MATH 2270

Final Exam - Fall 2008

Name: Answer Key

1. (18 points) Let $A = \begin{pmatrix} 1 & -2 & -10 \\ -2 & 2 & 12 \\ 4 & 4 & 8 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ -2 \\ 20 \end{pmatrix}$.

(a) Solve the linear system $A\vec{x} = \vec{b}$ for $\vec{x}$ using Gauss-Jordan elimination. Clearly show each step and indicate your answer. If the system is inconsistent, write *inconsistent*.

$$\left(\begin{array}{ccc|c} 1 & -2 & -10 & -1 \\ -2 & 2 & 12 & -2 \\ 4 & 4 & 8 & 20 \end{array}\right) \begin{array}{l} \\ +2\mathrm{I} \\ -4\mathrm{I} \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -10 & -1 \\ 0 & -2 & -8 & -4 \\ 0 & 12 & 48 & 24 \end{array}\right) \begin{array}{l} \\ \div -2 \\ \ \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -10 & -1 \\ 0 & 1 & 4 & 2 \\ 0 & 12 & 48 & 24 \end{array}\right) \begin{array}{l} +2\mathrm{II} \\ \\ -12\mathrm{II} \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore \quad \begin{array}{l} x_1 - 2x_3 = 3 \\ x_2 + 4x_3 = 2 \end{array} \quad \Bigg| \quad \begin{array}{l} x_1 = 2x_3 + 3 \\ x_2 = -4x_3 + 2 \end{array}$$

let $x_3 = t$,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

(b) What is the reduced row echelon form of A?

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) What is rank(A)?

2 (2 columns have leading ones)

(d) Is the matrix A invertible?

No (there is a column w/out a leading 1)

(e) Find a basis for the kernel of A.

$$\left\{ \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \right\}$$

(f) What is the nullity of A?

1

(g) Find a basis for the image of A.

First two columns of A:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \right\}$$

2. (8 points) Consider the 3×3 matrix $A = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$ and suppose $A^T A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

$$A^T A = \begin{pmatrix} - \vec{v}_1 - \\ - \vec{v}_2 - \\ - \vec{v}_3 - \end{pmatrix} \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$$

(a) Find $\|\vec{v}_2\|$.

$\|\vec{v}_2\|^2 = \vec{v}_2 \cdot \vec{v}_2 = 2$

$\therefore \boxed{\|\vec{v}_2\| = \sqrt{2}}$

(b) Find $\vec{v}_1 \cdot \vec{v}_3$.

$\vec{v}_1 \cdot \vec{v}_3 = 1$

(c) Which (if any) of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are orthogonal?

$\vec{v}_1$ and $\vec{v}_2$ are orthogonal

(d) What are the possible values of $\det(A)$?

$\det \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 2(4-1) + 1\cdot(0-2) = 6-2 = \boxed{4}$

$\therefore \det(A^TA) = \det(A^T)\det(A) = \det(A)^2 = 4 \Rightarrow$

$\boxed{\det(A) = \pm 2}$

3. (10 points) Let V be the vector space spanned by the polynomials $\mathfrak{B} = \{1, 2x, 3x^2\}$ and let $T : V \to V$ be the linear transformation given by $T(f) = f'' - f' + f$.

(a) Find the matrix of T with respect to the basis $\mathfrak{B}$.

$$B = \begin{pmatrix} | & | & | \\ T(f_1)_{\mathfrak{B}} & T(f_2)_{\mathfrak{B}} & T(f_3)_{\mathfrak{B}} \\ | & | & | \end{pmatrix},$$

$$T(f_1) = T(1) = 0 - 0 + 1 = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$T(f_2) = T(2x) = 0 - 2 + 2x = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}_{\mathfrak{B}}$$

$$T(f_3) = 6 - 6x + 3x^2 = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}_{\mathfrak{B}}$$

$$\therefore \quad B = \begin{pmatrix} 1 & -2 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Is T injective?

Yes (B has trivial kernel)

(c) Is T surjective?

Yes (B has full rank)

(d) Suppose now $T(f) = f' - f''$. Find a basis for the kernel of T.

$$B = \begin{pmatrix} 0 & 2 & -6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}. \quad \therefore \quad \ker(B) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

In other words, ker (T) is the span of the constant function $\boxed{P(x) = 1}$.

4. (8 points) Find the QR factorization of the matrix

$$M = \begin{array}{c} \begin{pmatrix} 1 & 4 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \vec{v}_1 \ \ \vec{v}_2 \end{array}.$$

Clearly show each step.

- $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{\vec{v}_1}{2}$

- $\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$

$= \vec{v}_2 - \dfrac{(\vec{v}_2 \cdot \vec{v}_1)\,\vec{v}_1}{4}$

$= \vec{v}_2 - \dfrac{4\vec{v}_1}{4} = \vec{v}_2 - \vec{v}_1 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

- $\vec{u}_2 = \dfrac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \dfrac{\vec{v}_2^{\perp}}{\sqrt{12}}$.

$\therefore \quad Q = \begin{pmatrix} 1/2 & 3/\sqrt{12} \\ 1/2 & -1/\sqrt{12} \\ 1/2 & -1/\sqrt{12} \\ 1/2 & -1/\sqrt{12} \end{pmatrix}$

$\|\vec{v}_1\|$

$R = \begin{pmatrix} 2 & 2 \\ 0 & \sqrt{12} \end{pmatrix}$ $\quad (\vec{v}_2 \cdot \vec{u}_1) = 2$

$\|\vec{v}_2^{\perp}\|$

5. (10 points) Let $V = \mathbb{R}^{2\times 2}$ be an inner product space with inner product $\langle C, D\rangle = \operatorname{trace}\left(C^T D\right)$ and let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(a) Find the norm of A in V.

$$\|A\|^2 = \langle A, A\rangle = \operatorname{trace}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \operatorname{trace}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$$

$$\therefore \quad \|A\| = \sqrt{4} = \boxed{2}$$

(b) Find the orthogonal projection of $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ onto the subspace of V spanned by A.

$$B^{\parallel} = \frac{1}{2}\langle B, A\rangle \frac{A}{2}$$

$$= \tfrac{1}{2} \cdot \operatorname{trace}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{A}{2}$$

$$= \tfrac{1}{2} \operatorname{trace}\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} \frac{A}{2} = \tfrac{1}{2}(4) \cdot \frac{A}{2} = \boxed{A}$$

$$= \boxed{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}$$

(c) Find a nonzero matrix B in V such that $\langle B, A\rangle = 0$.

$$B^{\perp} = B - B^{\parallel}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix}}$$

Check:

$$\langle B^{\perp}, A\rangle = \operatorname{trace}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \operatorname{trace}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 0 \checkmark$$

(d) Suppose S is an orthogonal matrix. Prove $\langle SA, SA\rangle = \langle A, A\rangle$.

$$\underline{\langle SA, SA\rangle} = \operatorname{trace}\left((SA)^T SA\right)$$

$$= \operatorname{trace}\left(A^T \underbrace{S^T S}_{I_2 \text{ since } S \text{ orthogonal}} A\right)$$

$$= \operatorname{trace}\left(A^T A\right)$$

$$= \underline{\langle A, A\rangle} \checkmark$$

$\blacksquare$

6. (10 points) Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find a diagonal matrix D and an orthogonal matrix S such that $D = S^T AS$. Show your work.

eigenvalues: $f_A(\lambda) = \det(A - \lambda I) = \det \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)\left((1-\lambda)^2 - 1\right) - (-\lambda) + \lambda = (1-\lambda)(\lambda^2 - 2\lambda) + 2\lambda$

$= -\lambda^3 + \lambda^2 + 2\lambda^2 - \cancel{2\lambda} + \cancel{2\lambda} = \boxed{\lambda^2(3-\lambda)}$

$\therefore$ eigenvalues are $0, 0, 3$

$E_0 = \ker(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$

$E_3 = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\therefore \quad S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

7. (4 points) For $\alpha, \beta, \gamma \in \mathbb{R}$, let

$$A = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 2 & \beta & 0 \\ 0 & 0 & 2 & \gamma \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(a) What are the eigenvalues (with multiplicities) of A?

1, 2, 2, 3

(b) For which values of α, β, and γ is the matrix A diagonalizable?

$\beta = 0$, α, γ arbitrary

8. (12 points) Multiple choice.

(a) The vectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ \alpha \end{pmatrix}$ form a basis of $\mathbb{R}^3$ for all values of α except

i. $\alpha = -2$.

ii. $\boxed{\alpha = -1.}$

iii. $\alpha = 0$.

iv. $\alpha = 1$.

v. $\alpha = 2$.

$\det \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & \alpha \end{pmatrix} = 1(-2) - 1(0) + \alpha(-2)$

want: $-2 - 2\alpha = 0$

$2\alpha = -2$

$\boxed{\alpha = -1}$

(b) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, what is the value of $\text{rank}(A) - \det(A)$?

i. -2.

ii. -1.

iii. 0.

iv. 1.

$\boxed{\text{v. 2.}}$

$\det(A) = 1\cdot(45-48) - 4(18-24) + 7(12-15)$

$= 1(-3) - 4(-6) + 7(-3)$

$= -3 + 24 - 21 = \boxed{0}$

$\text{rank}(A) = \boxed{2}$

$\therefore\ 2 - 0 = \boxed{2}$

(c) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$T\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

will also satisfy $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

i. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\boxed{\text{ii. } \begin{pmatrix} 1 \\ 0 \end{pmatrix}.}$

iii. $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

iv. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

v. Cannot be determined from the given information.

Observe

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\therefore$

$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = T\begin{pmatrix} 1 \\ 2 \end{pmatrix} + T\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

(d) Recall a matrix A is said to be *skew-symmetric* if $A^T = -A$. What is the dimension of the space of all 4×4 skew-symmetric matrices?

i. 4.

ii. 6.

iii. 8.

iv. 16.

v. None of the above.

$$\begin{pmatrix} 0 & * & * & * \\ & 0 & * & * \\ & & 0 & * \\ & & & 0 \end{pmatrix}$$

determined