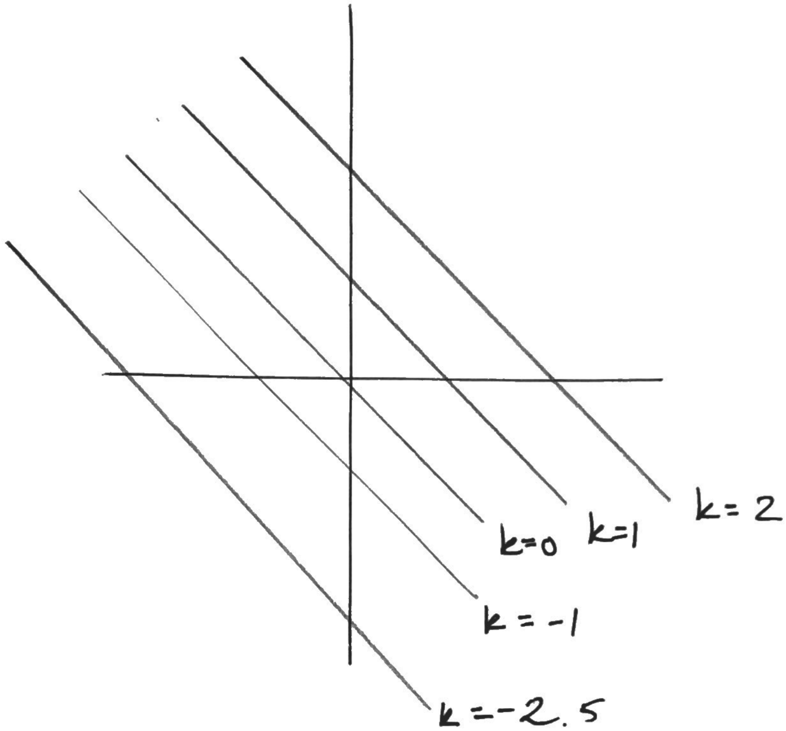
Chapter 3: Functions of Several Real Variables

Section 1: More on Graphing: Contour Plots

Recall that one methods for understanding multivariable functions is to graph them. Notice that for a function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

the level sets determine curves in $\mathbb{R}^2$.
For example, $f(x,y) = x+y$ has as level sets the lines $x+y=k$ $k \in \mathbb{R}$ fixed.
If we project the level sets onto the (x,y)-plane, we get a contour plot of f. For the function above, the contour plot looks like:



As $x+y=k \Rightarrow$ ~~$y=-x+(-k)$~~ $y=-x+k$.

Contour plots are not just for functions $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$. We could instead replace 2 with 3, 4, or n, and proceed in the same fashion.

Consider $f(x,y,z)=x^2+y^2+z^2$. Then the level sets are spheres of radius $\sqrt{k}$ for $k \geq 0$ and empty for $k<0$.

Something else we can change is the domain of the function!

<u>Def</u>: A <u>region</u> D in $\mathbb{R}^n$ is a connected open set. <u>Open</u> means that for all points $p \in D$ there exists $r > 0$ such that

$$B_r(p) = \{ x \in \mathbb{R}^n : |x - p| < r \}$$

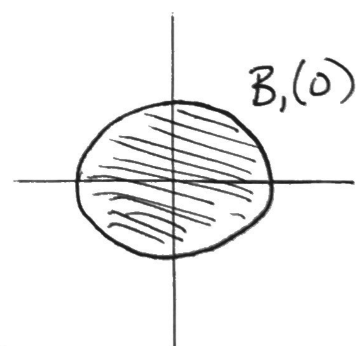
is contained entirely in D.

this set is called the <u>ball of radius</u> r at p.

<u>Examples of regions</u>:

i) $D = \mathbb{R}^n$

ii) $D = B_1(0)$



iii) $D = \bigcup_{n \in \mathbb{N}} B_1((1/n, 0))$

We can define functions on regions as

$$f : D \longrightarrow \mathbb{R}$$

where $f(\underset{\sim}{x})$ is only defined for $\underset{\sim}{x} \in D$.

Example: Determine the domain for the function $f(x,y,z) = \ln(z-y) + xy\sin z$.

$\ln(z-y)$ is defined for all

$$z > y.$$

Therefore

$$D = \{(x,y,z) \in \mathbb{R}^3 : z > y\}$$

is the domain of f.

Similar to the single variable case, we want to understand the behavior of f on the boundary ∂D of the domain. To do this, we need the language of limits.

Def: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a multivariable function. If $(a,b) \in \mathbb{R}^2$ we put

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

if for any $\varepsilon > 0$ $\exists\, \delta > 0$ such that

$$\|(x,y) - (a,b)\| < \delta \Rightarrow |f(x,y) - L| < \varepsilon.$$

<u>Lemma</u>: Suppose $f: D \to \mathbb{R}$ is a function.
If there exist two paths C_1, C_2 containing (a,b) such that

$$\lim_{\substack{(x,y)\to(a,b)\\ C_1}} f(x,y) = L_1 \quad \text{and} \quad \lim_{\substack{(x,y)\to(a,b)\\ C_2}} f(x,y) = L_2$$

then

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ does not exist!}$$

Lets figure out if the following limits exist :

i) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Limit along y-axis $= -1$

Limit along x-axis $= 1$

So Limit DNE !

ii) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Along x-axis, $\lim \longrightarrow 0$

Along y-axis, $\lim \longrightarrow 0$

Along $x=y$, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$.

Hence, the limit DNE.

iii) $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ (Hint Squeeze Theorem)

Along x-axis $\lim \longrightarrow 0$

Along y-axis $\lim \longrightarrow 0$

Along $y=x^2$ $\lim \longrightarrow 0$

So we start to believe that the limit exists.
To show this notice that $x^2 \leq x^2+y^2$ so

$$0 \leq \frac{3x^2|y|}{x^2+y^2} \leq 3|y|$$

Squeeze theorem $\Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$ ☺

Now that we have the notion of limits we can define when multivariable functions are to be continuous at a point.

Def: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function.

We say that f is continuous at a point $\underset{\sim}{x} \in \mathbb{R}^n$ if and only if for any $\varepsilon > 0$ $\exists \delta > 0$ such that whenever $\| \underset{\sim}{y} - \underset{\sim}{x} \| < \delta$

$$|f(\underset{\sim}{x}) - f(\underset{\sim}{y})| < \varepsilon .$$

Corollary: If f is continuous at $\underset{\sim}{x}$ then

$$\lim_{\underset{\sim}{y} \to \underset{\sim}{x}} f(\underset{\sim}{y}) = f(\underset{\sim}{x}) .$$

A function $f : \overset{\subseteq \mathbb{R}^n}{D} \longrightarrow \mathbb{R}$ is continuous if for all points $d \in D$, f is continuous at d.

Fact: A polynomial is any function of the form $\sum_{\alpha} x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$ $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index of integers. These are all continuous.

<u>Example:</u> Use continuity to find the following limits :

i) $\lim_{(x,y,z)\to(1,1,0)} xyz^2 + x^3 + y^3 + x^4y^2z$

↳ this is a polynomial, hence continuous. Thus

$$\lim = 1\cdot 1\cdot 0 + 1^3 + 1^3 + 1^4\cdot 1^2\cdot 0$$

$$= 2$$

ii) $\lim_{(x,y,z)\to(-1,2,3)} xy + yz + xz$

↳ Polynomial ⇒ continuous.

$$\lim = (-1)2 + 2\cdot 3 + (-1)3$$

$$= -2 + 6 - 3$$

$$= 1.$$

Now we want to determine if a given function is continuous. This will amount to taking limits.

Consider the functions

(1) $$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

As we have seen, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ – DNE.

So, there is no way that $f(x,y)$ can be continuous at $\underline{0} \in \mathbb{R}^2$.

(2) $$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (0,0) \end{cases}$$

The computation above implies that $f(x,y)$ is continuous at $(0,0)$ as

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

<u>Fact</u>: Every function from Calculus 1 (1311) which was continuous is continuous when we introduce multiple variables.

Classes of Continuous functions:

(i) Polynomials

(ii) Rational Functions

(iii) Trigonometric Functions

The next topic we want to discuss is differentiation. In order to do this, we want to recall some single variable calculus.

There is a series of inclusions:

$$\{\text{functions}\} \supseteq \{\text{cts. functions}\} \supseteq \{\text{Diffble functions}\}$$

For instance

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is not continuous.

$$f(x) = \begin{cases} |x| & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous, but not differentiable.

Section 2.2: Differentiation

When passing to multivariable functions it is unclear (at first glance) what being differentiable means. For this reason, we start by "cheating" a bit.

Consider $f(x,y,z) = xyz$. If we hold y and z constant, we reduce f to a function of a single variable. Therefore, we can take the derivative:

$\boxed{\text{PD1}}$ denote by

$$f_x(x_0,y_0,z_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0, z_0) - f(x_0,y_0,z_0)}{h}$$

this is the <u>partial derivative</u> of f with respect to x.

$\boxed{\text{PD2}}$ Suppose we have a surface, then f_x can be realized geometrically:

Now we give a general definition:

<u>Def</u>: Let $f : \mathbb{R}^n \to \mathbb{R}$. Then the <u>partial derivative</u> of f with respect to the x_i variable at $\underset{\sim}{y} = (y_1, y_2, \dots, y_n)$ is

$$\frac{\partial f}{\partial x_i} = f_{x_i}(y_1, \dots, y_n) = \lim_{h \to 0} \frac{f(y_1, \dots, y_i + h, \dots, y_n) - f(y_1, \dots, y_n)}{h}$$

if the right hand side exists.

We say that f is differentiable at $\underset{\sim}{y} = (y_1, \dots, y_n)$ if $\frac{\partial f}{\partial x_i}$ exists for all $1 \le i \le n$.

<u>Def</u>: We can arrange the partial derivatives into a matrix called the <u>Jacobian of f</u>.

$$(D_{\underset{\sim}{y}} f) v = \left[\frac{\partial f}{\partial x_1}(\underset{\sim}{y}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\underset{\sim}{y}) \right] \cdot v = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underset{\sim}{y}) v_i$$

for $v \in \mathbb{R}^n$.

<u>Lemma</u>: f is differentiable at $\underset{\sim}{y}$ iff $D_{\underset{\sim}{y}} f$ is defined/exists.

Examples:

i) Let $f(x,y) = xy$

$$\Rightarrow \quad f_x(x,y) = y$$

$$f_y(x,y) = x$$

$$Df = \begin{bmatrix} y & x \end{bmatrix}$$

ii) $g(x,y,z) = \sin(xy) + z^2 e^x$

$$\Rightarrow \quad g_x(x,y,z) = y\cos(xy) + z^2 e^x$$

$$g_y(x,y,z) = x\cos(xy)$$

$$g_z(x,y,z) = 2ze^x$$

iii) $h(x,y,z,\omega) = xy^2 z + \omega x^3 + \omega^2 z^2 + \sinh(x\omega)$

$$\Rightarrow \quad h_x(x,y,z,\omega) = y^2 z + 3\omega x^2 + \omega\cosh(x\omega)$$

$$h_y(x,y,z,\omega) = 2xyz$$

$$h_z(x,y,z,\omega) = xy^2 + 2\omega^2 z$$

$$h_\omega(x,y,z,\omega) = x^3 + 2\omega z^2 + x\cosh(x\omega)$$

Higher Order partial derivatives:

We can take multiple partial derivatives in sequence. These are denoted as:

$$f_{xy} = \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial x}$$

$$f_{xxx} = \left(\frac{\partial}{\partial x}\right)^3 f$$

$$f_{xy^2} = \left(\frac{\partial}{\partial y}\right)^2 \frac{\partial f}{\partial x}$$

$$f_{xyx} = \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial x}$$

A good question to ask is if these higher order partial derivatives depend on the choice of order. For instance, is

$$f_{xy} = f_{yx}$$

The answer to this question depends on the behavior of f and its derivatives near the desired point.

Theorem (Clairaut's Theorem):

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. If all second order partial derivatives of f exist and are continuous, then

$$\frac{\partial^2}{\partial x_i \partial x_j} f = \frac{\partial^2}{\partial x_j \partial x_i} f$$

for all $1 \leq i, j \leq n$.

Example:

i) $f(x,y) = \frac{x}{y}$

$$f_x = \frac{1}{y} \qquad f_y = \frac{-x}{y^2}$$

$$f_{xy} = -\frac{1}{y^2} = f_{yx} = \frac{-1}{y^2}$$

ii) $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ this is a famous counter-example.

$$f_x = \frac{(3x^2y - y^3)(x^2+y^2) - 2x\,(xy(x^2-y^2))}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_y = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \rightsquigarrow f_{xy}(0,0) = -1$$

$$f_{yx}(0,0) = 1$$

Extrema of Functions:

Theorem (Fermat): If f has a local minimum or local maximum at a point (a,b) and the first partials exist, then

$$f_x = f_y = 0 \quad \text{at } (a,b).$$

This mirrors the one-dimensional case!

Q: How do we determine if a point is a local min/max?

A: Use the second derivatives!

Definition 1: Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function. The gradient of f is the vector

$$\nabla f = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right]$$

Definition 2: The Hessian matrix of f is

$$\underset{\sim}{H} f = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{1 \le i,j \le n}$$

As we will see, $\underset{\sim}{H} f = D(\nabla f)$.

<u>Proposition</u>: Suppose the second partial derivatives of f are continuous. on a disc around (a,b), and suppose $f_x(a,b) = 0 = f_y(a,b)$.

Then if $\det \underset{\sim}{H}f = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

i) if $\det \underset{\sim}{H}f > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum

ii) if $\det \underset{\sim}{H}f > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum

iii) if $\det \underset{\sim}{H}f < 0$ then $f(a,b)$ is neither a local max or min.

In this case, $f(a,b)$ is called a <u>saddle point</u>

<u>Example</u>: Let $f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x(x,y) = 4x^3 - 4y$ $\qquad$ $f_y(x,y) = 4y^3 - 4x$

$\Rightarrow$ critical pts at

$x^3 - y = 0$ $\qquad$ $y^3 - x = 0$

Combining this we get

$$x^9 - x = 0 \Rightarrow x(x^8 - 1) = x(x^4+1)(x^4-1)$$
$$= x(x^4+1)(x^2-1)(x^2+1)$$

So the real roots are

$$x = 0, 1, -1$$

all other roots are purely complex.

So the critical points are

$$(0,0) \qquad (1,1) \qquad (-1,-1)$$

$$f_{xx} = 12x^2 \qquad f_{yy} = 12y^2$$

$$f_{xy} = -4 = f_{yx}$$

$$\Rightarrow \det \underset{\sim}{H}f = 144x^2y^2 - 16$$

At $(0,0)$, $\det \underset{\sim}{H}f(0,0) = -16 \Rightarrow f(0,0)$ saddle point.

At $(1,1)$, $\det \underset{\sim}{H}f(1,1) = 128 > 0$, $f_{xx}(1,1) = 12 > 0$
so $f(1,1)$ is a local minimum.

At $(-1,-1)$, $\det \underset{\sim}{H}f(-1,-1) = 128 > 0$, $f_{xx}(-1,-1) = 12 > 0$
so $f(1,1)$ is a local minimum.

So far we have discussed local extrema. Can we say anything about global extrema?

<u>Theorem</u>: Let $f: D \xrightarrow{} \mathbb{R}$, $D \subseteq \mathbb{R}^n$, be continuous. If D is compact (closed and bounded) then f attains a maximum and minimum value on D.

Extrema under constraints:

Suppose we have two functions $f: D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^3$, and $g: D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^3$. We can find the <u>maximum of f along g</u> via the method of Lagrange multipliers.

<u>Def/Outline</u>: To find the max/min of $f(x,y,z)$ along $g(x,y,z) = k$ (it is necessary that $\nabla g \neq \underline{0}$ on this level set)

(a) Find all values of x, y, z, λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and

$$g(x,y,z) = k.$$

(b) Evaluate f at the pts from (a).

Example: A rectangular box with no lid is made of $12m^2$ of cardboard. Find the max. volume.

$$V = \overset{\text{length}}{x}\;\underset{\text{width}}{y}\;\overset{\text{Height}}{z}$$

subject to

$$g(x,y,z) = 2xz + 2yz + xy = 12.$$

Lagrange $\Rightarrow$

i) $V_x = \lambda g_x$

ii) $V_y = \lambda g_y$ $\qquad\qquad g(x,y,z) = 12.$

iii) $V_z = \lambda g_z$

$$\rightsquigarrow \quad yz = \lambda(2z + y)$$

$$xz = \lambda(2z + x)$$

$$xy = \lambda(2x + 2y)$$

Some trick: multiply each equation by the remaining variable.

$$\left.\begin{aligned} xyz &= \lambda(2xz + xy) \\ xyz &= \lambda(2yz + xy) \end{aligned}\right\} \Rightarrow x = y$$

$$xyz = \lambda(2xz + 2yz) \longrightarrow y = 2z$$

$x = y = 2z$

So

$4z^2 + 4z^2 + 4z^2 = 12$

$\Rightarrow z^2 = 1 \rightarrow x = y = 2.$

Another example:

Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $g(x,y) = x^2+y^2 = 1$.

$f_x = 2x \qquad\qquad g_x = 2x$

$f_y = 4y \qquad\qquad g_y = 2y$

Lagrange $\Rightarrow$

$$2x = \lambda \cdot 2x \qquad (1)$$

$$4y = \lambda 2y \qquad (2)$$

$$x^2+y^2 = 1 \qquad (3)$$

(1) $\Rightarrow$ $x=0$ <u>or</u> $\lambda = 1$

(a) if $x=0$, then $\lambda = 2$ and $y = \pm 1$ (by (3))

(b) if $\lambda = 1$, then $y=0$ and $x = \pm 1$ (by (3))

So the critical points are

$(0,1) \quad (0,-1) \qquad (1,0) \quad (-1,0)$

$f(0,\pm 1) = 2 \qquad\qquad f(\pm 1, 0) = 1$

max $\qquad\qquad$ min

Section 3.3: Vector Valued functions, Parametrization, and Directional derivatives.

So far, we have generalized single variable calculus by enlarging/changing the domain of the function. What happens if we change the co-domain of f? That is, how do functions

$$f: \mathbb{R} \longrightarrow \mathbb{R}^m$$

behave? To best understand these functions, we first consider an important class of projection operators:

<u>Def</u>: Define $\pi_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ by

$$\underset{\sim}{x} = (x_1, \ldots, x_i, \ldots, x_n) \longmapsto x_i$$

these are the projection operators.

<u>Fact</u>: i) $\pi_i(\underset{\sim}{x} + \underset{\sim}{y}) = \pi_i(\underset{\sim}{x}) + \pi_i(\underset{\sim}{y})$. (Linearity)

ii) They are differentiable

Def: A vector valued function is any function $f : \mathbb{R} \longrightarrow \mathbb{R}^m$. The choice of $\mathbb{R}$ here is arbitrary. We could have equally investigated/defined vector valued functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

We say a vector valued function is <u>Continuous</u> if

$$\pi_i \circ f \text{ is continuous}$$

for all $1 \leq i \leq m$. Notice that this amounts to a function

$$\pi_i \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

and thus continuity is defined as before. As a result of these operators

$$f(x_1, \dots, x_n) = \langle \pi_1 \circ f(x_1, \dots, x_n), \dots, \pi_m \circ f(x_1, \dots, x_n) \rangle$$

$$:= \langle f_1, \dots, f_m \rangle$$

We call f_i the component function of f. Additionally f is differentiable if

$$f_i \text{ is differentiable}$$

for all $1 \leq i \leq m$.

We shall break down our study of vector valued functions into different cases depending on the values m, n

---

$\boxed{m=1}$ This is the case of single variable calculus and functions of many variables as in the earlier parts of Chapter 3.

$\boxed{m=2,\ n=1}$ A function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is called a plane curve. We denote

$$f(t) = \langle x(t), y(t) \rangle$$

In this regime, we can compute a few different quantities:

i) Arc - Length : Let $[a,b] \subseteq \mathbb{R}$ be a closed interval. Then the arc-length of f on $[a,b]$ is

$$s(b) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

If we fix a, then on (a,∞) we define a function

$$s(t) = \int_a^t \| f'(t) \|\, dt$$

where

ii) $f'(t) = \langle x'(t), y'(t) \rangle$

Note that for this to be well-defined f needs to be smooth/differentiable.

iii) $\int f(t) = \left\langle \int x(t), \int y(t) \right\rangle$

So operations are generally taken component-wise.

Ex: $f(t) = \langle 2\sin(t), t \rangle$. What is the graph of f?

Using the function $s(t)$, we can re-parametrize f in terms of s. This is called parametrizing f by arc-length.

Using $s(t) = \int_a^t \| f'(t) \| \, dt$

we will get a relationship between s and t and thus can substitutes

$$f(t) \rightsquigarrow f(t(s))$$

⊗ See 63 for an example.

$\boxed{m=3, n=1}$ In this case, $r : \mathbb{R} \longrightarrow \mathbb{R}^3$ is called a space curve.

We write

$$r(t) = \langle x(t), y(t), z(t) \rangle$$

and define the arc-length similarly

$$s(b) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

In this case, we can define a series of other curves from $r(t)$. These go by the name T, N, B.

$\boxed{T}$ The <u>tangent curve</u> to $r(t)$ is $r'(t)$. In particular, the <u>unit tangent curve/vector</u> is

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

<u>Def</u>: The <u>curvature</u> of r at t is

$$\kappa = \left\| \frac{dT}{ds} \right\| = \left\| \frac{dT}{dt} \cdot \frac{dt}{ds} \right\|$$

$$= \left\| \frac{dT/dt}{ds/dt} \right\| \quad \text{via chain rule.}$$

$$\frac{ds}{dt} = \frac{d}{dt} \int_a^t \|r'(t)\| \, dt = \|r'(t)\|.$$

So

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|}.$$

<u>Example</u>: $r(t) = \langle \cos(t), \sin(t), t \rangle$ this is a helix.

Let's compute $\kappa(t)$.

$$T(t) = \frac{\langle -\sin(t), \cos(t), 1 \rangle}{\sqrt{\cos^2(t) + \sin^2(t) + 1}} = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$T'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\Rightarrow \|T'(t)\| = \frac{1}{\sqrt{2}}$$

$$\|r'(t)\| = \sqrt{2}$$

$$\Rightarrow \kappa(t) = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$

In particular, this says that a helix has constant curvature $1/2$.

<u>Question</u>: Is $T(t)$ actually perpendicular to $r(t)$?

Lemma: If $|r(t)| = c$ constant then $r(t)$ is perpendicular to $r'(t)$ for all $t \in \mathbb{R}$.

Proof: $r(t) \cdot r(t) = c^2 = |r(t)|^2$.

Using the facts that

$$0 = \frac{d}{dt}[r(t) \cdot r(t)] = r'(t) \cdot r(t) + r(t) \cdot r'(t)$$

$\Rightarrow r'(t) \cdot r(t) = 0.$ ∎

Corollary: T is not perpendicular to r, but T is perpendicular to T'.

Example: Finding an arc-length parametrization:

$$r(t) = \langle \cos(t), \sin(t), t \rangle$$

Starting point: $a = (1,0,0) \Leftrightarrow t = 0$.

$$s(t) = \int_0^t \|r'(u)\| du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t$$

$\Rightarrow t = \frac{s}{\sqrt{2}}$

$$r(t(s)) = \left\langle \cos\left(s/\sqrt{2}\right), \sin\left(s/\sqrt{2}\right), s/\sqrt{2} \right\rangle$$

Using the corollary, we now have motivation for

Def: The unit normal vector/curve

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

By the corollary $N(t) \perp T(t)$ for all $t \in \mathbb{R}$.

The binormal vector/curve is

$$B(t) = T(t) \times N(t)$$

By the property of the cross product

$$B(t) \perp T(t) \perp N(t)$$

Example: Compute $B(t)$ and $N(t)$ for the helix.

$r(t) = \langle \cos t, \sin t, t \rangle \Rightarrow T(t) = \langle -\sin t, \cos t, 1 \rangle \cdot \frac{1}{\sqrt{2}}$

$T'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle \Rightarrow N(t) = \langle -\cos t, -\sin t, 0 \rangle$

$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$

We now have 3 vectors in $\mathbb{R}^3$. These define two different planes that are of interest:

<u>Normal Plane</u>: Let $X_p = \{tN + sB : t, s \in \mathbb{R}\}$ where $p = r(t_0)$ for some $t_0 \in \mathbb{R}$.

This is given by "the set of all vectors at p perpendicular to $T(t_0)$"

In particular, if $T(t_0) = \langle a, b, c \rangle$ then

$$X_p : ax + by + cz = d$$

where $d = ap_1 + bp_2 + cp_3$.

<u>Osculating Plane</u>: Let $O_p = \{tN + sT : s, t \in \mathbb{R}\}$ where $p = r(t_0)$ for some $t_0 \in \mathbb{R}$.

"Think about this as a tangent plane for curves"

<u>Caution</u>: a curve has a 1-dim tangent space, so this is not techically true

This plane is given by B.

Application: Motion in Space

Let $\underset{\sim}{r}(t)$ be the position of a particle moving through space.

$$\underset{\sim}{r}'(t) = \underset{\sim}{v}(t) := \text{velocity}$$

$$\underset{\sim}{r}''(t) = \underset{\sim}{a}(t) := \text{acceleration}$$

Newton's Second Law:

Let the particle have mass m.
Then, the force of a particle is

$$F(t) = m\,\underset{\sim}{a}(t).$$

Understanding $\underset{\sim}{a}(t)$:

$$T(t) = \frac{\underset{\sim}{v}(t)}{\|\underset{\sim}{v}(t)\|} \Rightarrow \underset{\sim}{v}(t) = \|\underset{\sim}{v}(t)\|\, T$$

So, by the product rule

$$\underset{\sim}{v}'(t) = \underset{\sim}{a}(t) = \|\underset{\sim}{v}(t)\|'\, T + \|\underset{\sim}{v}(t)\|\, T'$$
$$= v'(t)\, T + \kappa v^2(t)\, N$$

To compute these coefficients, notice that

$$\|\underset{\sim}{v}(t)\|' = v'(t) = \underset{\sim}{a}(t) \cdot T(t)$$

and

$$\kappa v^2(t) = \underset{\sim}{a}(t) \cdot N(t).$$

However, these definitions are computationally intensive. To fix this, we want to give κ in terms of only r and its derivatives!

<u>Lemma:</u> $$\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

<u>Proof:</u> Lets drop the dependence on t momentarily. Then as $T = \frac{r'}{\|r'\|}$ and $\|r'\| = \frac{ds}{dt}$ we have

$$r' = \|r'\| T = \frac{ds}{dt} T$$

Applying the product rule, we get

$$r'' = \frac{d^2s}{dt^2} T + \frac{ds}{dt} \cdot T'$$

Now

$$r' \times r'' = \left(\frac{ds}{dt}\right)^2 (T \times T')$$

Therefore by orthogonality,

$$\|r' \times r''\| = \left(\frac{ds}{dt}\right)^2 \|T\| \cdot \|T'\| = \left(\frac{ds}{dt}\right)^2 \|T'\|$$

Therefore,

$$\|T'\| = \frac{\|r' \times r''\|}{\left(\frac{ds}{dt}\right)^2}$$

$$= \frac{\|r' \times r''\|}{\|r'\|^2} \qquad \blacksquare$$

Now

$$v'(t) = \frac{\underset{\sim}{v}(t) \cdot \underset{\sim}{a}(t)}{v(t)} = \frac{r'(t) \cdot r''(t)}{\|r'(t)\|}$$

$$\kappa(t)\, v^2(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} \cdot \|r'(t)\|^2 = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|}$$

Example: $r(t) = \langle \cos t, \sin t, t \rangle$

- $\underset{\sim}{v}(t) = \langle -\sin t, \cos t, 1 \rangle = r'(t)$

  $\underset{\sim}{a}(t) = \langle -\cos t, -\sin t, 0 \rangle = r''(t)$

$$a_T = \frac{r'(t) \cdot r''(t)}{\|r'(t)\|} = 0$$

$$a_N = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

- $$\begin{array}{r} \langle -\sin t, \cos t, 1 \rangle \\ \times \langle -\cos t, -\sin t, 0 \rangle \end{array} = \langle \sin t, -\cos t, 1 \rangle$$

Parametric Surfaces:

- Now we move on to the case

$\boxed{m=3,\ n=2}$

In this case

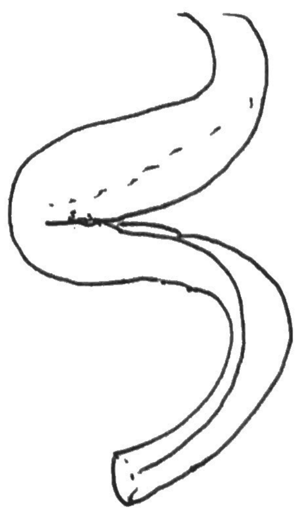
$$\underset{\sim}{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v) \rangle$$

where $(u,v) \in D \subseteq \mathbb{R}^2$.

To understand such objects, we consider "grid curves" (aka. u,v-slices)

Example: $r(u,v) = \langle (2+\sin v)\cos u,\ (2+\sin v)\sin u,\ u+\cos v \rangle$

on the region $0 \le u \le 4\pi$, $0 \le v \le 2\pi$.



Each v-slice is a helix

and each u-slice is a circle-ish.

One important class of parametrized surfaces:

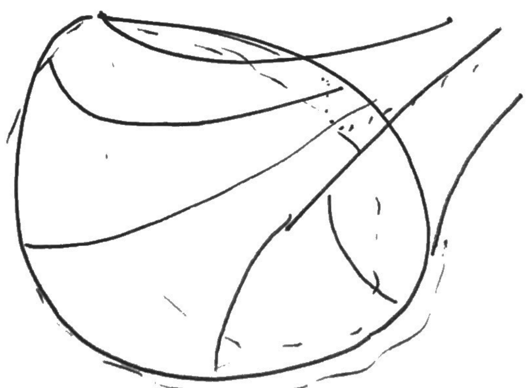
Surfaces of Revolution:

Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function.

Then define

S_f to be the surface with the following coordinates.

$$S_f = r(x,\theta)=\langle x, f(x)\cos\theta, f(x)\sin\theta\rangle.$$



a trumpet!

Directional Derivatives:

We will now rephrase some constructions in the previous sections in terms of vector valued functions. In particular, we can now define directional derivatives! These are a generalization of partial derivatives in some sense.

Def: Let $n \geq 1$. Define

$$C^{\alpha}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \longrightarrow \mathbb{R} \ : \ \partial^{\beta} f \text{ exists and is continuous} \right\}$$

for $\alpha \in \mathbb{N} \cup \{0\}$, where $\partial^{\beta} f$ is defined as $\frac{\partial^{\beta_1}}{(\partial x_1)^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{(\partial x_n)^{\beta_n}} f$ with $\sum \beta_i = \alpha$.

Example: $\alpha = 1$, $C^1(\mathbb{R}^n)$ consists of all functions $\mathbb{R}^n \to \mathbb{R}$ which have continuous first order partial derivatives.

Question: Why do we care?

Answer: We want to understand the domain of the operator (don't think about the definition of this just yet),

$$\text{grad } f = \nabla f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle .$$

For grad( ) to be defined, we need for $\frac{\partial f}{\partial x_i}$ to be defined for all $1 \le i \le n$. Therefore,

grad(h) is defined for all $h \in C^1(\mathbb{R}^n)$.

Let us understand grad and $C^\alpha(\mathbb{R}^n)$ better.

Theorem: $C^\alpha(\mathbb{R}^n)$ is closed under pointwise addition and $\mathbb{R}$-multiplication. This makes $C^\alpha(\mathbb{R}^n)$ a vector space over $\mathbb{R}$. Further grad is linear with respect to this addition.

<u>Proof</u>: Let $f, g \in C^{\alpha}(\mathbb{R}^n)$. Put

$$f(\underset{\sim}{x}) + g(\underset{\sim}{x}) = (f+g)(\underset{\sim}{x})$$

and

$$(f \cdot g)(\underset{\sim}{x}) = f(\underset{\sim}{x}) \cdot g(\underset{\sim}{x})$$

Then $f+g, f\cdot g \in C^{\alpha}(\mathbb{R}^n)$ as derivatives are linear and the product rule holds. Clearly, $\lambda f \in C^{\alpha}(\mathbb{R}^n)$

Now,

$$\begin{aligned} \operatorname{grad}(f+g) &= \langle f_{x_1} + g_{x_1}, \ldots, f_{x_n} + g_{x_n} \rangle \\ &= \langle f_{x_1}, \ldots, f_{x_n} \rangle + \langle g_{x_1}, \ldots, g_{x_n} \rangle \\ &= \operatorname{grad} f + \operatorname{grad} g. \end{aligned}$$

■

Now, what is the codomain?

* This will not be tested

$$\operatorname{grad} f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Denote by $C^{\alpha}(\mathbb{R}^n, \mathbb{R}^m)$ the set of functions $\mathbb{R}^n \to \mathbb{R}^m$ which are α-differentiable.

Thus,

$$\mathrm{grad} : C^{\alpha}(\mathbb{R}^n) \longrightarrow C^{\alpha-1}(\mathbb{R}^n, \mathbb{R}^n)$$

In particular,

$$\mathrm{grad} : C^{1}(\mathbb{R}^n) \longrightarrow C^{0}(\mathbb{R}^n, \mathbb{R}^n)$$

We can extend the domain of grad to $\alpha = \infty$. In this case

$$\mathrm{grad} : C^{\infty}(\mathbb{R}^n) \longrightarrow C^{\infty}(\mathbb{R}^n, \mathbb{R}^n).$$

Now that we have this, let us understand how grad moves in each coordinate direction. That is,

$$(\mathrm{grad}\ f) \cdot \underbrace{\langle 1, \dots, 0 \rangle}_{e_1} = \frac{\partial f}{\partial x_1}$$

Def: ~~The~~ Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^1 function. For $u \in \mathbb{R}^n$, ~~we~~ we denote the Directional derivative of f in the direction of u is

$$d\cancel{f}f_u(p) = (\mathrm{grad}\, f)(p) \cdot u = Df_p(u).$$

Question: When is $df_u(p)$ maximized?

Answer:

Theorem: $df_u(p)$ is maximized when grad f and u point in the same direction.

Proof: $|df_u(p)| = \|\text{grad } f\| \cdot \|u\| \cos\theta$.

As they point in the same direction, $\theta = 0, \pi$

Hence, $|df_u(p)| = \|\text{grad } f\| \cdot \|u\|$.

$\blacksquare$

Now, this motivates another definition of the tangent plane to a surface.

Let $f(x,y)$ be a function. Then for $(x, y, f(x,y)) \in \mathbb{R}^3$, we can find the equation for the tangent plane to be

$$(\star) \quad (z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

As a generalization of linear approximations, we see that for (x,y) sufficiently close to (x_0,y_0),

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

Example: Estimate xe^{xy} near $(1.1,-0.1)$.

In a neighbourhood of $(1,0)$,

$$f(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)\,y$$

$$= 1 + (x-1) + y = x+y.$$

Thus, $f(1.1,-.1) \approx 1$.

Actual value: $.9854$

Special Case: Tangent plane to a Parametric surface, $\underset{\sim}{r}(u,v)$.

$$\underset{\sim}{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$\underset{\sim}{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

Then if $\underset{\sim}{r}_u \times \underset{\sim}{r}_v \neq 0$, then S is smooth and the tangent plane is given by $\underset{\sim}{r}_u \times \underset{\sim}{r}_v$ as the normal vector.

Note that $\underset{\sim}{r}_u \times \underset{\sim}{r}_v = 0 \Leftrightarrow \underset{\sim}{r}_u$ points in the same direction as $\underset{\sim}{r}_v$.

(*) See pg. 82

The final topic for differentiation is the Chain Rule!

To set this up, we first take two parametrizations $x(t), y(t)$ for the variables in $\mathbb{R}^2$. In reality this is a single parametrization of a plane curve. The question we want to answer is: if f is differentiable, what is

$$\frac{df}{dt} \text{ in terms of } \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{dx}{dt}, \frac{dy}{dt}?$$

Theorem: (Chain Rule in $\mathbb{R}^2$)

Let $r(t) = \langle x(t), y(t) \rangle$ be a plane curve. If $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable, then

$$\frac{d}{dt}(f \circ r) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Idea of proof: Changes in t change both x any y. So derivatives vary in both directions.

Example: Let $r(t) = \langle a\cos t, a\sin t \rangle$ and $f(x,y) = e^{x^2+y^2}$. Then $f \circ r(t) = e^{a^2}$. So $\frac{d}{dt}(f \circ r) = 0$.

But,

$$\frac{d}{dt}(f \circ r) = 2x e^{x^2+y^2}(-a\sin t) + 2y\, e^{x^2+y^2}\, a\cos t.$$

$$= 2(a\cos t)e^{a^2}(-a\sin t) + 2(a\sin t)e^{a^2} a\cos t$$

$$= 0.$$

Theorem: (Chain Rule in $\mathbb{R}^n$)

- Let $r(t) = \langle x_1(t), \dots, x_n(t) \rangle$ be a curve in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ differentiable (C^1). Then

$$\frac{d}{dt}(f \circ r) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}$$

Question: What happens if instead of $r : \mathbb{R} \to \mathbb{R}^n$, we have $r : \mathbb{R}^2 \to \mathbb{R}^n$ or $r : \mathbb{R}^3 \to \mathbb{R}^n$.

- **Answer**:

Theorem: (Chain Rule, General)
Let $r : \mathbb{R}^m \to \mathbb{R}^n$ differentiable and $f : \mathbb{R}^n \to \mathbb{R}$ differentiable. Then

$$\frac{\partial}{\partial t_i}(f \circ r) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial t_i}$$

for $1 \le i \le m$.

- **Fun Fact**: This theorem fails in infinite dimensional spaces.

Implicit differentiation:

- Suppose we have a function $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $F(x,y)=0$ defines y in terms of x (i.e. is an implicit function). Then by the Chain Rule,

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0$$

As $\frac{\partial x}{\partial x}=1$, we see that

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

- For $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $z=f(x,y)$ such that $F(x,y,f(x,y))=0$, we have analogous results:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

For general functions, we have a similar result called the

- Implicit Function Theorem

Example: Let $x^3 + y^3 = 6xy$. Find y'.

$$F(x,y) = x^3 + y^3 - 6xy = 0.$$

Then

$$y' = \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

(*) The Jacobian Revisited:

$\boxed{\text{This will be tested!!}}$

Recall that for $f : \mathbb{R}^n \to \mathbb{R}$ we defined

$$Df_p = \left[f_{x_1}(p) \ \cdots \ f_{x_n}(p) \right].$$

To fully understand what this object is doing, we need to talk about two operations from Linear Algebra:

i) Matrix Multiplication

ii) Determinants.

(i) Matrices and their operations:

Def: A <u>matrix</u> is an array of real numbers

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = (a_{ij})_{1 \le i,j \le n}$$

← Square matrix

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

← Rectangular matrix/ non-square/ <u>$m \times n$-matrix</u>

The collection of <u>all matrices</u> of a given size is denoted

$$M_{mn}(\mathbb{R}).$$

<u>Theorem</u>: There exists an addition and $\mathbb{R}$-multiplication making $M_{mn}(\mathbb{R})$ an $\mathbb{R}$-vector space.
If $m=n$, there exists an additional operation called matrix multiplication which makes

$$M_n(\mathbb{R}) := M_{nn}(\mathbb{R}) \quad \text{an } \mathbb{R}\text{-algebra}.$$

Proof / Construction:

- Let $A, B \in M_{mn}(\mathbb{R})$. Then

$$A+B = (a_{ij}+b_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} = \begin{bmatrix} a_{11}+b_{11} & \cdots\cdots & a_{1n}+b_{1n} \\ \vdots & & \\ a_{m1}+b_{m1} & \cdots\cdots & a_{mn}+b_{mn} \end{bmatrix} \in M_{mn}(\mathbb{R}).$$

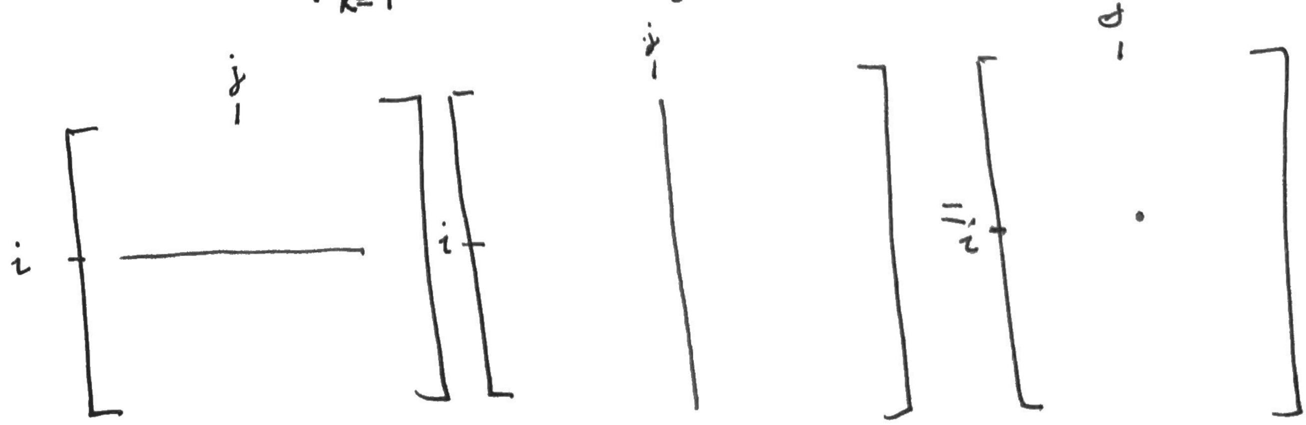
For $\lambda \in \mathbb{R}$,

$$\lambda A = \begin{bmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix} \in M_{mn}(\mathbb{R}).$$

- $\Rightarrow$ $\mathbb{R}$-vector space.

If $m=n$, define

$$AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{1\le i,j\le n}$$



$\in M_{nn}(\mathbb{R}) = M_n(\mathbb{R})$.

- $\Rightarrow$ $\mathbb{R}$-algebra.

Example:

- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

In general, $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in M_n(\mathbb{R})$ is called the identity matrix.

We can use this definition of matrix multiplication to define it not only for square matrices!

If $A \in M_{mn}(\mathbb{R})$ and $B \in M_{nk}(\mathbb{R})$ then

$$AB \in M_{mk}(\mathbb{R})$$ (we get rid of the middle index)

In particular, if $A \in M_{mn}(\mathbb{R})$ and $v \in \mathbb{R}^n = M_{n1}(\mathbb{R})$, we have that

$$Av \in M_{m1}(\mathbb{R}) = \mathbb{R}^m.$$

It can be shown that for $v, w \in \mathbb{R}^n$,

$$A(v+w) = Av + Aw.$$

In particular, <u>all</u> matrices are linear transformations.

Lets see how this applies to the Jacobian!

If $f: \mathbb{R}^m \to \mathbb{R}^m$ or $f: \overset{\subseteq \mathbb{R}^m}{D} \to \mathbb{R}^m$ then

<u>Def:</u> The Jacobian of f is

$$Df_p = \left(\frac{\partial f_i}{\partial x_j}(p) \right)_{\substack{1 \le i \le n \\ 1 \le j \le m}} \in M_{mn}(\mathbb{R})$$

For a vector $v \in T_pD = \mathbb{R}^n$ (this is the <u>tangent space</u> of D at p)

we get $Df_p(v) = (Df_p)v$ as matrices.

In particular, $(Df_p)v \in \mathbb{R}^m = T_{f(p)}\mathbb{R}^m$.

So Jacobians carry tangent spaces to tangent spaces. This resolves the question of how Df plays the role of "the" derivative of f.

(ii) Determinants

- For a square matrix $A \in M_n(\mathbb{R})$, we can assign a unique(!) number called the determinant.

Def: The <u>determinant</u> of A is

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} A_{1i} \det(A[1,j])$$

where $A[i,j]$ is the matrix obtained by deleting the i-th row and j-th column from A. Thus,

- if $A \in M_2(\mathbb{R})$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \det A = ad - bc.$$

<u>Example:</u> $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\det A = 1 \cdot 1 - (-1 \cdot 1) = 2$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det B = 1(1-0) + 0 \cdot (0 \cdot 1 - 1 \cdot 1) + 1(0-1)$$
$$= 0.$$

Theorem: If $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, then

$$|\det A| = \mathrm{Vol}(Ae_1, \dots, Ae_n).$$

This tells us that the determinant determines a geometric quantity! This will play a large factor when we start integration.

Chapter 4: Integration

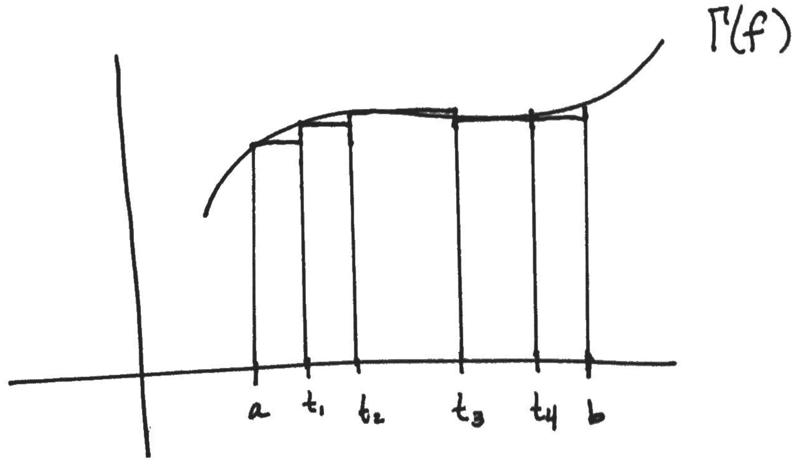
Section 1: Riemann/Darboux Theory of integration.

Recall from single variable calculus that if f is continuous on $[a,b]$ then

$$\int_a^b f\,dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(t_i^*)(t_{i+1}-t_i)$$

where $\{t_0, \dots, t_{n+1}\}$ is a partition of $[a,b]$ and $t_i^* \in [t_i, t_{i+1}]$.

Pictorally:



We can generalize this to $\mathbb{R}^n$ for all $n < \infty$. For the purposes of this class, $n = 2, 3$ will be used exclusively.

If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then the "good guess" for the integral of f, over $[a,b]\times[c,d]$ is

$$\lim_{m,n\to\infty} \sum_{i=0}^{m} \sum_{j=0}^{n} f(s_i^*, t_j^*)(s_{i+1} - s_i)(t_{i+1} - t_i)$$

If this limit exists, we say f is <u>integrable</u> on $R = [a,b] \times [c,d]$.

We denote this quantity

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

In particular, if $D \subseteq \mathbb{R}^2$ is a region, then

$\iint_D f(x,y)\,dA$ can be defined as a limit of integrals over rectangles.

⊛ Draw a picture.

Explain π.

<u>Claim:</u> Integration is a linear function

$$\iint : C^0(\mathbb{R}^n) \longrightarrow \mathbb{R}.$$

<u>Proof Sketch:</u> Notice that

$$\iint_R f+g \, dA = \iint_R f + \iint_R g$$

as the finite sums split across sums.

Further for $\lambda \in \mathbb{R}$,

$$\iint_R \lambda f \, dA = \lambda \iint_R f \, dA$$

for the same reason. ∎

<u>Some fun facts:</u>

(1) If $f(x,y) \geq 0$ then the volume under $\Gamma(f)$ in $\mathbb{R}^3$ is given by

$$V = \iint_R f(x,y) \, dx dy.$$

(2) The average value of a single variable function is

$$f_{av} = \frac{1}{b-a} \int_a^b f \, dx$$

For multivariable functions we generalize to

$$f_{ave} = \frac{1}{A(D)} \iint_D f \, dA .$$

where $A(D)$ is the area.

(3) Monotonicity: If $g \leq f$ for all $\underset{\sim}{x} \in D$ then

$$\iint_D g \, dA \leq \iint_D f \, dA .$$

Question: How do we evaluate a double (or iterated) integral?

Answer: Work inside to out.

If $R = [a,b] \times [c,d]$ is a rectangle then

- $$\iint_R f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

We could have chosen the other order to integrate in however! So, is

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \stackrel{?}{=} \int_a^b \int_c^d f(x,y)\,dy\,dx$$

<u>Theorem (Fubini-Tonelli)</u>:

- If f is continuous and R is a rectangle, then

$$\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy. \quad \smile$$

<u>Example</u>: Evaluate $\iint_R x - 3y^2\,dA$ with $R = [0,2]\times[1,2]$ using Fubini's theorem.

$$\int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy = \int_1^2 \frac{x^2}{2} - 3xy^2 \Big|_0^2 \,dy$$

$$= \int_1^2 2 - 6y^2\,dy$$

$$= 2y - 2y^3 \Big|_1^2 = -12 - 0 = -12$$

Example 2: Diffenent orders of integration can be significantly mone difficult to compute.

Consider $\iint_R y\sin(xy)\,dA$, $R=[1,2]\times[0,\pi]$

Way 1:

$$\int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy = \int_0^{\pi} y\cdot\frac{1}{y}\left[-\cos(xy)\Big|_1^2\right]dy$$

$$= \int_0^{\pi} \cos y - \cos 2y \; dy$$

$$= \sin y - \frac{1}{2}\sin(2y)\Big|_0^{\pi}$$

$$= 0$$

Way 2: $\int_1^2\int_0^{\pi} y\sin(xy)\,dy\,dx =$

Inner Integral: Integration by Parts

$u = y$ $\qquad dv = \sin(xy)\,dy$

$du = dy$ $\qquad v = \dfrac{-\cos(xy)}{x}$

So $\displaystyle\int_0^{\pi} y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Big|_0^{\pi} + \frac{1}{x}\int_0^{\pi}\cos(xy)\,dy$

$$= \frac{-\pi\cos(\pi x)}{x} + \frac{0}{x} + \left[\frac{1}{x^2}\sin(xy)\right]_0^{\pi}$$

$$= \frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}$$

So $$\int_0^{\pi} y\sin(xy) = \frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}$$

Now we need to integrate this with respect to x.

Notice that

$$\int \frac{-\pi\cos(\pi x)}{x} = -\frac{\sin(\pi x)}{x} - \int \frac{\sin(\pi x)}{x^2}\,dx$$

Thus,

$$\int\left(\frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\right) = \frac{-\sin(\pi x)}{x} - \int\frac{\sin(\pi x)}{x^2} + \int\frac{\sin(\pi x)}{x^2}$$

Hence,

$$\int_1^2\int_0^{\pi} y\sin(xy)\,dy\,dx = \frac{-\sin(2\pi)}{2} + \sin(\pi) = 0.$$

Example 3: Computing volumes / interesting domains.

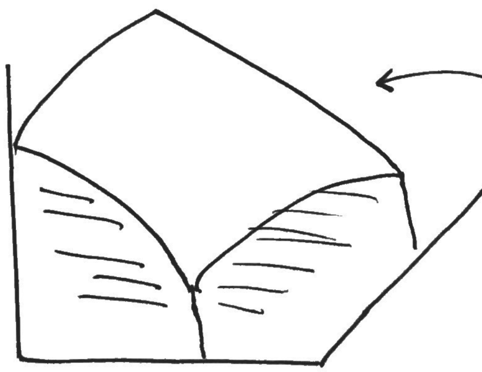
- Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$, $y=2$, and the three coordinate planes.

Step 1: Set-up

Ask: What is this region? In $\mathbb{R}^2$? In $\mathbb{R}^3$?

$$R = [0,2] \times [0,2]$$

and we are looking for the volume



$$\underset{f(x,y)}{\underset{\parallel}{z}} = 16 - x^2 - 2y^2$$

Step 2: Integrate!

$$\int_0^2 \int_0^2 16 - x^2 - 2y^2 \, dx\, dy = \int_0^2 16x - \tfrac{1}{3}x^3 - 2xy^2 \Big|_0^2 \, dy$$

$$= \int_0^2 32 - \tfrac{8}{3} - 4y^2 = 32y - \tfrac{8}{3}y - \tfrac{4}{3}y^3 \Big|_0^2$$

$$= 64 - \tfrac{16}{3} - \tfrac{32}{3} = 64 - \tfrac{32}{3}.$$

Section 4.2: Integration over general regions and changing coordinates.

Def: (Regions of type I, II)

Let $D \subseteq \mathbb{R}^2$ be a region. We say that D is of type I if

$$D = \{(x,y) \mid a \leq x \leq b,\ f(x) \leq y \leq g(x)\}$$

D is of type II if

$$D = \{(x,y) \mid p(y) \leq x \leq q(y),\ c \leq y \leq d\}$$

Theorem/Definition: If $f : D \to \mathbb{R}$ is continuous and

i) D is of type I then

$$\iint_D f(x,y)\, dA = \int_a^b \int_{f(x)}^{g(x)} f(x,y)\, dy\, dx$$

ii) D is of type II then

$$\iint_D f(x,y)\, dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\, dx\, dy$$

Example 1: Evaluate $\iint_D (x+2y)\, dA$ over the region D bounded by $y = 2x^2$ and $y = 1+x^2$.

Step 1: Identify the type of the region.

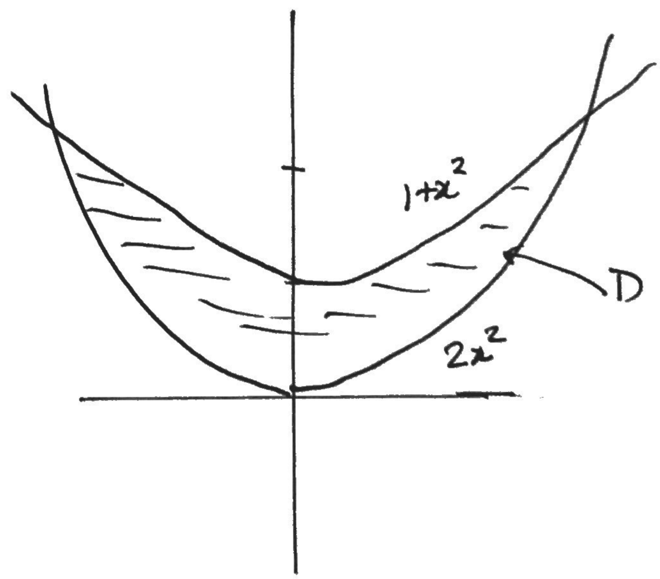
As y is given as functions of x, this is a type I region.

Step 2: Determine the range of x (resp. y).

$$2x^2 = 1+x^2 \iff x^2 = 1 \iff x = \pm 1$$

@ $x = 1$ $\quad y = 2$

@ $x = -1$ $\quad y = 2$



Step 3: Integrate

$$\int_{-1}^{1} \int_{2x^2}^{1+x^2} x+2y \, dy\, dx = \int_{-1}^{1} \left[xy + y^2 \right]_{2x^2}^{1+x^2} dx = \longrightarrow$$

$$= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2 \right] dx$$

$$= \int_{-1}^{1} x(1-x^2) + x^4 + 2x^2 + 1 - 4x^4 \; dx$$

$$= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1 \; dx$$

$$= \left[\frac{-3}{5}x^5 - \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{x^2}{2} + x \right]_{-1}^{1}$$

$$= \frac{32}{15}$$

Example 2: Compute $\iint_D x^2+y^2 \, dA$ over the region D bounded by the line $y=2x$ and $y=x^2$.

Step 1: D is a region of type I and II.

Step 2: Range of $x = (0, 2)$.

Step 3: $\int_0^2 \int_{x^2}^{2x} x^2+y^2 \, dy \, dx = \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^{2x} dx$

$$= \int_0^2 \left[2x^3 - x^4 + \frac{8}{3}x^3 - \frac{x^6}{3} \right] dx = \frac{x^4}{2} - \frac{x^5}{5} + \frac{2}{3}x^4 - \frac{x^7}{21} \Big|_0^2$$

$$= \frac{216}{35}$$

(*) See 107

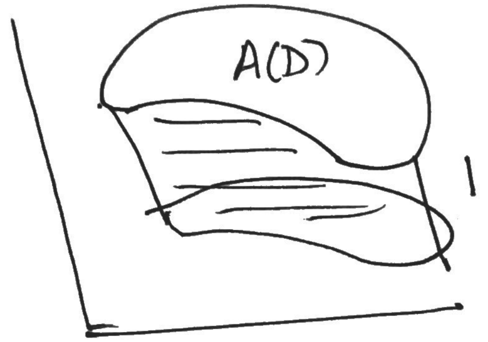
Applications of double integrals:

① Area of a region:

Suppose $D \subseteq \mathbb{R}^2$ is a region. If we denote by $A(D)$ the area, then

$$A(D) = \iint_D 1 \, dA$$

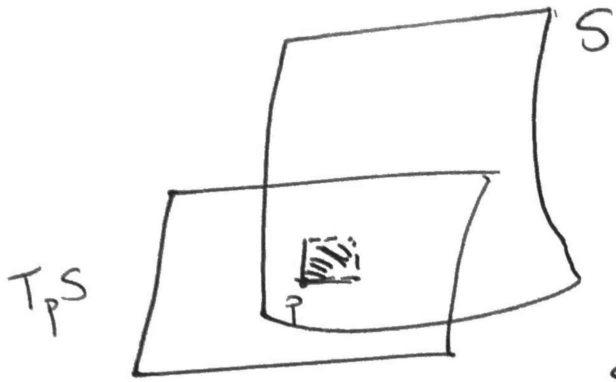
Why?



Ans: Volume of a Cylinder/prism is $A \cdot h$.

② Surface Area of a Parametric Surface.

Let $\underset{\sim}{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ be a parametric surface. For a small rectangular region on S, notice that



the small rectangle is well approximated by the area in T_pS. So area is given by $\|\underset{\sim}{x}_u \times \underset{\sim}{x}_v\|$ and thus.

$$SA(D) = \iint_D \|\underset{\sim}{r}_u \times \underset{\sim}{r}_v\| \, du\,dv$$

③ Surface area of a graph

Using the above we put $z = f(x,y)$

$$A_{\Gamma(f)}(D) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx\,dy$$

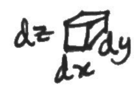
Why? Any graph has a parametrization

$$\Gamma(f) = \langle x, y, f(x,y) \rangle.$$

Triple Integrals:

- Let $\omega = f(x,y,z)$. We can take a so-called <u>triple integral</u> of ω over a solid region $\Omega \subseteq \mathbb{R}^3$.

$$\iiint_\Omega \omega \, dV$$ where dV represents the volume form



dV

If Ω is a rectangle $[a,b]\times[c,d]\times[p,q]$ we have the following version of Fubini.

<u>Theorem (Fubini-Tonelli)</u>: For $f:\Omega \to \mathbb{R}$ continuous. and Ω defined as above:

$$\iiint_\Omega f \, dV = \int_p^q \int_c^d \int_a^b f \; dx\,dy\,dz = \int_a^b \int_c^d \int_p^q f \; dz\,dy\,dx.$$

Computations work inside to out:

Computing the volume of Ω.

Similar to the 2-D case,

$$V(\Omega) = \iiint_{\Omega} 1 \, dV.$$

Changing Coordinates: The Jacobian Revisited:

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map such that $f(u,v) = \langle x(u,v), y(u,v) \rangle$ defines a change of coordinates. Then for a function $g : \mathbb{R}^2 \to \mathbb{R}$

$$\iint_{\Omega} g(x,y) \, dA = \iint_{f(\Omega)} g(x(u,v), y(u,v)) \, |\det Df| \, du dv$$

Where does the $\det Df$ come from?

Recall the theorem

$$|\det A| = \mathrm{Vol}(Ae_1, \dots, Ae_n).$$

In particular, $du dv$ is a volume form on subsets of $\mathbb{R}^2$ generated by linearization in the tangent space.

So a change of coordinates amounts to writing

$$dx\,dy = |\det Df|\,du\,dv$$

which arises from the relations

$$x = x(u,v)$$
$$y = y(u,v).$$

Compare this to the single variable case where u-substitution replaces changing coordinates.

In fact, we can generalize this immediately to $\mathbb{R}^n$. Namely

$$\int \cdots \int_{\Omega} f(x_1, \ldots x_n)\, dV = \int \cdots \int_{f(\Omega)} f(x_1(\ldots), x_2(\ldots), \ldots, x_n(\ldots))|\det Df|\, dV$$

In the cases we care about let's determine what det Df is for spherical and polar coordinates!

<u>Polar</u>: $f(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$

$$Df = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boxed{\text{Det } Df = r}$$

Spherical:

$$f(\rho,\theta,\phi) = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi \rangle$$

$$Df = \begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

$$\Rightarrow \det Df = -\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin^2\theta\sin\phi\cos^2\phi + 0$$
$$- \rho^2\cos^2\theta\sin\phi\cos^2\phi - 0 - \rho^2\sin^2\theta\sin^3\phi$$

$$= -\rho^2\left(\sin^3\phi + \sin\phi\cos^2\phi\right)$$

$$= -\rho^2\sin\phi\left(\sin^2\phi + \cos^2\phi\right)$$

$$= \boxed{-\rho^2\sin\phi}$$

Therefore we get the identities:

- **Polar:**

$$\iiint_{\Omega} f(x,y,z)\, dV = \iiint_{P(\Omega)} f(r,\theta,z)\cdot r \, dV$$

Spherical:

$$\iiint_{\Omega} f(x,y,z)\, dV = \iiint_{S(\Omega)} f(\rho,\theta,\phi)\cdot \rho^2 \sin\phi \, dV$$

where $P(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$

and $S(\rho,\theta,\phi) = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle$

are the change of coordinates.

Examples: ① Integrate x^2+y^2 over the region

$D = \{(x,y) : x^2+y^2 \leq 1\}$

$$\iint_D f(x,y)\, dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2 \, dy\, dx$$

$$= \int_0^{2\pi}\int_0^1 r^2 \cdot r \, dr\, d\theta$$

$$= \frac{\pi}{2}$$

(2) Integrate over a strange region:

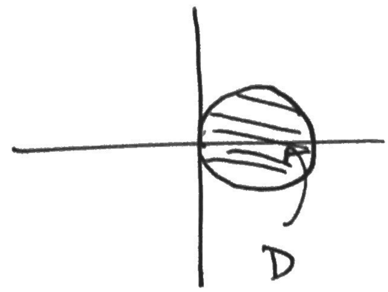
Find the volume of the solid under the paraboloid $z = x^2+y^2$, ~~and~~ above the xy-plane, and inside the cylinder $x^2+y^2 = 2x$.

Step 1:

What is D?

$$x^2+y^2 = 2x \implies r^2 = 2r\cos\theta \iff r = 2\cos\theta$$

$$\Rightarrow D = \{(r,\theta) : \theta \in [\tfrac{-\pi}{2}, \tfrac{\pi}{2}], 0 \le r \le 2\cos\theta\}$$



Step 2:

Thus

$$V = \iint_D f(x,y)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^4(\theta)\,d\theta$$

$$= 8 \int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 8\left[\left(\tfrac{1}{4}\cos^3\theta\sin\theta\right)\Big|_0^{\pi/2} + \tfrac{3}{4}\int_0^{\pi/2} \cos^2\theta\,d\theta\right]$$

$$= 6 \int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 6 \int_0^{\pi/2} \frac{1 + \cancel{2}\cos 2\theta}{2}\,d\theta$$

$$= \frac{3\pi}{2}.$$

⊛ Using Fubini to change boundaries:

- Suppose we are in a region D which is both type I and type II. Fubini–Tonelli implies

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dx\,dy$$

We want to relate p and q to g and h.

<u>Claim:</u> $a = p(c)$, $b = q(d)$, $c = g(a)$, $d = h(b)$

- <u>Pf sketch:</u>

$$D = \{(x,y) : a \le x \le b,\ g(x) \le y \le h(x)\}$$

$$= \{(x,y) : p(y) \le x \le q(y),\ c \le y \le d\}$$

Smallest valued of x is $p(c) = a$ and largest equals $q(d) = b$.

<u>Example:</u> Integrate e^{-x^2} over the triangle $0 \le x \le 1$, $0 \le y \le x$.

- Write out both double integrals!

$$\int_0^1 \int_y^1 e^{-x^2}\,dx\,dy = \int_0^1 \int_0^x e^{-x^2}\,dy\,dx$$

Type II — not solvable by elementary functions

Type I — very easy.

§ 4.3 Vector Calculus, ~~Differentials~~, and Stokes Theorem (three instances)

Now that we can integrate multivariable functions of 2 and 3 variables, we may ask the question "Is it possible to integrate vector valued functions?" At this point in your mathematical career, the answer is no! You need the tools of measure theory to make sense of it all. For this reason we will construct / discover a way to turn a vector-valued function into something integrable.

Def: A vector field on $\mathbb{R}^n$ is a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Notice that for any multivariable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, grad f is a vector field on $\mathbb{R}^n$. In particular, if F is a vector field and

$$F = \text{grad } f \quad \text{for some } f,$$

then F is called a gradient field.

A function $f \in C^2(\mathbb{R}^2)$ gives rise to a vector field $F = \text{grad } f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle P, Q \rangle$ which satisfies

$$Q_x - P_y = 0 \quad !$$

Why? Clairaut!

We want to understand gradient fields and when a given vector field is a gradient field. To do so fully requires a new type of integration. Thus, before we go there we will spend a bit more time defining some quantities associated to a vector field.

__Def__: If $F = \langle P, Q \rangle$ is a diff'ble vector field on $\mathbb{R}^2$. Then

$$\text{curl}(F) = Q_x - P_y .$$

If $F = \langle P, Q, R \rangle$ is a differentiable vector field on $\mathbb{R}^3$, then

$$\text{curl}(F) = \nabla \times F = \hat{i}(R_y - Q_z) + \hat{j}(P_z - R_x) + \hat{k}(Q_x - P_y)$$

Example/Computations:

① $F(x,y) = \langle \overset{P}{e^{xy}}, \overset{Q}{x^2+y^2} \rangle$

$$\text{curl}(F) = \underbrace{2x}_{Q_x} - \underbrace{xe^{xy}}_{P_y}$$

② $F(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$$\text{curl}(F) = \langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \rangle$$

$$= \underset{\sim}{0}.$$

③ $F(x,y,z) = \langle xy, e^z, \ln(xyz) \rangle$

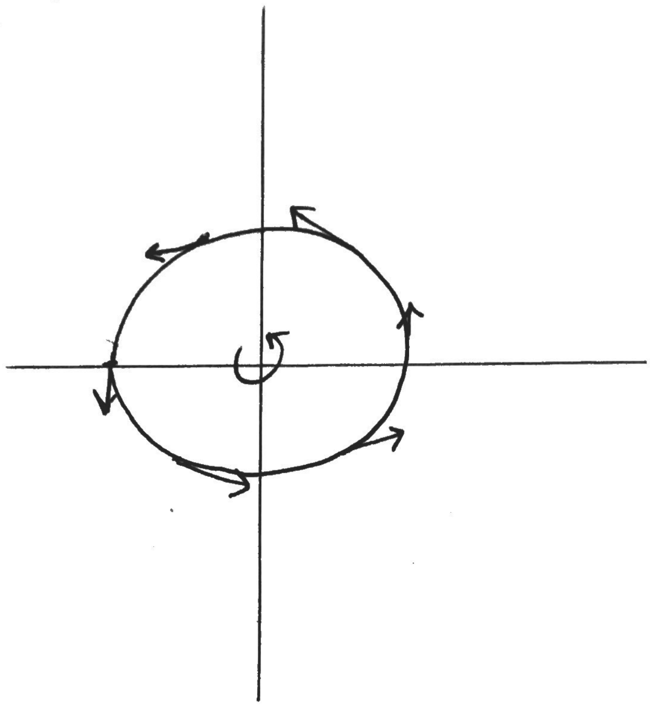
$$\text{curl}(F) = \langle \frac{1}{y} - e^z, 0 - \frac{1}{x}, 0 - x \rangle$$

$$= \langle \frac{1}{y} - e^z, -\frac{1}{x}, -x \rangle$$

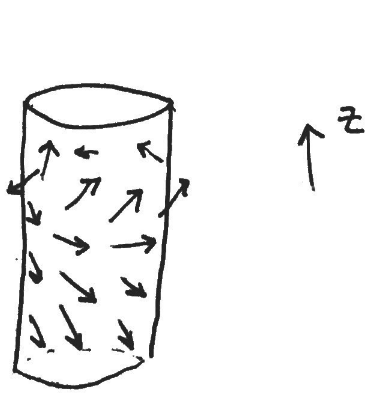
What is curl(F) measuring?

Any 3D vector field restricts to a cylinder and thus we can ask how much the vector field "wraps around" the cylinder. In particular:

2D proj:



3D proj:



The second quantity we can associate to vector fields is ~~exclusive~~ scalar valued!

<u>Def:</u> Let F be a differentiable vector field on $\mathbb{R}^n$. Then the <u>divergence</u> is

$$\operatorname{div}(F) = \nabla \cdot F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

<u>Theorem:</u> If F is in $C^2(\mathbb{R}^3, \mathbb{R}^3)$, then

$$\operatorname{div}(\operatorname{curl}(F)) = 0.$$

<u>Proof:</u> Clairaut's Theorem.

<u>Application:</u> When is a vectorfield $F = \operatorname{curl}(G)$?

We cannot answer this positively, but we can say $F \neq \operatorname{curl}(G) \Leftarrow \operatorname{div}(F) \neq 0$!

Now that we have these objects, we can start discussing the new integration.

Line Integrals:

Line integrals are an integral (HA!) part of mathematics. In particular, they form the analog of single integrals but for vector valued functions. As a glimpse into the future, we will define line integrals, and then use them to prove/investigate/understand the following theorems:

Green's Theorem: $\oint_C F \cdot dr = \iint_D \operatorname{curl}(F) \cdot \hat{k} \, dA$

if F is a 2D vector field with continuous partials.
$C = \partial D$

Stoke's Theorem: $\oint_C F \cdot dr = \iint_S \operatorname{curl}(F) \cdot dS$

if F is a 3D vector field with continuous partials, and S is a closed oriented surface with boundary curve C.

Divergence Theorem: $\iint_S F \cdot dS = \iiint_E \operatorname{div}(F) \, dV$

if F is a 3D vector field with continuous partials and E is a closed ~~oriented surfa~~ region with boundary surface S.

We will re-formulate these in terms of differential forms after we complete the chapter.

We shall start the investigation of line integrals where we left off with curves. Recall that the arc-length of a curve $C = r(t)$ was

$$s(t) = \int_0^t \| r'(u) \| \, du .$$

We shall re-write such a quantity as

$$\int_C 1 \cdot ds = |C| . \quad \text{length of } C.$$

In particular, if $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous, then

<u>Def:</u>

$$\oint_C f \, ds = \int_a^b f(r(t)) \| r'(t) \| dt$$

is the <u>line integral</u> of f over C. The left hand side works regardless of parametrization.

<u>Example:</u> Compute $\int_C (2 + x^2 y) \, ds$ for C ~~the~~ positive semicircle $x^2 + y^2 = 1$.

$$\int_C (2 + x^2 y) \, ds = \int_0^\pi (2 + \cos^2 t \sin t) \cdot \sqrt{\sin^2 t + \cos^2 t} \, dt$$

$$= \int_0^\pi 2 + \cos^2 t \sin t \, dt$$

$$= 2\pi + \int_0^\pi \cos^2 t \sin t \, dt$$

$$= 2\pi - \frac{1}{3} \cos^3 t \Big|_0^\pi = 2\pi + \frac{2}{3}$$

Now we can generalize this to vector fields.

<u>Def</u>: Let F be a continuous vector field and C a smooth curve given by a parametrization $r(t)$. Then the <u>line integral</u> of F over C ~~and~~ is given by

$$\oint_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

<u>Example</u>: $F(x,y) = \langle -x, y \rangle$, $C = S^1$ = unit circle.

$r(t) = \langle \cos t, \sin t \rangle$

$r'(t) = \langle -\sin t, \cos t \rangle$

$F(r(t)) = \langle -\cos t, \sin t \rangle$

$$\oint_C F \cdot dr = \int_0^{2\pi} \langle -\cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} \sin t \cos t + \sin t \cos t \, dt$$

$$= \int_0^{2\pi} \sin 2t \, dt$$

$$= -\frac{1}{2} \cos 2t \Big|_0^{2\pi}$$

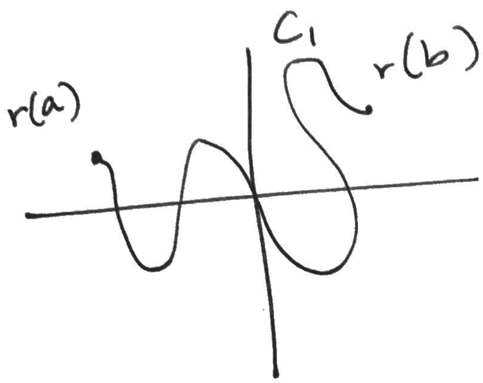
$$= 0$$

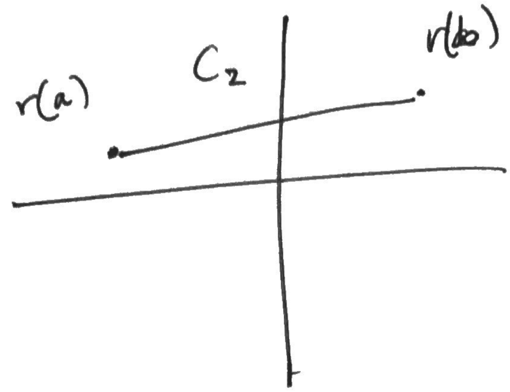
Theorem (Fundamental Theorem of Line Integrals):

Let $F = \text{grad } f$ be a gradient field for $f \in C^1(\mathbb{R}^3)$. Then if C is a curve given by $r(t)$ on $[a,b]$, then

$$\oint_C F \cdot dr = \oint_C \text{grad } f \cdot dr = f(r(b)) - f(r(a)).$$

Picture: $F = \text{grad } f$





$$\oint_{C_1} F \cdot dr = \oint_{C_2} F \cdot dr$$

The choice of curve doesn't matter for gradient fields!

Def: If F is a continuous vector field on D and $r_1(t), r_2(t)$ are any curves, we say F is independent of path if

$$\oint_{C_1} F \cdot dr = \oint_{C_2} F \cdot dr \quad \text{for all } r_1, r_2.$$

Theorem: If F is independent of path on $\mathcal{U} \subseteq \mathbb{R}^3$ open connected, then

$$\oint_C F \cdot dr = 0.$$

The converse is true trivially.

Def: A vector field is conservative if it is independent of path for loops.

Theorem: A vector field F is a gradient field if and only if F is conservative.

Pf idea: $f(x,y) = \displaystyle\int_{(a,b)}^{(x,y)} F \cdot dr.$

"Integrate to get what you want."

Corollary: F is conservative iff F is continuously differentiable and

$$\text{curl}(F) = 0!$$

Example: (Finding the potential function)

If $F(x,y,z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$

find f such that $F = \text{grad } f$.

Step 1: Integrate $F_1 = y^2$ w.r.t. x.

$$\rightsquigarrow f = xy^2 + g(y,z)$$

Step 2: $f_y = 2xy + g_y(y,z)$

$$\rightsquigarrow g_y(y,z) = e^{3z}$$

$$\Rightarrow f = xy^2 + ye^{3z} + h(z)$$

Step 3:

$$\rightsquigarrow f_z = 3ye^{3z} + h'(z)$$

$$\Rightarrow h'(z) = 0$$

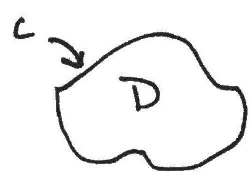
$$\Rightarrow f = xy^2 + ye^{3z} + k \qquad k \in \mathbb{R}.$$

The first integral theorem:

Theorem (Green): Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field. Extend F to $\tilde{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\tilde{F}(x,y,z) = \langle \underset{\in \mathbb{R}^2}{F(x,y)}, 0 \rangle.$$

Then for C a piecewise smooth curve bounding D

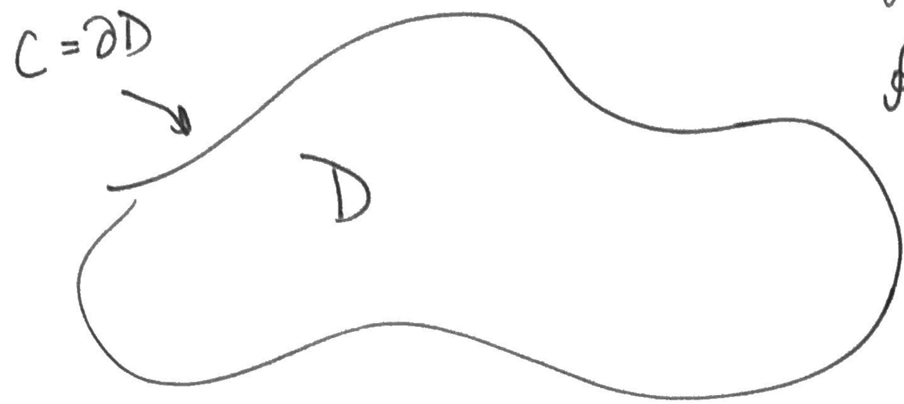


$$\oint_C F \cdot dr = \iint_D (\text{curl}(\tilde{F})) \cdot \hat{k} \, dA.$$

What is this saying?

A different formulation

$$\oint_C F \cdot n \, ds = \iint_D \operatorname{div}(F) \, dA$$



C traversed counter-clockwise then the line integral is computing a volume of some kind. In particular, the curl is controlled by the line integral around the boundary!

Example: Find the area enclosed by the ellipse $\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} = D$

$\iint_D 1 \, dA = A(D)$. To use Green's Theorem we need to find F such that $\operatorname{curl}(\vec{F}) = 1$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$\Rightarrow$ several possibilities

i) $P(x,y) = 0$
$Q(x,y) = x$

ii) $P(x,y) = -y$
$Q(x,y) = 0$

iii) $P(x,y) = -\frac{1}{2}y$
$Q(x,y) = \frac{1}{2}x$

Any of these will suffice, but lets pick iii).

Then

$$\iint_D 1 \, dA = \oint_C F \cdot dr$$

$$= \frac{1}{2}\int_0^{2\pi} \langle \cancel{-acost}\ bsint, \cancel{bsint}\ acost \rangle \cdot \langle -asint, +bcost \rangle \, dt$$

$$= \frac{1}{2}\int_0^{2\pi} absin^2t + abcos^2t \, dt$$

$$= \frac{1}{2} ab \cdot 2\pi = ab\pi \quad \blacksquare$$

<u>Example 2</u>: A line integral which is impossible without Green's Theorem.

Let $F = \langle 3y - e^{\sin x}, 7x + \sqrt{y^4+1} \rangle$. Find $\oint_C F \cdot dr$ for $C := x^2+y^2=9$. $\Leftrightarrow C := r(t) = \langle 3cost, 3sint \rangle$.

Yuck! $\int_0^{2\pi} \langle 3(3sint) - e^{\sin(3\cos t)}, 7(3\cos t) + \sqrt{(3\sin t)^4+1} \rangle \cdot \langle -3sint, 3cost \rangle dt$

$$= \iint_D \underbrace{7}_{Q_x} - 3 \, dA$$

$$= \iint_D 4 \, dA$$

$$= 4 \cdot \pi \cdot 3^2 = 36\pi. \quad \blacksquare$$

Now we want to generalize Green's Theorem to 3D vector fields. To do so, we need to generalize line integration to surface integration.

Let S be a parametric surface $\underset{\sim}{r}(u,v)$, and $f:\mathbb{R}^3 \to \mathbb{R}$ a multivariable continuous function..

<u>Def</u>: The <u>surface integral</u> of f along S is

$$\iint_S f(x,y,z)\, d\underset{\sim}{S} = \iint_D f(r(u,v)) \underbrace{\| \underset{\sim}{r}_u \times \underset{\sim}{r}_v \| \, dA}_{\text{surface area!}}.$$

If $S = \Gamma(g)$ for some $g:\mathbb{R}^2 \to \mathbb{R}$, then

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dA$$

<u>Example</u>: Integrate y over the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

$$\frac{\partial z}{\partial x} = 1 \qquad \frac{\partial z}{\partial y} = 2y$$

$$\Rightarrow \iint_S y\, dS = \iint_D y\sqrt{2+4y^2}\, dA$$

$$= \int_0^1 \int_0^2 y\sqrt{2+4y^2}\, dy\, dx = \sqrt{2}\int_0^1\int_0^2 y\sqrt{1+2y^2}\, dy\, dx$$

$= \sqrt{2}\int_0^1 \int \frac{1}{4}\sqrt{u}\,du\,dx$ u-sub in first int.

$= \sqrt{2}\int_0^1 \frac{1}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^3\,dx$ change bounds

$= \sqrt{2}\int_0^1 \frac{1}{4}\cdot\frac{2}{3}(1+2y^2)^{3/2}\Big|_0^2\,dx$ re-sub.

$= \sqrt{2}\cdot\frac{1}{6}\cdot(9^{3/2}-1)$

$= \sqrt{2}\cdot\frac{1}{6}\cdot(27-1)$

$= \frac{13\sqrt{2}}{3}$ ■

Fact: If $S = S_1 \cup \ldots \cup S_k$ is a union of disjoint surfaces (or surfaces whose intersection has dim ≤ 1) then

$$\iint_S f\,dS = \sum_{i=1}^{k} \iint_{S_i} f\,dS.$$

Oriented Surfaces:

Def: A surface is orientable iff there exists a consistent choice of normal vector which varies continuously.

Ex: S^2 2-sphere   Non-Ex: Möbius Band

This is the only context we can integrate vector fields!

- $\boxed{\text{Why?}}$ When we integrate, we need a real value, if S is not oriented/orientable there is no canonical choice of vector/way to turn a vector field into a single #.

<u>Def</u>: Let $(S, \underset{\sim}{n})$ be an oriented surface, F a continuous vector field on S. Then

$$\underbrace{\iint_S F \cdot d\underset{\sim}{S}}_{\text{bold } S.} = \iint_S F \cdot \underset{\sim}{n}\, dS \quad \Big\}\ \text{Flux of } F \text{ on } S.$$

- If $(S, \underset{\sim}{n})$ is parametric, then

$$\iint_S F \cdot d\underset{\sim}{S} = \iint_D F \cdot \underbrace{(\underset{\sim}{r_u} \times \underset{\sim}{r_v})}_{\text{normal vector!}}\, dA.$$

<u>Example</u>: $F = \langle 0, 1, z^2 \rangle$. Compute $\iint_S F \cdot d\underset{\sim}{S}$ for $S := 2x + y + z = 1$ over the region $D = [0,2] \times [0,2]$.

$$\iint_S F \cdot d\underset{\sim}{S} = \iint_D F \cdot (\underset{\sim}{r_u} \times \underset{\sim}{r_v}) \cdot dA = \iint_D F \cdot \langle 2, 1, 1 \rangle\, dA$$

$$= \int_0^2 \int_0^2 z^2 + 1\, dA$$

$z = 1-2x-y \Rightarrow F(r(x,y)) = \langle 0, 1, 1-2x-y \rangle$

So

$$\int_0^2\int_0^2 z^2+1\,dA = \int_0^2\int_0^2 (1-2x-y)^2+1\,dA$$

$$= \int_0^2\int_0^2 2-4x-2y+4x^2+4xy+y^2\,dx\,dy$$

$$= \int_0^2 \left[2x-2x^2-2xy+\tfrac{4}{3}x^3+2x^2y+xy^2\Big|_0^2\right]dy$$

$$= \int_0^2 \frac{32}{3}-4+4y+2y^2\,dy$$

$$= \frac{64}{3}-8+\left[2y^2+\frac{2}{3}y^3\right]_0^2$$

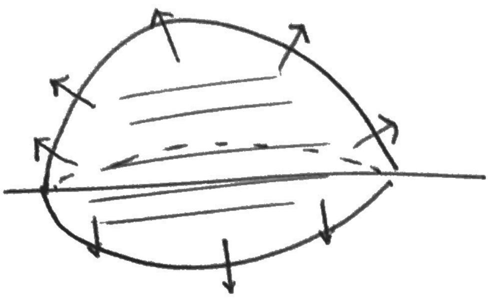
$$= \frac{64}{3}\cancel{-8}+\cancel{8}+\frac{16}{3}$$

$$= \frac{64}{3}+\frac{16}{3}$$

$$= \frac{80}{3}.$$

:)

Example 2: Consider $F = \langle y, x, z \rangle$. Compute $\iint_S F \, d\underset{\sim}{S}$ where S is the surface bounding the region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



Oriented outward!

Step 1: $D = \{(x,y) : x^2 + y^2 \leq 1\}$ and

$$F(r(x,y)) = \langle y, x, 1 - x^2 - y^2 \rangle.$$

Step 2: $\underset{\sim}{r}_x \times \underset{\sim}{r}_y = -\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}$

$$\Rightarrow \underset{\sim}{r}_x \times \underset{\sim}{r}_y = -(-2x)\hat{i} - (-2y)\hat{j} + \hat{k}$$

$$= \langle 2x, 2y, 1 \rangle$$

Step 3: $\iint_S F \cdot d\underset{\sim}{S} = \iint_S F \cdot \overset{\underset{\sim}{n}}{\underset{\sim}{r}_x \times \underset{\sim}{r}_y} \, dS$

$$= \iint_D F \cdot \underset{\sim}{r}_x \times \underset{\sim}{r}_y \, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xy + 2xy + 1 - x^2 - y^2 \, dy\, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 + 4xy - x^2 - y^2 \, dy\, dx$$

Polar

$$= \int_0^{2\pi} \int_0^1 \left(1 + 4r^2 \cos\theta \sin\theta - r^2 \cos^2\theta - r^2 \sin\theta\right) r \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^3 + 4r^3 \cos\theta \sin\theta \, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4} + r^4 \cos\theta \sin\theta \right)\Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} + \cos\theta \sin\theta \, d\theta$$

$$= \frac{\pi}{2} + \frac{1}{2}\sin^2\theta \Big|_0^{2\pi}$$

$$= \frac{\pi}{2} + 0 = \boxed{\frac{\pi}{2}}$$

:)

The Second and Third big theorems:

We will now state the final named theorems in the course. These completely determine integration of vector fields on regions in $\mathbb{R}^3$.

<u>Stoke's Theorem:</u> Let S be an oriented piecewise smooth surface with a piece-wise smooth boundary curve C with positive orientation. Let F be a C^1 vector field in a neighbourhood of S. Then

$$\int_C F \cdot d\underset{\sim}{r} = \iint_S \mathrm{curl}(F) \cdot d\underset{\sim}{S} = \iint_S \mathrm{curl}(F) \cdot \underset{\sim}{n}\, dS$$

Compare this to Green's Theorem:
if $S \subseteq \mathbb{R}^2$, then stoke's degenerates to

$$\int_C F \cdot d\underset{\sim}{r} = \iint_S \mathrm{curl}(F) \cdot d\underset{\sim}{S} = \iint_S \mathrm{curl}(F) \cdot \hat{k}\, dA$$

So these are "the same" theorem.

<u>Divergence Theorem:</u> Let E be a simple solid region and S the boundary surface of E given with positive orientation. Let F be C^1 vector field. Then

$$\iint_S F \cdot d\underset{\sim}{S} = \iiint_E \mathrm{div}\, F\, dV$$

In total, we get the following triangle of integral theorems

$$1 \xrightarrow{\text{FTC}} 1$$

$$1 \xrightarrow{\text{FTL}} 2 \xrightarrow{\text{Green}} 1$$

$$1 \xrightarrow{\text{FTL}} 3 \xrightarrow{\text{Stokes}} 3 \xrightarrow{\text{Diver.}} 1$$

all of these theorems are particular instances of the following theorem on exterior differentials.

Theorem (Stokes): Let M be an orientable manifold with boundary ∂M oriented positively. Then for any differential form ω

$$\int_{\partial M} \omega = \int_M d\omega$$

where $d\omega$ is the exterior derivative.

In particular, (grad. curl, div) are the exterior derivatives in $\mathbb{R}^3$.

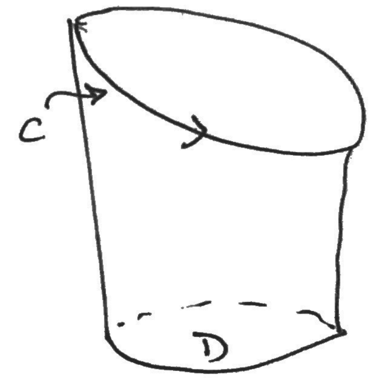
Examples:

① Stokes for Line Integrals

Evaluate $\int_C F \cdot d\underset{\sim}{r}$ for $F = \langle -y^2, x, z^2 \rangle$ and C the curve of intersection between the plane $y+z=2$ and the cylinder $x^2+y^2=1$.

$\boxed{\text{Key Idea}}$ Finding parametrizations of C is annoying. Use surface integrals instead!

Step 1: $\text{curl}(F) = \nabla \times F$



$$= (1+2y)\,\hat{k}$$

Step 2: $\oint_C F \cdot d\underset{\sim}{r} \overset{\text{Stokes}}{=} \iint_S \text{Curl}(F) \cdot d\underset{\sim}{S}$

$$= \iint_D (1+2y)\, dA$$

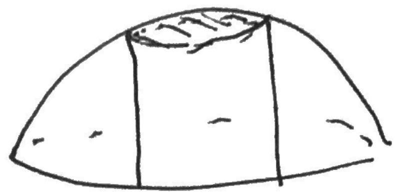
$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \underbrace{\frac{2}{3}\sin\theta}_{0}\, d\theta$$

$$= \pi$$

② Surface integrals via line integrals

Compute $\iint_S \operatorname{curl}(F) \cdot d\underset{\sim}{S}$ where $F = \langle xz, yz, xy \rangle$ and S is the surface generated by the intersection of $x^2+y^2+z^2=4$ and $x^2+y^2=1$ lying above the xy-plane. Together with the disc closing the surface.



Step 1: Compute C:

$$x^2+y^2+z^2=4$$
$$- \quad x^2+y^2=1$$
$$\Rightarrow z^2 = 3 \Rightarrow z = \sqrt{3}$$

$$\Rightarrow C = \langle \cos\theta, \sin\theta, \sqrt{3} \rangle = r(\theta)$$

Step 2: $F(r(\theta)) = \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \cos\theta\sin\theta \rangle$

$$F(r(\theta)) \cdot r'(\theta) = -\sqrt{3}\sin\theta\cos\theta + \sqrt{3}\sin\theta\cos\theta + 0$$
$$= 0$$

So $\iint_S \operatorname{curl}(F)\, d\underset{\sim}{S} = \int_C F \cdot d\underset{\sim}{r} = \int_0^{2\pi} 0 \, d\theta$

$$= 0.$$

(3) Divergence to find Flux

Find the flux of $F = \langle z, y, x \rangle$ over the unit sphere.

Step 1: $\text{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$= 0 + 1 + 0$$

$$= 1$$

Step 2:
So $\iint_S F \cdot d\underset{\sim}{S} = \iiint_E 1 \, dV$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{4\pi}{3}$$

(4) A surface integral made easier via Div.

Evaluate $\iint_S F \cdot d\underset{\sim}{S}$ where

$$F = \langle xy, \; y^2 + e^{xz^2}, \; \sin(xy) \rangle$$

and S is the surface bounding the region defined by the $z = 1 - x^2$ and the plane $z = 0$, $y = 0$, $y + z = 2$.

Step 1: $\operatorname{div}(F) = y + 2y + 0$

$$= 3y$$

Step 2: The region E.

$$E = \{(x,y,z) : -1 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 2 - z\}$$

Step 3: $\iint_S F \cdot d\underset{\sim}{S} = \iiint_E 3y \, dV$

$$= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} 3y \; dy\, dz\, dx$$

$$= \int_{-1}^{1} \int_{0}^{1-x^2} \frac{3}{2} y^2 \Big|_{0}^{2-z} dz\, dx$$

$$= \int_{-1}^{1} \int_{0}^{1-x^2} \frac{3}{2}(2-z)^2 \; dz\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} -\frac{(2-z)^3}{3} \Big|_{0}^{1-x^2} dx$$

$$= -\frac{1}{2}\int_{-1}^{1} \left[(x^2+1)^3 - 8 \right] dx$$

$$= -\int_0^1 \left(x^6 + 3x^4 + 3x^2 - 7 \right) dx = \frac{184}{35}$$

This completes the main material for the course. We will now cover some basic linear algebra to get some exposure to abstract mathematics. This will not only be helpful for your later math courses, but also hopefully open your mind to new approaches.