

DUALITY FOR THE UNIVERSAL COVER OF
 $\text{Spin}(2n + 1, 2n)$

by

Scott Philip Crofts

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Chair: Peter Trapa

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Date

Peter Trapa
Chair, Supervisory Committee

Approved for the Major Department

Aaron Betram
Chair/Dean

Approved for the Graduate Council

David S. Chapman
Dean of The Graduate School

ABSTRACT

Let $\mathbb{G} = \text{Spin}(4n+1, \mathbb{C})$ be the connected, simply connected complex Lie group of type B_{2n} and let $G = \text{Spin}(2n+1, 2n)$ denote its (connected) split real form. Then G has fundamental group \mathbb{Z}_2 and we denote the corresponding nonalgebraic double cover by $\tilde{G} = \widetilde{\text{Spin}}(2n+1, 2n)$. The main purpose of this dissertation is to describe a latent symmetry in the set genuine representation theoretic parameters for \tilde{G} at certain half-integral infinitesimal characters. This symmetry is then used to establish a duality of the corresponding generalized Hecke modules and ultimately results in a character multiplicity duality for the genuine characters of \tilde{G} .

To Prachee, Las Vegas, and The Chicago Bears

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CHAPTER 1

INTRODUCTION

1.1 Duality for Verma Modules

Before stating the main result for which we are aiming (Theorem 2.4.1), we begin by recalling a familiar case. Let \mathbb{G} be a simple complex algebraic group with Lie algebra \mathfrak{g} and write $U(\mathfrak{g})$ for its universal enveloping algebra. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system. Write $W = W(\mathfrak{g}, \mathfrak{h})$ for the Weyl group of Δ and fix a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Then \mathfrak{b} induces a choice $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ of positive roots for Δ and we set ρ to be the half sum of the elements of Δ^+ . Finally, for $w \in W$ we define

$$M_w = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{w\rho - \rho}$$

to be the Verma module of highest weight $w\rho - \rho$ and we write L_w for its unique irreducible quotient.

A fundamental problem in the representation theory of Verma modules is to determine the composition factors (with multiplicities) of M_w . Each such factor is known to be of the form L_y , for some $y \in W$. On the level of formal characters we seek a decomposition of the form

$$\text{ch}(M_w) = \sum_{y \in W} n_{y,w} \text{ch}(L_y) \quad n_{y,w} \in \{0, 1, 2, \dots\}$$

where it remains to compute the numbers $n_{y,w}$. A related problem asks for a type of inverse decomposition

$$\text{ch}(L_w) = \sum_{y \in W} N_{y,w} \text{ch}(M_y) \quad N_{y,w} \in \mathbb{Z}$$

of an irreducible module in terms of Verma modules. A conjectural algorithm for computing the multiplicities $n_{y,w}$ and $N_{y,w}$ was first given by Kazhdan and Lusztig in [10]. Their method uses the combinatorics of Hecke algebras to inductively build polynomials (now

known to be) related to singularities of the corresponding Schubert varieties. The celebrated Kazhdan-Lusztig conjecture asserts that evaluating these polynomials at one gives the desired multiplicities. It was later proven correct in the work of Brylinski-Kashiwara and Beilinson-Bernstein [7], [5].

A striking feature of the Kazhdan-Lusztig theory is the existence of a type symmetry in the multiplicities $n_{y,w}$ and $N_{y,w}$. Let w_0 denote the longest element of W and define the *duality map*

$$\begin{aligned} \Psi : W &\longrightarrow W \\ w &\longmapsto w_0 w. \end{aligned}$$

Then Ψ is an involution of W that implements the familiar up-down symmetry of the corresponding Coxeter graph. Moreover, Ψ defines a type of dual Hecke module whose combinatorics are formally reversed. On the level of multiplicities this dualization induces the equality

$$N_{y,w} = \pm n_{\Psi(w), \Psi(y)} \tag{1.1}$$

for all $y, w \in W$. In other words, the multiplicity of $\text{ch}(M_y)$ in $\text{ch}(L_w)$ is equal (up to sign) to the multiplicity of $\text{ch}(L_{\Psi(w)})$ in $\text{ch}(M_{\Psi(y)})$, where the sign is explicitly computable. Geometrically, this suggests the representation theory of Verma modules is dual to the singular structure of certain Schubert varieties. This observation is central to [2], with duality playing a key role in a reformulated version of the local Langlands conjecture.

1.2 Duality for Linear Groups

We now recall the analogous result for a large class of real Lie groups. Fix a real form $G \subset \mathbb{G}$ of \mathbb{G} and let $\Pi(G)$ denote the set of equivalence classes of irreducible admissible representations of G . For $\bar{\pi} \in \Pi(G)$, write π for the standard representation (Section 2.2) corresponding to $\bar{\pi}$. For simplicity (and by analogy with the case above), we restrict our attention to the finite set $\Pi_{\text{triv}}(G)$ of representations with trivial infinitesimal character. If $\bar{\pi} \in \Pi_{\text{triv}}(G)$, we again seek to understand the composition factors of π as

$$\pi = \sum_{\bar{\eta} \in \Pi_{\text{triv}}(G)} m(\bar{\eta}, \pi) \bar{\eta} \quad m(\bar{\eta}, \pi) \in \{0, 1, 2, \dots\}$$

with the sum interpreted in an appropriate Grothendieck group. Similarly, the inverse problem asks for an expression (again in a Grothendieck group) for $\bar{\pi}$ in terms of standard representations

$$\bar{\pi} = \sum_{\bar{\eta} \in \Pi_{\text{triv}}(G)} M(\eta, \bar{\pi}) \eta \quad M(\eta, \bar{\pi}) \in \mathbb{Z}.$$

Once again, it remains to compute the numbers $m(\bar{\eta}, \pi)$ and $M(\eta, \bar{\pi})$.

This problem was solved by Vogan for reductive linear Lie groups (see [14],[15],[18]) in an extension of the results cited above. In this setting, $\Pi_{\text{triv}}(G)$ is no longer parameterized by elements of a Weyl group, but rather a collection of geometric parameters (roughly, irreducible equivariant local systems on the flag manifold). These parameters form a natural basis for a Hecke module whose combinatorics embody a generalized Kazhdan-Lusztig algorithm. The resulting polynomials (sometimes called Kazhdan-Lusztig-Vogan polynomials) again give the desired multiplicities through evaluation at one.

In a remarkable final paper [17], Vogan describes a generalization of the duality in (1.1). If we write

$$\bar{\pi}_1 \sim \bar{\pi}_2 \iff m(\bar{\pi}_1, \pi_2) \neq 0$$

then \sim generates an equivalence relation whose equivalence classes are called *blocks*. Given a block $B = \{\bar{\pi}_1, \dots, \bar{\pi}_n\} \subset \Pi_{\text{triv}}(G)$, Vogan constructs a real form $G^\vee \subset \mathbb{G}^\vee$ of the complex dual group, a block $B^\vee = \{\bar{\eta}_1, \dots, \bar{\eta}_n\} \subset \Pi(G^\vee)$, and a bijection $\Psi : B \rightarrow B^\vee$. As before, the map Ψ commutes with the combinatorics of the Hecke module for B allowing one to define a dual Hecke module ([17], Definition 13.3) isomorphic to the one for B^\vee . On the level of multiplicities, this implies

$$M(\pi_i, \bar{\pi}_j) = \pm m(\Psi(\bar{\pi}_j), \Psi(\pi_i)) \tag{1.2}$$

([17], Theorem 1.15) as desired.

1.3 Duality for Nonlinear Groups

There are several partial results extending the above ideas to the setting of nonlinear groups. Recall $G \subset \mathbb{G}$ is a real form of a complex simple algebraic group \mathbb{G} . For simplicity,

we restrict our attention to *nonalgebraic* double covers \tilde{G} of G , that is nonlinear central extensions of the form

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{G} \rightarrow G \rightarrow 1.$$

In general, the methods cited above fail when applied to the representation theory of \tilde{G} . The main difficulty stems from the fact that disconnected Cartan subgroups in \tilde{G} need not be abelian. In such cases, results of Adams (see [1], Proposition 2.2) imply it suffices to understand the disconnected *center* of \tilde{H} . Since the identity component of \tilde{H} is automatically central, this reduces to understanding $\pi_0(\mathbf{Z}(\tilde{H}))$. Unfortunately computing $\pi_0(\mathbf{Z}(\tilde{H}))$ for each Cartan subgroup of \tilde{G} can be a subtle problem, requiring the development of new techniques.

In this direction, the Kazhdan-Lusztig-Vogan algorithm has been generalized for a large class of nonlinear groups by Renard and Trapa [13]. Their method builds extended Hecke algebra structures to track nonintegral wall crossings and computes KLV-polynomials for a potentially large number of infinitesimal characters simultaneously.

Alternatively, the duality of equation (1.2) has been extended to include nonlinear groups in certain cases. In [4] Adams and Trapa establish a duality theory for those \tilde{G} whose corresponding root system is simply laced. Their results take advantage of the fact that

$$\mathbf{Z}(\tilde{H}) = \mathbf{Z}(\tilde{G})\tilde{H}^\circ$$

for every Cartan subgroup $\tilde{H} \subset \tilde{G}$. If \tilde{G} is the metaplectic group (the nontrivial double cover the symplectic group), then all Cartan subgroups are abelian and we have

$$\pi_0(\mathbf{Z}(\tilde{H})) = \pi_0(\tilde{H}).$$

Using this fact, Renard and Trapa build a general duality theory for \tilde{G} in [12] by directly extending ideas of [17].

For simple classical groups, this leaves type B. For several reasons, this turns out to be a surprisingly subtle problem. First, there is no simple characterization of $\mathbf{Z}(\tilde{H})$ — in general there are often central elements of \tilde{H} that are not central in \tilde{G} . A second difficulty arises from the presence of short (type II) integral roots. The Hecke algebra action for such roots is complicated (see [13], Chapter 6) and thus so is their role in any form of a duality theory (for technical reasons these subtleties do not arise for the metaplectic group). The final difficulty is an apparent dependence on the parity of the rank — certain self duality results seem to hold only if the rank of \tilde{G} is even.

1.4 Duality for $\widetilde{\text{Spin}}(2n + 1, 2n)$

Let $\mathbb{G} = \text{Spin}(4n + 1, \mathbb{C})$ be the connected, simply connected complex Lie group of type B_{2n} and let $G = \text{Spin}(2n + 1, 2n)$ denote its (connected) split real form. The fundamental group of G is \mathbb{Z}_2 and we denote the corresponding nonalgebraic double cover by $\widetilde{G} = \widetilde{\text{Spin}}(2n + 1, 2n)$. In this dissertation we confront the above difficulties and establish a character multiplicities duality for \widetilde{G} at certain half integral infinitesimal characters (Section 6.2). Roughly speaking, a symmetric infinitesimal character is one for which the number of genuine parameters is as large as possible. The techniques developed are original and not simple generalizations of previous work. Moreover, it seems likely they can be extended to handle other nonalgebraic forms in type B.

After a brief discussion of notation and fundamental theory in Chapter 2, we redevelop the construction of linear parameters for $G = \text{Spin}(n + 1, n)$ in Part I (Chapters 3–5). Part II (Chapters 6–9) focuses on nonlinear parameters and in particular the structure of nonabelian Cartan subgroups (Chapters 7 and 8). Finally, in Part III we specialize to even rank and discuss the construction of the duality map Ψ for \widetilde{G} . The difficulty is defining a map on the level of nonlinear parameters with the appropriate representation theoretic properties (Theorems 12.7.3, 12.7.4, and 12.7.5). Once these issues are overcome in Chapters 11 and 12, the desired character multiplicity duality is again a formal consequence of the induced Hecke module duality described above (Theorem 12.8.4).

1.5 Future Directions

Unlike Vogan’s duality theory for linear groups, the results cited above establish nonlinear duality on a case-by-case basis using special properties of the groups being considered. We expect a more uniform approach to the duality theory of nonalgebraic double covers is possible. In order to develop such a theory, we must first fill the remaining gaps not covered by previous results.

The largest existing gap is odd rank groups of type B. Problems with this case arise immediately, even for the split real forms (it is shown here a type of self-duality exists only in the even rank case). At half integral infinitesimal character, such groups each possess two genuine discrete series and two genuine principal series. Unfortunately, the central action for these representations is not compatible with any kind of intrinsic definition of duality. In particular, a dual block (possessing an opposite central action) must exist elsewhere, if it exists at all. It is conjectured that such a block exists for a *disconnected* nonlinear covering

group of the same type. Moreover, we are hopeful that allowing disconnected nonlinear coverings will ultimately lead to a complete duality theory of type B.

The other remaining gap is F_4 (for technical reasons G_2 is included in the results of [4]). It is expected a duality theory exists and should be related to (or perhaps even a consequence of) the even rank duality for type B established here. With a complete (albeit ad hoc) duality theory of simple nonlinear double covers at hand, the possibility of finding a uniform approach (of the kind described for linear groups by Vogan) exists. Moreover, it is reasonable to expect the duality for type B to take a leading role since this is the only classical case that encompasses all of the nonlinear phenomenon of [13].

More broadly, we remark that some flavor of duality plays a key role in other important results in the field. A second possibility then is to interpret nonlinear duality as a bridge for extending existing theory to nonlinear groups. As mentioned above, Vogan duality is central to the results of [2] suggesting a uniform duality theory might allow an extension of the Langlands formalism to nonlinear groups. Foundations for this approach are described by Adams and Trapa in [4]. Another example is the theory of character lifting. Adams and Herb describe character lifting for nonlinear simply laced groups in [3]. Their work possesses formal properties closely related to the duality of [4] suggesting one could use a nonlinear duality theory to create a general notion of character lifting from linear to nonlinear groups.

CHAPTER 2

NOTATION AND PRELIMINARIES

2.1 Notation

Throughout this dissertation $\mathbb{G} = \text{Spin}(2n+1, \mathbb{C})$ will denote the connected, simply connected complex simple Lie group of type B_n and $G = \text{Spin}(n+1, n)$ will denote its (connected) split real form. Corresponding Lie algebras will be denoted by \mathfrak{g} and $\mathfrak{g}_{\mathbb{R}}$ with similar notation used for subgroups and subalgebras. Let Θ be a Cartan involution for G and $K = G^{\Theta}$ be the corresponding maximal compact subgroup. If H is a Θ -stable Cartan subgroup of G , write $H = TA$ for its decomposition into compact and vector pieces. Let $\Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system and $W(\mathfrak{g}, \mathfrak{h})$ the algebraic Weyl group. Finally, the Killing form (\cdot, \cdot) is a natural inner product on \mathfrak{h} allowing us to identify $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$. For some calculations it will be convenient to use (\cdot, \cdot) to view roots and coroots as living in the same vector space.

Many of the results in this dissertation depend on the formal properties of $\Delta(\mathfrak{g}, \mathfrak{h})$ and $W(\mathfrak{g}, \mathfrak{h})$, and not on how these objects were constructed. For this reason we find it convenient to treat these objects abstractly whenever possible, invoking the connection to the above Lie groups only when necessary. Therefore we fix once and for all an *abstract* Cartan subalgebra $\mathfrak{h}^a \subset \mathfrak{g}$ and view our calculations as taking place in $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ and $W = W(\mathfrak{g}, \mathfrak{h}^a)$. We then use conjugation in \mathfrak{g} to compare roots and Weyl groups from different Cartan subalgebras in the usual way.

2.2 The Set \mathcal{D}_{χ}

The discussion in this section is valid whenever G is a reductive linear group with abelian Cartan subgroups. Let $\mathcal{HC}(\mathfrak{g}, K)$ denote the category of Harish-Chandra modules for G and suppose $\mathcal{Z}(\mathfrak{g})$ is the center of its universal enveloping algebra. Each irreducible object $X \in \mathcal{HC}(\mathfrak{g}, K)$ has a corresponding infinitesimal character χ of $\mathcal{Z}(\mathfrak{g})$ and we will only consider X for which χ is nonsingular. For any maximal torus $\mathbb{H} \subset \mathbb{G}$, the Harish-Chandra isomorphism

$$\rho : \mathcal{Z}(\mathfrak{g}) \rightarrow S(\mathfrak{h})^{W(\mathfrak{g}, \mathfrak{h})}$$

allows us to identify infinitesimal characters with $W(\mathfrak{g}, \mathfrak{h})$ orbits in \mathfrak{h}^* .

Write $\mathcal{HC}(\mathfrak{g}, K)_\chi$ for the full subcategory of $\mathcal{HC}(\mathfrak{g}, K)$ consisting of modules with infinitesimal character χ and let $\mathcal{KHC}(\mathfrak{g}, K)_\chi$ denote its Grothendieck group. The irreducible objects in $\mathcal{HC}(\mathfrak{g}, K)_\chi$ (or equivalently their distribution characters) form a natural basis of $\mathcal{KHC}(\mathfrak{g}, K)_\chi$ that we wish to understand. Typically this is achieved with the help of a second basis given by (distribution characters of) equivalence classes of *standard modules* [16]. Loosely, standard modules are representations induced from discrete series on cuspidal parabolic subgroups of G and for the purposes of this dissertation we assume these objects are known. Each standard module in $\mathcal{HC}(\mathfrak{g}, K)_\chi$ has a canonical irreducible subquotient creating a one-to-one correspondence between the standard and irreducible modules.

We will frequently need an explicit parameterization for the standard (or irreducible) objects in $\mathcal{HC}(\mathfrak{g}, K)_\chi$. To this end, let \mathcal{D}_χ be the set of K -conjugacy classes of triples (H, ϕ, Γ) , where $H = TA$ is a Θ -stable Cartan subgroup of G , ϕ is in the $W(\mathfrak{g}, \mathfrak{h})$ -orbit determined by χ (viewed in \mathfrak{h}^*), and Γ is a character of T whose differential is determined by ϕ . Specifically, we must have

$$d\Gamma = \phi|_{\mathfrak{t}} + \rho_i^\phi - 2\rho_{ic}^\phi$$

where ρ_i^ϕ and ρ_{ic}^ϕ are the half sums of positive (with respect to ϕ) imaginary and compact imaginary roots respectively. We refer to ([19], Chapter 3) for more details. Although technically redundant, we will occasionally write $(H, \phi, \Gamma)_\chi$ for a representative triple in \mathcal{D}_χ when we wish to emphasize the infinitesimal character χ .

The set \mathcal{D}_χ exactly parameterizes the standard (or irreducible) elements in $\mathcal{HC}(\mathfrak{g}, K)_\chi$ [16] and will be referred to as the set of *linear parameters* for G at χ . The following proposition gets us started understanding the set \mathcal{D}_χ .

Proposition 2.2.1 ([8]). *The are finitely many K -conjugacy classes of Θ -stable Cartan subgroups in G . Moreover there is a one-to-one correspondence between K -conjugacy classes of Θ -stable Cartan subgroups and G -conjugacy classes of Cartan subgroups.*

The proposition allows us to work with a finite set of Θ -stable Cartan representatives $\{H_i\}$. We get a corresponding finite partition

$$\mathcal{D}_\chi = \coprod_i \mathcal{D}_\chi^i$$

where \mathcal{D}_χ^i is the set of all triples in \mathcal{D}_χ whose first element is conjugate to H_i . Let Λ_i be the set of elements in \mathfrak{h}_i^* that are in the $W(\mathfrak{g}, \mathfrak{h}_i)$ -orbit determined by χ . Clearly the normalizer $N_K(H_i)$ acts on Λ_i and the stabilizer of any element is $Z_K(H_i)$. Let

$$W(G, H_i) = N_K(H_i)/Z_K(H_i)$$

be the *real Weyl group* of H_i . Then $|W(G, H_i)|$ is finite and $W(G, H_i)$ can be viewed as a subgroup of $W(\mathfrak{g}, \mathfrak{h})$. Since $|\Lambda_i| = |W(\mathfrak{g}, \mathfrak{h})|$ it follows

$$|W(\mathfrak{g}, \mathfrak{h})/W(G, H_i)|$$

is the number of distinct pairs (H_i, ϕ) up to K -conjugacy. Conjugacy class of such pairs are parameterized in Chapter 9.

Finally, the differential of Γ must be compatible with $\phi \in \mathfrak{h}_i^*$ and depending on the pair (H_i, ϕ) , a compatible Γ need not exist. Assuming one does, there are as many choices for Γ as there are connected components in H_i , and it is well known that $\pi_0(H_i) \cong \mathbb{Z}_2^k$ for some k . Therefore \mathcal{D}_χ^i is finite and can be viewed as a product of $W(G, H_i)$ -orbits in Λ_i with a finite 2-group (assuming ϕ is sufficiently integral).

The upshot of this discussion is that \mathcal{D}_χ is a finite set that can be parameterized in terms of certain structure theoretic information for G . In Chapters 3, 4, and 5 we recall how to determine the number of conjugacy classes of Cartan subgroups in G , their component groups, and their corresponding real Weyl groups.

2.3 The Set $\widetilde{\mathcal{D}}_\chi$

The complex group \mathbb{G} is simply connected, however the real form $G = \text{Spin}(n+1, n)$ has fundamental group \mathbb{Z}_2 . In particular, the unique nonlinear double cover $\widetilde{G} = \widetilde{\text{Spin}}(n+1, n)$ is a central extension of G and we have a short exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \widetilde{G} \rightarrow G \rightarrow 1.$$

Let $\pi : \widetilde{G} \rightarrow G$ be the projection map and follow the usual convention that preimages of subgroups under π are denoted by adding a tilde. For example, $\widetilde{K} := \pi^{-1}(K)$ is a maximal compact subgroup of \widetilde{G} .

A Cartan subgroup $\tilde{H} \subset \tilde{G}$ is defined to be the centralizer of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. As the notation suggests, $\tilde{H} = \pi^{-1}(H)$ where H is a Cartan subgroup of G and the real Weyl group

$$W(\tilde{G}, \tilde{H}) = N_{\tilde{G}}(\tilde{H})/Z_{\tilde{G}}(\tilde{H})$$

is naturally isomorphic to $W(G, H)$. These facts allow us to reduce many questions about Cartan subgroups in \tilde{G} to equivalent ones about Cartan subgroups in G .

Let $\mathcal{HC}(\mathfrak{g}, \tilde{K})$ denote the category of Harish-Chandra modules for \tilde{G} and fix a triple $(\tilde{H}, \phi, \tilde{\Gamma})$ for \tilde{G} as in Section 2.2. We say a module (respectively triple) is *genuine* if the action of -1 (respectively $\tilde{\Gamma}(-1)$) is nontrivial. Let $\mathcal{HC}(\mathfrak{g}, \tilde{K})^{\text{gen}} \subset \mathcal{HC}(\mathfrak{g}, \tilde{K})$ denote the full subcategory of genuine modules in $\mathcal{HC}(\mathfrak{g}, \tilde{K})$. The following proposition implies the irreducible genuine objects in $\mathcal{HC}(\mathfrak{g}, \tilde{K})^{\text{gen}}$ are parameterized in the same fashion as G (see [13] for the appropriate general statement).

Proposition 2.3.1 ([13], Proposition 6.1). *The genuine standard (or irreducible) objects in $\mathcal{HC}(\mathfrak{g}, \tilde{K})^{\text{gen}}$ are parameterized by \tilde{K} -conjugacy classes of genuine triples $(\tilde{H}, \phi, \tilde{\Gamma})$.*

For nonsingular χ , let $\mathcal{HC}(\mathfrak{g}, \tilde{K})_{\chi}^{\text{gen}}$ denote the full subcategory of genuine modules with infinitesimal character χ . Our goal is to understand the set $\tilde{\mathcal{D}}_{\chi}$ of \tilde{K} -conjugacy classes of genuine triples $(\tilde{H}, \phi, \tilde{\Gamma})$, where the orbit of $\phi \in \mathfrak{h}^*$ is determined by χ . The set $\tilde{\mathcal{D}}_{\chi}$ will be referred to as the set of *genuine parameters* for \tilde{G} at infinitesimal character χ . We will mainly be interested in certain symmetric half-integral infinitesimal characters whose description we postpone for now.

Although understanding $\tilde{\mathcal{D}}_{\chi}$ will ultimately require different techniques, for some aspects our results for G are sufficient. For example, since \tilde{G} is a central extension of G , the \tilde{K} -conjugacy classes of Cartan subgroups in \tilde{G} are in one-to-one correspondence with the K -conjugacy classes of Cartan subgroups in G . These conjugacy classes will be parameterized in Chapter 3.

Given a Cartan subgroup \tilde{H}_i , the infinitesimal character χ determines a $W(\mathfrak{g}, \mathfrak{h}_i)$ -orbit $\Lambda_i \subset \mathfrak{h}_i^*$. The group $W(\tilde{G}, \tilde{H}_i)$ stabilizes \tilde{H}_i and thus acts on Λ_i with finitely many orbits. In the linear case, assuming χ was sufficiently integral each such orbit corresponded to at least one irreducible module. In the nonlinear case, stronger (half) integrality conditions on χ prevent this from happening in general. We address these issues in Chapter 6.

Finally we note that Cartan subgroups of \tilde{G} need not be abelian. Hence their representations may be more complicated than the characters of Cartan subgroups in G . In

particular, if we fix a Cartan subgroup $\tilde{H} \subset \tilde{G}$ and $\phi \in \mathfrak{h}^*$, we cannot necessarily count genuine triples $(\tilde{H}, \phi, \tilde{\Gamma})$ by simply counting the connected components of \tilde{H} . We are thus forced to confront nonabelian Cartan subgroups and their (genuine) representations. We handle these issues in Chapters 7 and 8.

2.4 Character Multiplicity Duality

If $\gamma \in \tilde{\mathcal{D}}_\chi$ is a genuine parameter for \tilde{G} , we will denote by $\text{std}(\gamma)$ and $\text{irr}(\bar{\gamma})$ the corresponding standard and irreducible modules in $\mathcal{HC}(\mathfrak{g}, \tilde{K})_\chi^{\text{gen}}$ respectively. Write $m(\bar{\gamma}, \delta) \in \mathbb{N}$ for the number of times $\text{irr}(\bar{\gamma})$ appears as a subquotient of $\text{std}(\delta)$. In the group $\mathcal{KHC}(\mathfrak{g}, \tilde{K})_\chi^{\text{gen}}$ we have

$$\text{std}(\delta) = \sum_{\gamma \in \tilde{\mathcal{D}}_\chi} m(\bar{\gamma}, \delta) \text{irr}(\bar{\gamma}).$$

Similarly write $M(\gamma, \bar{\delta}) \in \mathbb{Z}$ for the multiplicity of $\text{std}(\gamma)$ in $\text{irr}(\bar{\delta})$ and

$$\text{irr}(\bar{\delta}) = \sum_{\gamma \in \tilde{\mathcal{D}}_\chi} M(\gamma, \bar{\delta}) \text{std}(\gamma)$$

in $\mathcal{KHC}(\mathfrak{g}, \tilde{K})_\chi^{\text{gen}}$. The uniqueness of the above expressions implies

$$\sum_{\pi \in \tilde{\mathcal{D}}_\chi} M(\gamma, \bar{\pi}) m(\bar{\pi}, \delta) = \begin{cases} 1 & \gamma = \delta \\ 0 & \gamma \neq \delta \end{cases}$$

so that the matrices m and M are inverses. The integers $M(\gamma, \bar{\delta})$ are thus of fundamental importance. We can now state the main theorem we aim to prove (see Theorem 12.8.5).

Theorem 2.4.1. *Let λ be a symmetric infinitesimal character (Section 6.2) and suppose the rank of $\tilde{G} = \widetilde{\text{Spin}}(n+1, n)$ is even. Fix a genuine central character $\tilde{\Gamma}$ of $Z(\tilde{G})$ and let $\mathcal{B} = \{\gamma_1, \dots, \gamma_r\} \subset \tilde{\mathcal{D}}_\lambda$ be the collection of genuine parameters in $\tilde{\mathcal{D}}_\lambda$ with central character $\tilde{\Gamma}$. If $\mathcal{B}' = \tilde{\mathcal{D}}_\lambda \setminus \mathcal{B}$, then there is a bijection $\Psi : \mathcal{B} \rightarrow \mathcal{B}'$ such that*

$$M(\gamma_i, \bar{\gamma}_j) = \epsilon_{ij} m(\overline{\Psi(\gamma_j)}, \Psi(\gamma_i))$$

where $\epsilon_{ij} = \pm 1$.

Central characters are discussed in Section 11.4 and the map Ψ will be defined on various sets throughout the course of these notes (Definitions 3.2.4, 12.7.1, and Proposition 11.5.2). Theorem 2.4.1 will be proved in Section 12.8 and is essentially a formal consequence of the structure and representation theoretic properties of the map Ψ (Theorems 12.7.3, 12.7.4, and 12.7.5).

PART I

LINEAR PARAMETERS

CHAPTER 3

INVOLUTIONS IN W

We begin with the structure theory necessary for describing the set \mathcal{D}_χ . Let H be a Θ -stable Cartan subgroup of $G = \text{Spin}(n+1, n)$. The complexified Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is also Θ -stable and Θ acts on $\Delta(\mathfrak{g}, \mathfrak{h})$. Since G contains a compact Cartan subgroup, the action of Θ on $\Delta(\mathfrak{g}, \mathfrak{h})$ is equivalent to the regular action of an order two element $\tau \in W(\mathfrak{g}, \mathfrak{h})$ [8]. Choose a conjugation map

$$i : \mathfrak{h} \rightarrow \mathfrak{h}^a$$

(Section 2.1) and write $i(\tau) = \tau_{\mathfrak{h}}^a \in W(\mathfrak{g}, \mathfrak{h}^a)$ for the induced involution on $\Delta(\mathfrak{g}, \mathfrak{h}^a)$. Then $\tau_{\mathfrak{h}}^a$ and $\tau_{\mathfrak{h}'}^a$ are conjugate in $W(\mathfrak{g}, \mathfrak{h}^a)$ if and only if H and H' are K -conjugate in G [8]. Therefore we have a well-defined injection from K -conjugacy classes of Θ -stable Cartan subgroups of G into conjugacy classes of $W(\mathfrak{g}, \mathfrak{h}^a)$ consisting of elements of order two. Since the group G is split, this map is also surjective and conjugacy classes of involutions in $W(\mathfrak{g}, \mathfrak{h}^a)$ exactly parameterize K -conjugacy classes of Cartan subgroups of G . In this section we study involutions in an abstract Weyl group W of type B_n .

3.1 Abstract Root System and Weyl Group

Let $\Delta \subset (\mathfrak{h}^a)^*$ be a root system of type B_n with the usual choice of coordinates and inner product on $(\mathfrak{h}^a)^*$, i.e.,

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$$

with $(e_i, e_j) = \delta_{ij}$. We have

$$|\Delta| = 4 \binom{n}{2} + 2n = 4 \frac{n(n-1)}{2} + 2n = 2n^2 - 2n + 2n = 2n^2.$$

Let Δ^+ denote the usual choice of positive roots

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$$

and Π the corresponding set of simple roots

$$\Pi = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_n\}.$$

Note that all of the simple roots are long except one. If W denotes the Weyl group, then

$$W \cong \mathbb{Z}_2^n \rtimes S_n \cong \langle s_\alpha \mid \alpha \in \Pi \rangle$$

where s_α is the root reflection in the simple root α . We will write an element $w \in W$ as $(\epsilon_1 \epsilon_2 \dots \epsilon_n, \sigma)$, where $\epsilon_i \in \{0, 1\}$ and $\sigma \in S_n$. The ϵ_i appearing in such an expression will be referred to as *bits*.

Definition 3.1.1. Given $w \in W$ define

$$\begin{aligned} S_w &= \{\epsilon_i \mid \sigma(i) = i \text{ and } \epsilon_i = 0\} \\ R_w &= \{\epsilon_i \mid \sigma(i) = i \text{ and } \epsilon_i = 1\} \\ C_w &= \{\epsilon_i \mid \sigma(i) \neq i\} \end{aligned}$$

and

$$\begin{aligned} n_s^w &= |S_w| \\ n_r^w &= |R_w| \\ n_c^w &= |C_w|. \end{aligned}$$

Then n_s^w counts the number of bits fixed by σ that are equal to zero, n_r^w counts the number fixed bits equal to one, and n_c^w counts the number of bits not fixed by σ . Note that $n = n_s^w + n_r^w + n_c^w$. When the element $w \in W$ is clear from context, we may write n_s for n_s^w and so forth.

It is often important to know when w possesses certain properties. For this purpose we define the following *indicator bits*

$$\begin{aligned} \epsilon_s^w &= \begin{cases} 0 & n_s^w = 0 \\ 1 & n_s^w \neq 0 \end{cases} \\ \epsilon_r^w &= \begin{cases} 0 & n_r^w = 0 \\ 1 & n_r^w \neq 0 \end{cases} \\ \epsilon_p^w &= \begin{cases} 0 & n_s^w \text{ is even} \\ 1 & n_s^w \text{ is odd} \end{cases} \\ \epsilon_m^w &= \begin{cases} 0 & n_r^w \text{ is even} \\ 1 & n_r^w \text{ is odd} \end{cases} . \end{aligned}$$

The root reflection corresponding to the simple root $\alpha = e_i - e_{i+1}$ is $(0 \dots 0, s_\alpha)$, where $s_\alpha = (i \ i+1)$. Therefore the long simple roots correspond to simple transpositions and thus generate the S_n piece of W . The remaining simple root is short and maps to $(0 \dots 01, 1)$. In general let $\mathcal{I} \subset W$ denote the elements of order two in W . The elements of \mathcal{I} are called *involutions*. In the next section we discuss convenient ways of representing involutions.

3.2 Representations of Involutions

Since we have chosen coordinates for our root system, we get a corresponding faithful matrix representation of the Weyl group. At times it will be convenient to treat elements of \mathcal{I} as matrices, so we quickly describe the correspondence. Given an involution $\theta = (\epsilon_1 \epsilon_2 \dots \epsilon_n, \sigma) \in \mathcal{I}$ the corresponding matrix M_θ is

$$M_\theta^{ij} = \begin{cases} 1, & \sigma(i) = j \text{ and } \epsilon_i = \epsilon_j = 0 \\ -1, & \sigma(i) = j \text{ and } \epsilon_i = \epsilon_j = 1 \\ 0, & \text{otherwise} \end{cases}.$$

The matrices arising this way are clearly symmetric and it is easy to deduce the following equalities

$$\begin{aligned} n_s^\theta &= |\{i \mid M_\theta^{ii} = 1\}| \\ n_r^\theta &= |\{i \mid M_\theta^{ii} = -1\}| \\ n_c^\theta &= |\{i \mid M_\theta^{ii} = 0\}|. \end{aligned}$$

Example 3.2.1. If $\theta = (0001, (23))$ then

$$M_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Although the matrix representation of an involution is straightforward, it is somewhat cumbersome for large values of n . We now introduce a way of representing involutions in terms of *diagrams*. The idea is to represent the action of an involution on the standard basis of $(\mathfrak{h}^a)^*$ (equivalently the short roots in Δ^+) in terms of a picture. Let $\theta \in \mathcal{I}$ and let e_i denote the i th standard basis vector. A diagram will be a sequence of symbols (read left to right) where the i th symbol represents the action of θ on e_i .

Example 3.2.2. The diagram for $\theta = (0001, (23))$ is given by

$$+ 3 2 - .$$

Here the symbol $+$ represents that e_1 is fixed by θ and the symbol $-$ represents that e_4 is sent to $-e_4$ by θ . The vectors e_3 and e_2 are interchanged by θ and this is represented by placing the number 3 in position 2 and the number 2 in position 3.

It is also possible for θ to interchange and negate two standard basis vectors. Negation of interchanged vectors will be represented in diagrams by placing parentheses around the corresponding numbers.

Example 3.2.3. If $\theta = (1110, (23))$ then

$$M_\theta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the corresponding diagram is

$$- (3) (2) + .$$

Given $\theta \in \mathcal{I}$, let D_θ denote its corresponding diagram. Then the number of $+$ signs appearing in D_θ is n_s^θ , the number of $-$ signs appearing in D_θ is n_r^θ , and the number of numbers appearing in D_θ is n_c^θ (Definition 3.1.1). We now come to an important definition.

Definition 3.2.4. Let $\theta = (\epsilon_1 \epsilon_2 \dots \epsilon_n, \sigma) \in \mathcal{I}$ and define

$$\Psi(\theta) = (\tilde{\epsilon}_1 \tilde{\epsilon}_2 \dots \tilde{\epsilon}_n, \sigma)$$

where $\tilde{\epsilon}_i = \epsilon_i + 1$. Then Ψ is an involution of \mathcal{I} called the *bit flip* involution. For matrices we have $-M_\theta = M_{\Psi(\theta)}$ and for diagrams Ψ interchanges $+$ and $-$ signs as well as changes the parentheses on transpositions. The examples above represent two diagrams interchanged by Ψ .

In particular, Ψ is a natural involution on the set \mathcal{I} . Extending Ψ to other representation theoretic data is a major focus of this dissertation.

3.3 Properties of Involutions

We have seen it is important to understand the W -conjugacy classes of \mathcal{I} . We begin with the following proposition whose proof is left to the reader.

Proposition 3.3.1. *Two involutions $\theta_1 = (\epsilon_1\epsilon_2\dots\epsilon_n, \sigma_1)$ and $\theta_2 = (\epsilon_1'\epsilon_2'\dots\epsilon_n', \sigma_2')$ are conjugate in W if and only if σ_1 and σ_2 are conjugate in S_n and $n_s^{\theta_1} = n_s^{\theta_2}$ (equivalently $n_r^{\theta_1} = n_r^{\theta_2}$). In other words the conjugacy class of θ is uniquely determined by the numbers n_c^θ and n_s^θ .*

In particular, each conjugacy class of involutions in S_n has a corresponding set of conjugacy classes in \mathcal{I} . Fix $\theta = (\epsilon_1\epsilon_2\dots\epsilon_n, \sigma) \in \mathcal{I}$. If $\sigma = 1$, Proposition 3.3.1 implies the conjugacy class of θ is determined by the number of nonzero ϵ_i . Clearly there are $n + 1$ possibilities and thus $n + 1$ corresponding conjugacy classes. If σ is an involution that permutes $2k$ bits, then Proposition 3.3.1 implies the number of corresponding conjugacy classes is $n - 2k + 1$ and we have the following corollary.

Corollary 3.3.2. *In the setting above*

$$|\mathcal{I}/W| = \sum_{k=0}^{\lfloor n/2 \rfloor} n - 2k + 1.$$

If n is even, this number is a perfect square.

Recall the bit flip involution $\Psi : \mathcal{I} \rightarrow \mathcal{I}$ (Definition 3.2.4). Since $n_c^{\Psi(\theta)} = n_c^\theta$, Proposition 3.3.1 implies Ψ descends to an involution (still denoted Ψ) on \mathcal{I}/W . In particular, Ψ sends the conjugacy class determined by (n_c^θ, n_s^θ) to the one determined by (n_c^θ, n_r^θ) . Note that Ψ will have fixed points in \mathcal{I}/W if and only if n is even (in which case it will have $\frac{n}{2} + 1$ of them).

Example 3.3.3. Consider the case $n = 4$. Corollary 3.3.2 implies that $|\mathcal{I}/W| = 9$. We can view \mathcal{I}/W as

$$\begin{array}{ccc} & & (1111, 1) \\ & & (1111, (12)) \quad (1110, 1) \\ (0000, (12)(34)) & (1110, (12)) & (1100, 1) \\ & (0000, (12)) & (0001, 1) \\ & & (0000, 1) \end{array}$$

where an element of \mathcal{I}/W is labeled by a representative in \mathcal{I} . Here each column corresponds to a different conjugacy class of involutions in S_n and Ψ reflects the diagram vertically about the middle row. The three elements on the middle row represent conjugacy classes in \mathcal{I} fixed by Ψ .

In terms of diagrams, Proposition 3.3.1 implies the conjugacy class of θ is determined by the number of numbers (n_c^θ) and the number of + signs (n_s^θ) that appear. Therefore a general element of \mathcal{I} is conjugate to one whose diagram is of the form

$$2 \ 1 \ 4 \ 3 \cdots + \ + \ \cdots + \ - \ - \cdots - .$$

Definition 3.3.4. Let $\theta \in \mathcal{I}$ and recall θ acts on $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$. We define

$$\begin{aligned} \Delta_i^\theta &= \{\alpha \in \Delta \mid \theta(\alpha) = \alpha\} \\ \Delta_{\mathbb{R}}^\theta &= \{\alpha \in \Delta \mid \theta(\alpha) = -\alpha\} \\ \Delta_{\mathbb{C}}^\theta &= \{\alpha \in \Delta \mid \theta(\alpha) \neq \pm\alpha\} \end{aligned}$$

the *imaginary, real, and complex roots* respectively. The imaginary and real roots form subsystems of Δ and we denote their corresponding Weyl (sub)groups by W_i^θ and $W_{\mathbb{R}}^\theta$. The set $\Delta_{\mathbb{C}}^\theta$ is *not* a root system, however if we write

$$\begin{aligned} \rho_i &= \frac{1}{2} \sum_{\alpha \in (\Delta_i^\theta)^+} \alpha \\ \rho_r &= \frac{1}{2} \sum_{\alpha \in (\Delta_{\mathbb{R}}^\theta)^+} \alpha \\ \Delta_{\mathbb{C}^\perp}^\theta &= \{\alpha \in \Delta \mid (\alpha, \rho_i) = (\alpha, \rho_r) = 0\} \end{aligned}$$

then $\Delta_{\mathbb{C}^\perp}^\theta$ is a root system consisting of complex roots (see [17], Definition 3.10).

Proposition 3.3.5 ([17], Lemma 3.11). *In the setting of Definition 3.3.4, we can write*

$$\Delta_{\mathbb{C}^\perp}^\theta = \Delta_1 \cup \Delta_2$$

as an orthogonal disjoint union with $\theta(\Delta_1) = \Delta_2$. In particular,

$$W_{\mathbb{C}}^\theta = \{(w, \theta w) \mid w \in W(\Delta_1)\}$$

is a Weyl subgroup of W isomorphic to $W(\Delta_1)$.

Proposition 3.3.6. *Let $\theta \in \mathcal{I}$ and let $k = \frac{n_c^\theta}{2}$. Then we have*

$$\begin{aligned} W_i^\theta &\cong W(B_{n_s^\theta}) \times W(A_1)^k \\ W_{\mathbb{R}}^\theta &\cong W(B_{n_r^\theta}) \times W(A_1)^k \\ W_{\mathbb{C}}^\theta &\cong W(A_{k-1}) \end{aligned}$$

where $W(A_i)$ and $W(B_i)$ are Weyl groups for root systems of type A_i and B_i respectively.

Example 3.3.7. Consider again the case $n = 4$. Since the isomorphism classes of W_i^θ , $W_{\mathbb{R}}^\theta$, and $W_{\mathbb{C}}^\theta$ depend only on the conjugacy class of θ , we may view the action of Ψ by drawing a picture similar to the one of Example 3.3.3. We draw the same picture but replace the label θ for an element of \mathcal{I}/W by the triple $(R_i^\theta, R_{\mathbb{R}}^\theta, R_{\mathbb{C}}^\theta)$, where $R_i^\theta, R_{\mathbb{R}}^\theta, R_{\mathbb{C}}^\theta$ are the types of the imaginary, real, and complex Weyl groups for θ respectively.

$$\begin{array}{rcc}
 & & (B_0, B_4, 1) \\
 & (A_1, B_2 \times A_1, 1) & (B_1, B_3, 1) \\
 (A_1^2, A_1^2, A_1) & (A_1^2, A_1^2, 1) & (B_2, B_2, 1) \\
 & (B_2 \times A_1, A_1, 1) & (B_3, B_1, 1) \\
 & & (B_4, B_0, 1)
 \end{array}$$

Recall that Ψ is the vertical reflection about the middle row. It is clear that the involution Ψ interchanges Δ_i^θ and $\Delta_{\mathbb{R}}^\theta$ and thus interchanges W_i^θ and $W_{\mathbb{R}}^\theta$. Note the symmetry of elements in the middle row.

Let W^θ denote the centralizer in W of the element $\theta \in \mathcal{I}$. Corollary 3.3.2 and the following theorem allow us to count elements of order two in W .

Theorem 3.3.8 ([17], Proposition 3.12). *The subgroups W_i^θ and $W_{\mathbb{R}}^\theta$ are normal subgroups in W^θ . Moreover we have*

$$W^\theta \cong (W_i^\theta \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta.$$

The number of elements in W conjugate to θ is the number of elements in W divided by the number of elements in W^θ . Using Proposition 3.3.6 and Theorem 3.3.8 we easily obtain the following corollary.

Corollary 3.3.9. *If $\theta \in \mathcal{I}$, the number of involutions in W conjugate to θ is given by*

$$\frac{n!}{(n - n_c^\theta)! (\frac{n_c^\theta}{2})!} \binom{n - n_c^\theta}{n_s^\theta}.$$

Fix an even number $0 \leq k \leq n$ and consider the set

$$X_k = \left\{ \theta \in \mathcal{I} \mid n_c^\theta = k \right\}.$$

Proposition 3.3.1 implies Ψ preserves X_k and thus permutes its $n - k + 1$ conjugacy classes. Note the left term in Corollary 3.3.9 depends only on the numbers k and n and is therefore constant for fixed X_k . Hence the order two symmetry observed in the map Ψ appears numerically as the familiar symmetry of binomial coefficients.

Example 3.3.10. Again let $n = 4$. Then the X_k defined above correspond to the columns appearing in the diagram of Example 3.3.3 (with X_0 corresponding to the rightmost column). Recall that Ψ is the vertical reflection about the middle row and thus obviously preserves the columns. Here we draw the same picture but replace the label for an element of \mathcal{I}/W by the actual number of involutions in the conjugacy class

$$\begin{array}{cccc}
 & & & 1 \\
 & & & 12 & 4 \\
 & & 12 & 24 & 6 \\
 & & & 12 & 4 \\
 & & & & 1.
 \end{array}$$

The fact that the columns are numerically symmetric is a result of the binomial coefficient symmetry discussed above.

CHAPTER 4

STRUCTURE OF REAL CARTAN SUBGROUPS

Let H be a Θ -stable Cartan subgroup of $G = \text{Spin}(n+1, n)$. Then H is a real form of a complex torus and it is well known

$$H \cong (S^1)^a \times (\mathbb{R}^\times)^b \times (\mathbb{C}^\times)^c$$

for some numbers a, b, c with $a + b + 2c = n$. Moreover H determines a conjugacy class of involutions in \mathcal{I} (Chapter 3) via conjugation of Θ to $\Delta(\mathfrak{g}, \mathfrak{h}^a)$. It is a remarkable fact that a representative involution for H in \mathcal{I} can be used to recover the numbers a, b, c . In this section we describe this process and compute the structure of all Θ -stable Cartan subgroups in G . The calculations are carried out in the abstract root system and Weyl group of Chapter 3.

4.1 Lattices and Rank Two

Recall $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ is a root system of type B_n with Weyl group $W = W(\mathfrak{g}, \mathfrak{h}^a)$. We begin with some definitions.

Definition 4.1.1. Let $\alpha \in \Delta$ be a root and write

$$\begin{aligned} \alpha^\vee &= \frac{2\alpha}{(\alpha, \alpha)} \\ \Delta^\vee &= \{\alpha^\vee \mid \alpha \in \Delta\} \end{aligned}$$

for the *coroots* of Δ . In coordinates we have

$$\Delta^\vee = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}$$

and it is well known that Δ^\vee is a root system. The *root* and *coroot lattices* are the \mathbb{Z} -modules generated by Δ and Δ^\vee and will be denoted by $L(\Delta)$ and $L(\Delta^\vee)$ respectively. Similarly the *weight lattice* in $(\mathfrak{h}^a)^*$ is given by

$$X = \{\lambda \in (\mathfrak{h}^a)^* \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}.$$

Let $\theta \in \mathcal{I}$ be an involution and suppose $\alpha \in \Delta$. Then

$$\theta(\alpha^\vee) = \theta \left(\frac{2\alpha}{(\alpha, \alpha)} \right) = \frac{2\theta(\alpha)}{(\theta(\alpha), \theta(\alpha))} = \theta(\alpha)^\vee$$

and we have $\theta(\Delta^\vee) = \Delta^\vee$. Moreover, if $\lambda \in X$ then

$$(\theta(\lambda), \alpha^\vee) = (\lambda, \theta(\alpha^\vee)) = (\lambda, \theta(\alpha)^\vee) \in \mathbb{Z}$$

and $\theta(X) = X$.

Definition 4.1.2. In the setting above we define

$$\begin{aligned} X_+^\theta &= \{\lambda \in X \mid \theta(\lambda) = \lambda\} \\ X_-^\theta &= \{\lambda \in X \mid \theta(\lambda) = -\lambda\} \\ Y &= X/2X \end{aligned}$$

so that $Y \cong \mathbb{Z}_2^n$. Let $\pi : X \rightarrow Y$ denote the corresponding quotient map and write

$$\begin{aligned} Y_+^\theta &= \pi(X_+^\theta) \\ Y_-^\theta &= \pi(X_-^\theta) \\ Y_\pm^\theta &= Y_+^\theta \cap Y_-^\theta. \end{aligned}$$

Given $\theta \in \mathcal{I}$, let $H(\theta) \subset G$ be a Θ -stable Cartan subgroup for which there exists a conjugation map

$$i : \mathfrak{h} \rightarrow \mathfrak{h}^a$$

with $i(\Theta) = \theta$. The structure of $H(\theta)$ is given by the following remarkable theorem. A proof is sketched at the end of Section 8.3. See also [8].

Theorem 4.1.3. *In the setting above we have*

$$H(\theta) \cong (S^1)^a \times (\mathbb{R}^\times)^b \times (\mathbb{C}^\times)^c$$

where $c = \dim_{\mathbb{Z}_2}(Y_\pm^\theta)$, $a = \dim_{\mathbb{Z}_2}(Y_+^\theta) - c$, and $b = \dim_{\mathbb{Z}_2}(Y_-^\theta) - c$.

In the next section we use Theorem 4.1.3 to compute the structure of the Θ -stable Cartan subgroups in G . All of the interesting aspects of this calculation can be seen in rank two.

Example 4.1.4. In the setting above, suppose $n = 2$. Then we have

$$\begin{aligned} X &= \left\langle \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right) \right\rangle \\ 2X &= \left\langle \left(\begin{array}{c} 2 \\ 0 \end{array} \right), \left(\begin{array}{c} 1 \\ 1 \end{array} \right) \right\rangle. \end{aligned}$$

In particular, X consists of vectors whose coordinates are all in \mathbb{Z} or all in $\mathbb{Z} + \frac{1}{2}$ and $2X$ consists of vectors whose coordinates are all even or all odd.

Case I. Suppose $M_\theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (Section 3.2). Then $\dim_{\mathbb{Z}_2}(Y_+^\theta) = 2$ and $\dim_{\mathbb{Z}_2}(Y_\pm^\theta) = 0$ and we have $H(\theta) \cong (S^1)^2$.

Case II. Suppose $M_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\begin{aligned} X_+^\theta &= \left\langle \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \right\rangle \\ X_-^\theta &= \left\langle \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \right\rangle \end{aligned}$$

and $\dim_{\mathbb{Z}_2}(Y_+^\theta) = 1$ and $\dim_{\mathbb{Z}_2}(Y_-^\theta) = 1$. However

$$\left(\begin{array}{c} 1 \\ 0 \end{array} \right) - \left(\begin{array}{c} 0 \\ 1 \end{array} \right) = \left(\begin{array}{c} 1 \\ -1 \end{array} \right)$$

is an element of $2X$ since its coordinates are all odd. Therefore $\dim_{\mathbb{Z}_2}(Y_\pm^\theta) = 1$ and we see $H(\theta) \cong \mathbb{C}^\times$.

Case III. Suppose $M_\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned} X_+^\theta &= \left\langle \left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right) \right\rangle \\ X_-^\theta &= \left\langle \left(\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \end{array} \right) \right\rangle \end{aligned}$$

and $\dim_{\mathbb{Z}_2}(Y_+^\theta) = 1$ and $\dim_{\mathbb{Z}_2}(Y_-^\theta) = 1$. However

$$\left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right) - \left(\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right)$$

is *not* an element of $2X$ since one coordinate is even and one is odd. Therefore $\dim_{\mathbb{Z}_2}(Y_\pm^\theta) = 0$ and we see that $H(\theta) \cong S^1 \times \mathbb{R}^\times$.

Case IV. Suppose $M_\theta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\dim_{\mathbb{Z}_2}(Y_-^\theta) = 2$ and $\dim_{\mathbb{Z}_2}(Y_\pm^\theta) = 0$ so that $H(\theta) \cong (\mathbb{R}^\times)^2$.

It is easy to check these cases represent the four conjugacy classes of involutions W . In particular, we have computed the structure of all Cartan subgroups appearing in the linear group $\text{Spin}(3, 2)$.

4.2 The General Case

Let $G = \text{Spin}(n+1, n)$ with $n \geq 2$ and retain the notation from the previous section. It is easy to check

$$\begin{aligned} X &= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \vdots \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle \\ 2X &= \left\langle \begin{pmatrix} 2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \right\rangle. \end{aligned}$$

In particular, the same characterization of X and $2X$ from Example 4.1.4 holds in general. Let $\theta = (\epsilon_1 \epsilon_2 \dots \epsilon_n, \sigma) \in \mathcal{I}$ be an arbitrary involution suppose $H(\theta)$ is a corresponding Cartan subgroup. We proceed in the same manner as Example 4.1.4.

Case I. Suppose $\sigma = 1$. Up to conjugation we may assume D_θ is of the form

$$\begin{array}{ccccccc} + & + & \cdots & + & - & - & \cdots & - \\ & & & n_s^\theta & n_{s+1}^\theta & & & n_s^\theta + n_r^\theta \end{array}$$

where $n_s^\theta + n_r^\theta = n$ (Definition 3.1.1). If $n_s^\theta = n$ or $n_r^\theta = n$ we have $H(\theta) \cong (S^1)^n$ or $H(\theta) \cong (\mathbb{R}^\times)^n$ respectively. If $n_s^\theta \neq 0$ and $n_r^\theta \neq 0$ set

$$\begin{aligned} v_+ &= \sum_{i=1}^{n_s^\theta} e_i \\ v_- &= \sum_{i=n_s^\theta+1}^n e_i. \end{aligned}$$

Then $v_+ \in X_+^\theta$, $v_- \in X_-^\theta$, and we have $v_+ - v_- \in 2X$ since all coordinate entries are odd. Therefore $\dim_{\mathbb{Z}_2}(Y_\pm^\theta) = 1$ and

$$H(\theta) \cong (S^1)^{n_s^\theta-1} \times (\mathbb{R}^\times)^{n_r^\theta-1} \times \mathbb{C}^\times.$$

Case II. Suppose $\sigma \neq 1$, $k = \frac{n_c^\theta}{2}$, and $n_r^\theta = 0$. Up to conjugation we may assume D_θ is of the form

$$2 \ 1 \ 4 \ 3 \ \cdots \ + \ + \ \cdots \ + .$$

Then X_-^θ is generated by k vectors of the form $v_-^i = e_i - e_{i+1}$ for odd $1 \leq i < k$ and X_+^θ is similarly generated by k vectors of the form $v_+^i = e_i + e_{i+1}$. However $v_+^i - v_-^i = 2e_{i+1} \in 2X$ since all coordinate entries are even. Therefore every negative eigenvector projects to something in Y_\pm^θ and we have

$$H(\theta) \cong (S^1)^{n_s^\theta} \times (\mathbb{C}^\times)^k.$$

Case III. Suppose $\sigma \neq 1$, $k = \frac{n_c^\theta}{2}$, and $n_s^\theta = 0$. Up to conjugation we may assume D_θ is of the form

$$2 \ 1 \ 4 \ 3 \ \cdots \ - \ - \ \cdots \ - .$$

Applying the argument from Case II gives

$$H(\theta) \cong (\mathbb{R}^\times)^{n_r^\theta} \times (\mathbb{C}^\times)^k.$$

Case IV. Suppose $\sigma \neq 1$, $n_s^\theta \neq 0$, $n_r^\theta \neq 0$, and $k = \frac{n_c^\theta}{2}$. Up to conjugation we may assume D_θ is of the form

$$2 \ 1 \ 4 \ 3 \ \cdots \ + \ + \ \cdots \ + \ - \ - \ \cdots \ - .$$

Combining the arguments of Cases I and II gives

$$H(\theta) \cong (S^1)^{n_s^\theta - 1} \times (\mathbb{R}^\times)^{n_r^\theta - 1} \times (\mathbb{C}^\times)^{k+1}.$$

Case V. Suppose $\sigma \neq 1$ and $n_s^\theta = n_r^\theta = 0$. All such involutions are conjugate and exist if and only if n is even. Up to conjugation we may assume D_θ is of the form

$$2 \ 1 \ 4 \ 3 \ \cdots \ n \ [n - 1].$$

This case is more subtle than the previous cases because of the existence of half integral vectors. In particular, the vectors

$$\{e_1 + e_2, e_3 + e_4, \dots, e_{n-1} + e_n\}$$

are *not* linearly independent when viewed as elements of Y since their sum is a vector whose coordinate entries are all odd. Therefore we must consider the vectors

$$\begin{aligned} v_+ &= \sum_{i=1}^n \frac{e_i}{2} \\ v_- &= \sum_{i=1}^n (-1)^{i+1} \frac{e_i}{2}. \end{aligned}$$

Then $v_+ \in X_+^\theta$, $v_- \in X_-^\theta$, and

$$v_+ - v_- = \sum_{i=1}^{\frac{n}{2}} e_{2i} \notin 2X$$

since it has even and odd coordinate entries. In particular, there is one positive and one negative dimension in Y that is not contained in Y_\pm^θ and we conclude

$$H(\theta) \cong S^1 \times \mathbb{R}^\times \times (\mathbb{C}^\times)^{\frac{n}{2}-1}.$$

We summarize these results in the following corollary.

Corollary 4.2.1. *In the setting above, let $\theta \in \mathcal{I}$, $k = \frac{n^\theta}{2}$, and recall the indicator bits $\epsilon_s^\theta, \epsilon_r^\theta$ of Definition 3.1.1. Then the number of \mathbb{C}^\times factors for $H(\theta)$ is given by*

$$k + \epsilon_s^\theta + \epsilon_r^\theta - 1.$$

Proof. Check this in each of the above cases. □

Remark 4.2.2. For each $\theta \in \mathcal{I}$ we have computed the structure of $H(\theta)$. Note that it is possible for $H(\theta)$ and $H(\theta')$ to be isomorphic even if θ and θ' are not conjugate in W . For example, the involutions given by

$$+ \ - \ - \ - \ \text{and} \ (2) \ (1) \ - \ -$$

each correspond to Cartan subgroups with structure $(\mathbb{R}^\times)^2 \times \mathbb{C}^\times$. Interestingly, the preimages of these subgroups in \tilde{G} are *not* isomorphic.

4.3 The Action of Ψ

In Chapter 3 we attached several pieces of structural data to elements of \mathcal{I}/W . In each case we observed this data had a two-fold symmetry induced by the map Ψ . In this section

we have attached the structure of a real Cartan subgroup to elements of \mathcal{I}/W and we again find a symmetry induced by Ψ . To see this, let $\theta \in \mathcal{I}$ be a representative for an element of \mathcal{I}/W . We clearly have

$$\begin{aligned} X_+^{\Psi(\theta)} &= X_-^\theta \\ X_-^{\Psi(\theta)} &= X_+^\theta \end{aligned}$$

so that

$$\begin{aligned} Y_+^{\Psi(\theta)} &= Y_-^\theta \\ Y_-^{\Psi(\theta)} &= Y_+^\theta \\ Y_\pm^{\Psi(\theta)} &= Y_\pm^\theta \end{aligned}$$

and

$$\begin{aligned} \dim_{\mathbb{Z}_2}(Y_+^{\Psi(\theta)}) &= \dim_{\mathbb{Z}_2}(Y_-^\theta) \\ \dim_{\mathbb{Z}_2}(Y_-^{\Psi(\theta)}) &= \dim_{\mathbb{Z}_2}(Y_+^\theta) \\ \dim_{\mathbb{Z}_2}(Y_\pm^{\Psi(\theta)}) &= \dim_{\mathbb{Z}_2}(Y_\pm^\theta). \end{aligned}$$

The upshot is the real Cartan subgroups $H(\Psi(\theta))$ and $H(\theta)$ will have the same number of \mathbb{C}^\times factors and interchanged numbers of S^1 and \mathbb{R}^\times factors.

Example 4.3.1. Suppose $n = 4$ and let $\theta \in \mathcal{I}$ be a representative for an element of \mathcal{I}/W . We draw the same picture as in Example 3.3.3 replacing θ by the structure of $H(\theta)$.

$$\begin{array}{ccc} & & (\mathbb{R}^\times)^4 \\ & & (\mathbb{R}^\times)^2 \times \mathbb{C}^\times \\ S^1 \times \mathbb{R}^\times \times \mathbb{C}^\times & \begin{array}{cc} (\mathbb{R}^\times)^2 \times \mathbb{C}^\times & (\mathbb{R}^\times)^2 \times \mathbb{C}^\times \\ \mathbb{C}^\times \times \mathbb{C}^\times & S^1 \times \mathbb{R}^\times \times \mathbb{C}^\times \\ (S^1)^2 \times \mathbb{C}^\times & (S^1)^2 \times \mathbb{C}^\times \end{array} & (\mathbb{R}^\times)^4 \\ & & (S^1)^4 \end{array}$$

Notice that vertical reflection about the middle row interchanges the S^1 and \mathbb{R}^\times factors and the elements on the middle row are again symmetric.

CHAPTER 5

REAL WEYL GROUPS

Let H be a Θ -stable Cartan subgroup of G and recall the *real Weyl group*

$$W(G, H) = N_K(H)/Z_K(H)$$

from Section 2.2. In this section we determine the (isomorphism classes of) real Weyl groups for each conjugacy class of Cartan subgroups in G . Ultimately we again conjugate to our abstract setting and thus determine $W(G, H)$ as a subgroup of $W = W(\mathfrak{g}, \mathfrak{h}^a)$.

5.1 Gradings

The main subtlety in describing the group $W(G, H)$ concerns the imaginary roots. In this section we recall additional structure of the imaginary root system that leads to a convenient description of the real Weyl group for H ([17], Chapter 3).

Definition 5.1.1. Let Δ be a root system. A *grading* on Δ is a map

$$\varepsilon : \Delta \rightarrow \mathbb{Z}_2$$

such that

$$\varepsilon(\alpha) = \varepsilon(-\alpha)$$

for all $\alpha \in \Delta$. Moreover if α, β , and $\alpha + \beta$ are in Δ we require

$$\varepsilon(\alpha + \beta) = \varepsilon(\alpha) + \varepsilon(\beta).$$

To see how gradings arise naturally, let H be Θ -stable Cartan subgroup of G . If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is imaginary, its corresponding root space \mathfrak{g}_α is fixed by Θ and entirely contained in its positive or negative eigenspace. We say the imaginary root α is *compact* if \mathfrak{g}_α is contained in the positive eigenspace and *noncompact* if it is contained in the negative eigenspace.

Proposition 5.1.2 ([17]). *Let H be a Θ -stable Cartan subgroup of G and let $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ denote the imaginary roots for H with respect to Θ . Define a map $\varepsilon : \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathbb{Z}_2$ via*

$$\varepsilon(\alpha) = \begin{cases} 0 & \alpha \text{ compact} \\ 1 & \alpha \text{ noncompact} \end{cases}$$

Then ε is a grading on $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$.

Corollary 5.1.3. *The set $\Delta_{ic}^\Theta(\mathfrak{g}, \mathfrak{h})$ of imaginary compact roots is itself a root system.*

Definition 5.1.4. Let $H \subset G$ be a Θ -stable Cartan subgroup. Proposition 5.1.2 gives a natural grading η on the imaginary roots $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ determined by Θ . Choose a conjugation map

$$i : \mathfrak{h} \rightarrow \mathfrak{h}^a$$

and write ε for the induced *abstract grading* $\varepsilon(i(\alpha)) = \eta(\alpha)$ of $\Delta_i = \Delta_i(\mathfrak{g}, \mathfrak{h}^a)$. We say an abstract imaginary root α is compact if $\varepsilon(i(\alpha)) = 0$ and noncompact if $\varepsilon(i(\alpha)) = 1$.

We now return to our abstract root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ and Weyl group $W = W(\mathfrak{g}, \mathfrak{h}^a)$. Let θ be an involution in W and suppose ε is an abstract grading of Δ_i^θ . Definition 5.1.1 implies ε is determined by its values on a set of simple imaginary roots and, a priori, all 2^n choices are allowed. However we are interested in abstract gradings coming from the construction of Definition 5.1.4, and it turns out not all possibilities arise in practice.

To see this, recall Δ_i^θ is a root system of type $B_m \times A_1^k$, where $m = n_s^\theta$ and $k = \frac{n_c^\theta}{2}$ (Proposition 3.3.6). The long imaginary roots that generate the A_1^k factor can each be written as a sum of two short complex roots interchanged by θ . It is easy to see such roots must be noncompact. More generally, since the group G is (quasi)split, there must exist a positive system for Δ_i^θ in which *all* simple roots are noncompact [17]. A grading satisfying these conditions is said to be *principal* ([17], Definition 6.3). In these notes, abstract imaginary gradings are assumed to be principal unless otherwise stated.

It will be convenient to incorporate principal abstract gradings into our involution diagrams (Section 3.2). Let $\theta \in \mathcal{I}$ and suppose ε is a principal grading for Δ_i^θ . The above discussion suggests we need only modify the portion of the diagram corresponding to the B_m factor of Δ_i^θ . In a root system of type B_m , any grading is determined by its values on a set of positive short roots. Since these are exactly the roots represented by $+$ signs in the diagram for θ , a grading can be described by indicating the $+$ signs that represent noncompact roots. We do this by adding a circle around the corresponding $+$ signs.

Example 5.1.5. Suppose $m = 6$ and $\theta = 1$ so that all roots are imaginary. If

$$\varepsilon(e_i) = \begin{cases} 1 & 1 \leq i \leq 3 \\ 0 & 4 \leq i \leq 6 \end{cases}$$

is an abstract grading, the corresponding diagram $D_\theta(\varepsilon)$ is

$$\oplus \oplus \oplus + + + .$$

Remark 5.1.6. In terms of diagrams, it is easy to check that a grading is principal if and only if exactly half ($\frac{m+1}{2}$ if m is odd) of the $+$ signs are circled. So the grading in the above diagram is principal, however the grading

$$+ \oplus + - - -$$

is not.

5.2 Real Weyl Groups

Fix an abstract involution θ and suppose ε is an abstract grading of Δ_i^θ . Write Δ_{ic}^θ for the set of compact imaginary roots with respect to ε and let W_{ic}^θ be its Weyl group. If ε is principal, we have seen there is a positive system of Δ_i^θ for which all simple roots are noncompact and thus Δ_{ic}^θ is exactly the set of even height imaginary roots. In this setting, it is easy to write down their structure.

Proposition 5.2.1. *In the situation above*

$$W_{ic}^\theta \cong \begin{cases} \mathbb{Z}_2 & m = 2 \\ W(D_{\frac{m}{2}}) \times W(B_{\frac{m}{2}}) & m \text{ is even, } m > 2 \\ W(D_{\frac{m+1}{2}}) \times W(B_{\frac{m-1}{2}}) & m \text{ is odd, } m > 2 \end{cases}$$

where D_m represents a root system of type D .

Example 5.2.2. Let $n = 4$, $\theta = 1$, and suppose ε is a principal imaginary grading of the form

$$+ \oplus + \oplus .$$

Then the set of (positive) compact roots for ε is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Here the first two roots are positive for the D_2 system and the last four are positive for the B_2 system.

Recall $H(\theta)$ denotes a Cartan subgroup of G whose induced abstract involution is conjugate to θ (Section 4.1) and Theorem 3.3.8 implies the centralizer of θ in W has the form

$$W^\theta \cong (W_i^\theta \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta.$$

The following proposition shows that W^θ and $W(G, H(\theta))$ are closely related.

Proposition 5.2.3 ([17], Proposition 4.16). *The group $W(G, H(\theta))$ is a subgroup of W^θ and we have*

$$W(G, H(\theta)) \cong (\mathcal{R} \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta.$$

Here $\mathcal{R} \subset W_i^\theta$ is of the form

$$\mathcal{R} \cong A \rtimes W_{ic}^\theta$$

and A is an elementary abelian two-group.

Proposition 5.2.4. *The group A from Proposition 5.2.3 is of the form*

$$A \cong \mathbb{Z}_2^k$$

where k is the number of \mathbb{C}^\times factors in the Cartan $H(\theta)$ (Section 4.2).

Proof. This is proved in Section 9.3. □

Although we will make no use of them, we may as well show how the ideas of this section and the last can be used to enumerate linear parameters for G in a nontrivial case.

Example 5.2.5. Fix $n = 4$. We are now in a position to compute $|\mathcal{D}_\chi^i|$ for $\mathcal{KHC}(\mathfrak{g}, K)_\chi$, when χ is regular and integral. Let H_i be a Θ -stable Cartan subgroup of G . Then we have

$$|\mathcal{D}_\chi^i| = \frac{|W(\mathfrak{g}, \mathfrak{h})|}{|W(G, H_i)|} \times |\pi_0(H_i)|$$

and in terms of abstract groups we have

$$|\mathcal{D}_\chi^i| = \frac{|W(\mathfrak{g}, \mathfrak{h}^a)|}{|W(G, H(\theta_i))|} \times |\pi_0(H(\theta_i))|.$$

Also

$$|W(\mathfrak{g}, \mathfrak{h})| = 2^4 \times 4!.$$

The form of $W(G, H(\theta_i))$ is given by Propositions 5.2.3 and the necessary details are filled in by Propositions 5.2.4 and 5.2.1. The structure theory needed for this case was

computed in Examples 4.3.1 and 3.3.7. Hence we simply choose a representative involution θ and principal imaginary grading ε for each element of \mathcal{I}/W and compute.

Case I. Suppose $D_\theta(\varepsilon)$ is of the form

$$+ \oplus + \oplus .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 1 \times (2^1 \cdot 2!) \times (2^2 \cdot 2!) \\ |W_{\mathbb{R}}^\theta| &= 1 \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 1 \end{aligned}$$

so that

$$|\mathcal{D}_x^{H(\theta)}| = \frac{2^4 \cdot 4!}{2^5 \cdot 1 \cdot 1} \times 1 = 12.$$

Case II. Suppose $D_\theta(\varepsilon)$ is of the form

$$\oplus + \oplus - .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 2 \times (2^1 \cdot 2!) \times 2 \\ |W_{\mathbb{R}}^\theta| &= 2 \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 1 \end{aligned}$$

so that

$$|\mathcal{D}_x^{H(\theta)}| = \frac{2^4 \cdot 4!}{2^4 \cdot 2 \cdot 1} \times 1 = 12.$$

Case III. Suppose $D_\theta(\varepsilon)$ is of the form

$$+ \oplus - - .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 2 \times 2 \\ |W_{\mathbb{R}}^\theta| &= 2^2 \cdot 2! \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 2 \end{aligned}$$

so that

$$\left| \mathcal{D}_x^{H(\theta)} \right| = \frac{2^4 \cdot 4!}{2^2 \cdot 2^3 \cdot 1} \times 2 = 24.$$

Case IV. Suppose $D_\theta(\varepsilon)$ is of the form

$$\oplus \quad - \quad - \quad - \quad .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 2 \times 1 \\ |W_{\mathbb{R}}^\theta| &= 2^3 \cdot 3! \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 2^2 \end{aligned}$$

so that

$$\left| \mathcal{D}_x^{H(\theta)} \right| = \frac{2^4 \cdot 4!}{2 \cdot 48 \cdot 1} \times 2^2 = 16.$$

Case V. Suppose $D_\theta(\varepsilon)$ is of the form

$$- \quad - \quad - \quad - \quad .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 1 \\ |W_{\mathbb{R}}^\theta| &= 2^4 \cdot 4! \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 2^4 \end{aligned}$$

so that

$$\left| \mathcal{D}_x^{H(\theta)} \right| = \frac{2^4 \cdot 4!}{1 \cdot 2^4 \cdot 4! \cdot 1} \times 2^4 = 16.$$

Case VI. Suppose $D_\theta(\varepsilon)$ is of the form

$$2 \ 1 \ + \ \oplus \ .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 2 \times 2 \\ |W_{\mathbb{R}}^\theta| &= 2 \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 1 \end{aligned}$$

so that

$$|\mathcal{D}_X^{H(\theta)}| = \frac{2^4 \cdot 4!}{2^2 \cdot 2 \cdot 1} \times 1 = 48.$$

Case VII. Suppose $D_\theta(\varepsilon)$ is of the form

$$2 \ 1 \ \oplus \ - \ .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 2^2 \times 1 \\ |W_{\mathbb{R}}^\theta| &= 2 \cdot 2 \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 1 \end{aligned}$$

so that

$$|\mathcal{D}_X^{H(\theta)}| = \frac{2^4 \cdot 4!}{2^2 \cdot 2^2 \cdot 1} \times 1 = 24.$$

Case VIII. Suppose $D_\theta(\varepsilon)$ is of the form

$$2 \ 1 \ - \ - \ .$$

Then

$$\begin{aligned} |\mathcal{R}| &= 2 \times 1 \\ |W_{\mathbb{R}}^\theta| &= 2^2 \cdot 2! \cdot 2 \\ |W_{\mathbb{C}}^\theta| &= 1 \\ |\pi_0(H(\theta))| &= 2^2 \end{aligned}$$

so that

$$|\mathcal{D}_X^{H(\theta)}| = \frac{2^4 \cdot 4!}{2 \cdot 2^4 \cdot 1} \times 4 = 48.$$

Case IX. Suppose $D_\theta(\varepsilon)$ is of the form

$$2 \ 1 \ 4 \ 3.$$

Then

$$\begin{aligned} |\mathcal{R}| &= 2 \times 1 \\ |W_{\mathbb{R}}^\theta| &= 2^2 \\ |W_{\mathbb{C}}^\theta| &= 2 \\ |\pi_0(H(\theta))| &= 2 \end{aligned}$$

so that

$$|\mathcal{D}_x^{H(\theta)}| = \frac{2^4 \cdot 4!}{2 \cdot 2^2 \cdot 2} \times 2 = 48.$$

Finally we have

$$|\mathcal{D}_x| = 12 + 12 + 24 + 16 + 16 + 48 + 24 + 48 + 48 = 248.$$

PART II

NONLINEAR PARAMETERS

CHAPTER 6

HALF INTEGRAL INFINITESIMAL CHARACTER

In the previous sections we computed the structure theoretic information needed to parameterize representations of G at regular integral infinitesimal character. We now turn to the problem of parameterizing genuine representations of the nonlinear group \tilde{G} . In particular we focus on the necessary structure theoretic ingredients.

6.1 Conjugation to \mathfrak{h}^a

Conjugation to \mathfrak{h}^a has been our main tool for relating algebraic structures associated with different Cartan subalgebras in \mathfrak{g} . To this point, maps relating Cartan subalgebras have been specified only up to Weyl group conjugation. This was appropriate since we were primarily concerned with the structure theory of Cartan subgroups. However, from now on we will be working with finer structure and are thus forced to be more precise about our conjugation maps.

Fix a nonsingular element $\lambda \in (\mathfrak{h}^a)^*$. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} , $\phi \in \mathfrak{h}^*$, and suppose λ and ϕ define the same infinitesimal character (Section 2.2). Then there is an inner automorphism

$$i_{\lambda, \phi} : \mathbb{G} \rightarrow \mathbb{G}$$

whose differential induces a map (also denoted $i_{\lambda, \phi}$)

$$\begin{aligned} (\mathfrak{h}^a)^* &\rightarrow \mathfrak{h}^* \\ \lambda &\rightarrow \phi. \end{aligned}$$

The map $i_{\lambda, \phi}$ is not unique, however the restriction of any two such maps to $(\mathfrak{h}^a)^*$ is the same. Hence we have a well-defined family of maps $\{i_{\lambda, \phi'}\}$ taking $(\mathfrak{h}^a)^*$ to \mathfrak{h}^* , where $\phi' \in \mathfrak{h}^*$ is in the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of ϕ . We also write $i_{\lambda, \phi}$ for the induced maps on Weyl groups, root systems, and so forth. When the element λ is fixed or clear from context, we will often just write i_ϕ .

6.2 Half-Integral Infinitesimal Characters

Our first task is to make precise the specific infinitesimal characters in which we are interested. The Harish-Chandra isomorphism allows us to specify an infinitesimal character by selecting a $W(\mathfrak{g}, \mathfrak{h}^a)$ -orbit in $(\mathfrak{h}^a)^*$. Since we have fixed a positive system for $\Delta(\mathfrak{g}, \mathfrak{h}^a)$ (Section 3.1), a nonsingular infinitesimal character is uniquely determined by an element of the corresponding dominant chamber. For the purpose of conjugation, it will often be convenient to specify an infinitesimal character via a dominant element of $(\mathfrak{h}^a)^*$. In this context we may refer to a dominant nonsingular element of $(\mathfrak{h}^a)^*$ as an infinitesimal character.

Fix an infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$, a Θ -stable Cartan subgroup $\tilde{H} \subset \tilde{G}$, and suppose there exists a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ as in Chapter 2. Write $\eta : \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathbb{Z}_2$ for the corresponding grading of $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ (Proposition 5.1.2). We have the following important definition.

Definition 6.2.1. The *abstract triple* associated to $(\tilde{H}, \phi, \tilde{\Gamma})$ is the 3-tuple $(\theta, \varepsilon, \lambda)$, where

$$\begin{aligned}\theta &= i_\phi^{-1} \cdot \Theta \cdot i_\phi \\ \varepsilon(\alpha) &= \eta(i_\phi(\alpha))\end{aligned}$$

with α an abstract imaginary root for θ . In particular, θ is an involution of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ corresponding to Θ and ε is a principal grading of Δ_i^θ .

It turns out existence of the genuine representation $\tilde{\Gamma}$ places restrictions on the abstract λ that are allowed.

Proposition 6.2.2 ([4]). *Let $(\tilde{H}, \phi, \tilde{\Gamma})$ be a genuine triple and let $(\theta, \varepsilon, \lambda)$ be the corresponding abstract triple. If $\alpha \in \Delta$ is a long imaginary root for θ then*

$$(\lambda, \alpha^\vee) \in \begin{cases} \mathbb{Z} & \varepsilon(\alpha) = 0 \\ \mathbb{Z} + \frac{1}{2} & \varepsilon(\alpha) = 1 \end{cases} .$$

If α is a long complex root for θ , then we have

$$(\lambda, \alpha^\vee + \theta(\alpha^\vee)) \in \begin{cases} \mathbb{Z} & (\alpha, \theta(\alpha^\vee)) = 0 \\ \mathbb{Z} + \frac{1}{2} & (\alpha, \theta(\alpha^\vee)) \neq 0 \end{cases} .$$

Define a half-integer to be a number x such that $2x \in \mathbb{Z}$ and a *strict* half-integer to be an element of $\mathbb{Z} + \frac{1}{2}$. If α is a long imaginary root, the proposition implies that λ must pair with α to be a half-integer. Moreover, the pairing must be integral or strict

half-integral depending on whether α is compact or noncompact. Note the proposition gives no restriction for long real roots or any type of short root.

If $\alpha \in \Delta^+$ is a long positive root, then in coordinates $\alpha^\vee = e_i \pm e_j$ for some i, j . If α is imaginary for θ , then in order to satisfy the first part of the proposition we see that $2(\lambda_i \pm \lambda_j) \in \mathbb{Z}$ with $\lambda_i = (\lambda, e_i)$. If we further assume that all roots are imaginary (i.e., $\theta = 1$), then we see that all coordinates for λ must have this property. In particular, we make the following assumption.

Assumption 1. From now on our infinitesimal character λ is assumed to be half-integral when written in abstract coordinates.

We can restrict λ even more if we take into account our grading ε . Suppose e_i and e_j are imaginary roots. Then $e_i + e_j$ is also imaginary and we have

$$(\lambda, e_i + e_j) = (\lambda, e_i) + (\lambda, e_j) = \lambda_i + \lambda_j.$$

Now $e_i + e_j$ will be noncompact if and only if exactly one of $\{e_i, e_j\}$ is noncompact. In this case, Proposition 6.2.2 implies $\lambda_i + \lambda_j \in \mathbb{Z} + \frac{1}{2}$ and thus exactly one of $\{\lambda_i, \lambda_j\}$ must be a strict half-integer. Conversely, if $e_i + e_j$ is compact then we must have $\lambda_i + \lambda_j \in \mathbb{Z}$ and thus either $\lambda_i, \lambda_j \in \mathbb{Z}$ or $\lambda_i, \lambda_j \in \mathbb{Z} + \frac{1}{2}$. Therefore the coordinates of λ corresponding to the short compact roots must all be of the same type (integers or strict half-integers) and the coordinates corresponding to the short noncompact roots must all be of the opposite type.

Again consider the case $\theta = 1$ and recall the grading ε is principal. This means exactly half ($\frac{n+1}{2}$ if n is odd) of the short roots must be noncompact (Section 5.1). The above argument then implies exactly half ($\frac{n+1}{2}$ if n is odd) of the coordinates of λ should be either integers or strict half-integers. A half-integral infinitesimal character with this property is said to be *symmetric*.

Assumption 2. We assume our half-integral infinitesimal character λ is symmetric.

Example 6.2.3. Let $n = 4, \theta = 1$, and $\lambda = (3, 2, \frac{3}{2}, \frac{1}{2})$. Then λ is half-integral and symmetric. If ε is a grading defined by one of the following diagrams

$$\begin{array}{cccc} \oplus & \oplus & + & + \\ + & + & \oplus & \oplus \end{array}$$

then the triple $(\theta, \varepsilon, \lambda)$ satisfies the conditions of Proposition 6.2.2. Conversely, abstract triples defined via

$$\begin{array}{cccc}
\oplus & + & \oplus & + \\
+ & \oplus & + & \oplus \\
\oplus & + & + & \oplus \\
+ & \oplus & \oplus & +
\end{array}$$

do not.

Example 6.2.4. Now let $n = 6$ and $\lambda = (5, 4, 3, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$. Since Proposition 6.2.2 places no restrictions on real roots, gradings of the form

$$\begin{array}{cccccc}
\oplus & \oplus & - & - & + & + \\
+ & + & - & - & \oplus & \oplus
\end{array}$$

are allowed under Proposition 6.2.2. However the following gradings are not

$$\begin{array}{cccccc}
\oplus & + & \oplus & + & - & - \\
\oplus & + & - & - & \oplus & + \\
- & - & \oplus & + & \oplus & + \\
- & \oplus & + & \oplus & + & - .
\end{array}$$

6.3 Supportable Orbits

To each genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})$ for \tilde{G} , we have constructed a corresponding triple of abstract data $(\theta, \varepsilon, \lambda)$ satisfying the conditions of Proposition 6.2.2. An abstract triple satisfying these conditions is said to be *supportable*. Supportable triples are thus abstract manifestations of genuine triples.

Let $\lambda \in (\mathfrak{h}^a)^*$ be a fixed symmetric infinitesimal character and let θ be an involution of Δ . We would like to understand when there exists a principal imaginary grading ε such that the abstract triple $(\theta, \varepsilon, \lambda)$ is supportable. When such a grading exists we say λ is *θ -symmetric* or θ is *supportable* for λ .

To begin, recall the indicator bits ϵ_s^θ , ϵ_r^θ , ϵ_p^θ , and ϵ_m^θ (Definition 3.1.1) and assume $n_c^\theta = 0$. Since ε must be principal, we need to choose exactly $\frac{n_s^\theta + \epsilon_p^\theta}{2}$ of the short imaginary roots to be noncompact. This implies exactly $\frac{n_s^\theta + \epsilon_p^\theta}{2}$ of the imaginary coordinates of λ must be either integers or strict half-integers. The remaining imaginary coordinates must then be of the opposite type.

Example 6.3.1. Let $n = 4$ and $\lambda = (3, 2, \frac{3}{2}, \frac{1}{2})$. Then λ is θ -symmetric for the following involutions

$$\begin{array}{cccc}
+ & + & + & + \\
+ & + & + & - \\
- & + & + & + \\
+ & - & + & - \\
+ & - & - & - \\
- & - & - & -
\end{array}$$

but not for the involutions

$$\begin{array}{cccc} + & + & - & - \\ - & - & + & + . \end{array}$$

In the first case, the following diagrams represent the gradings for which the abstract triple $(\theta, \varepsilon, \lambda)$ is supportable

$$\begin{array}{cccc} \oplus & \oplus & + & + \text{ or } + & + & \oplus & \oplus \\ \oplus & \oplus & + & - & & & \\ - & + & \oplus & \oplus & & & \\ \oplus & - & + & - \text{ or } + & - & \oplus & - \\ \oplus & - & - & - & & & \\ - & - & - & - . & & & \end{array}$$

Remark 6.3.2. When θ is supportable for λ and n_s^θ is even, the situation is symmetric and there will always be two choices for ε . Note that symmetric and θ -symmetric are the same if $\theta = 1$.

Now suppose $n_c^\theta \neq 0$. Then we have

$$\theta(e_i) = e_j$$

for some i and j . This implies $e_i + e_j$ is noncompact imaginary and thus

$$(\lambda, e_i + e_j) = \lambda_i + \lambda_j$$

must be a strict half-integer. Therefore exactly one of $\{\lambda_i, \lambda_j\}$ must be a pure integer. Clearly λ can be θ -symmetric only if it satisfies this property for all such i, j .

Example 6.3.3. Once again we let $n = 4$ and $\lambda = (3, 2, \frac{3}{2}, \frac{1}{2})$. Then λ is θ -symmetric for the following involutions

$$\begin{array}{ccc} + & 3 & 2 & + \\ + & 3 & 2 & - \\ - & 3 & 2 & - \\ 4 & 3 & 2 & 1 \end{array}$$

but not for the involutions

$$\begin{array}{ccc} 2 & 1 & - & - \\ 2 & 1 & 4 & 3. \end{array}$$

Here are the gradings that work for the first case

$$\begin{array}{cccc} \oplus & 3 & 2 & + \text{ or } + & 3 & 2 & \oplus \\ \oplus & 3 & 2 & - & & & \\ - & 3 & 2 & - & & & \\ 4 & 3 & 2 & 1. & & & \end{array}$$

Fix a symmetric regular infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and let \tilde{H} be a Θ -stable Cartan subgroup of \tilde{G} . Let $\phi \in \mathfrak{h}^*$ and suppose λ and ϕ define the same infinitesimal character. We would like to know when there exists a genuine triple of the form $(\tilde{H}, \phi, \tilde{\Gamma})$. Conjugating the pair (\tilde{H}, ϕ) by $i_{\lambda, \phi}^{-1}$ gives an abstract triple $(\theta, \varepsilon, \lambda)$ and Proposition 6.2.2 provides a necessary condition on $(\theta, \varepsilon, \lambda)$. A wonderful fact is that this condition is also sufficient.

Proposition 6.3.4 ([4]). *In the situation above, a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})$ exists if and only if the abstract triple $(\theta, \varepsilon, \lambda)$ is supportable.*

Let Λ be the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of ϕ in \mathfrak{h}^* . Then for each $W(G, H)$ -orbit in Λ we have a corresponding abstract triple $(\theta, \varepsilon, \lambda)$. This is well-defined since $W(G, H)$ preserves θ and the set of compact roots (see Proposition 9.1.2). A $W(G, H)$ -orbit is said to be *supportable* if its corresponding abstract triple is supportable. The proposition implies there exists a genuine triple corresponding to a $W(G, H)$ -orbit if and only if the orbit is supportable. This is in contrast to the linear case where each $W(G, H)$ -orbit in Λ corresponds to a triple (H, ϕ, Γ) provided the infinitesimal character is sufficiently integral. $W(G, H)$ -orbits in Λ will be discussed in detail in Chapter 9.

Example 6.3.5. Let $n = 4$ and $\theta = 1$. In Example 5.2.5 we saw there were 12 orbits in Λ mapping onto $\binom{4}{2} = 6$ possible abstract diagrams of the form

$$\begin{array}{cccc} \oplus & \oplus & + & + \\ + & + & \oplus & \oplus \\ \oplus & + & \oplus & + \\ + & \oplus & + & \oplus \\ \oplus & + & + & \oplus \\ + & \oplus & \oplus & + . \end{array}$$

If we choose $\lambda = (3, 2, \frac{3}{2}, \frac{1}{2})$, then only the first two diagrams are supportable. Therefore there will be many orbits in Λ that do not correspond to a genuine representation of \tilde{G} .

CHAPTER 7

GENUINE TRIPLES FOR \tilde{H}^s

Fix a regular symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and let \tilde{H} be a Θ -stable Cartan subgroup in \tilde{G} . Let $\phi \in \mathfrak{h}^*$ and suppose λ and ϕ define the same infinitesimal character. Denote the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of ϕ in \mathfrak{h}^* by Λ . In trying to parameterize representations for \tilde{G} , the replacement for the integrality of λ is the supportability conditions given by Proposition 6.2.2. If σ is the $W(\tilde{G}, \tilde{H})$ -orbit (equivalently $W(G, H)$ -orbit) in Λ containing ϕ , we first check to see if σ is supportable. Assuming it is, we know there exists at least one genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})$ by Proposition 6.3.4. In this section and the next we develop the structure theory needed to determine how many such triples exist.

7.1 Genuine Central Characters

Let H be a Θ -stable Cartan subgroup of G and let \tilde{H} be its double cover in \tilde{G} . Write H_0 for the identity component of H and similarly write $(\tilde{H})_0$ for the identity component of \tilde{H} . Although \tilde{H} may not be abelian, we do have the following lemma.

Lemma 7.1.1. *$(\tilde{H})_0$ is central (and thus abelian) in \tilde{H} .*

Proof. Let g and h be elements in H . Since H is abelian, we have

$$[g, h] = ghg^{-1}h^{-1} = 1.$$

Therefore the commutator of any two elements in \tilde{H} lands in $\{\pm 1\}$ and thus elements of $(\tilde{H})_0$ must have trivial commutator. \square

Remark 7.1.2. Lemma 7.1.1 implies noncommuting elements in \tilde{H} live in distinct connected components.

Let $\Pi_g(\tilde{H})$ denote the set of (equivalence classes of) irreducible genuine representations of \tilde{H} and let $\Pi_g(Z(\tilde{H}))$ denote the same set for $Z(\tilde{H})$. The connection between these two sets is given by the following proposition.

Proposition 7.1.3 ([1], Proposition 2.2). *There is a bijection*

$$\pi : \Pi_g(Z(\tilde{H})) \rightarrow \Pi_g(\tilde{H})$$

sending an element $\chi \in \Pi_g(Z(\tilde{H}))$ to $\pi(\chi) \in \Pi_g(\tilde{H})$. Here $\pi(\chi)$ is the unique element of $\Pi_g(\tilde{H})$ satisfying $\pi(\chi)|_{Z(\tilde{H})} = n\chi$, where $n = \left| \tilde{H}/Z(\tilde{H}) \right|^{\frac{1}{2}}$ is the dimension of $\pi(\chi)$.

Choose a symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and suppose there exists a corresponding genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})$. Proposition 7.1.3 implies the possibilities for $\tilde{\Gamma}$ are parameterized by (genuine) central characters of \tilde{H} compatible with ϕ (Section 2.2). However, $(\tilde{H})_0$ is central in \tilde{H} by Lemma 7.1.1 and the behavior of $\tilde{\Gamma}$ on $(\tilde{H})_0$ is uniquely specified by ϕ . In particular, the distinct possibilities for $\tilde{\Gamma}$ are determined by the structure of the connected components of \tilde{H} . The analysis of this structure is complicated and will occupy the next several sections.

7.2 The Group M

We begin our analysis with some elementary structure theory in G . We follow closely the treatment in [16]. Let $H = TA$ be a Θ -stable Cartan subgroup of G and suppose $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ is a real root. The root spaces $\{\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}\}$ generate a subalgebra of \mathfrak{g} defined over \mathbb{R} and there exists a map

$$\phi_{\alpha} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}_{\mathbb{R}}$$

satisfying

$$\begin{aligned} \phi_{\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &\in \mathfrak{h}_{\mathbb{R}} \\ \phi_{\alpha} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &\in \mathfrak{g}_{\alpha}. \end{aligned}$$

Since H is Θ -stable, we also require ϕ_{α} to commute with Θ , i.e.,

$$\phi_{\alpha}(-{}^t X) = \Theta \phi_{\alpha}(X).$$

It turns out these requirements do not uniquely determine ϕ_{α} ; however they limit the possibilities enough for our purposes.

Because the group G is linear, the exponentiated map has the form

$$\Phi_\alpha : \mathrm{SL}(2, \mathbb{R}) \rightarrow G$$

and we define

$$m_\alpha = \Phi_\alpha \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \in T.$$

It turns out $m_\alpha = m_{-\alpha}$ has order two in G and does not depend on the choice of ϕ_α . Since H is abelian, any two of these elements commute. The following proposition determines their remaining multiplicative properties.

Proposition 7.2.1 ([16], Corollary 4.3.20). *Let α , β , and γ be real roots in $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ and suppose*

$$\gamma^\vee = \alpha^\vee + \beta^\vee.$$

Then

$$m_\gamma = m_\alpha m_\beta.$$

Fix a split Θ -stable Cartan subgroup $H^s \subset G$ and let \mathfrak{h}^s denote its complexified Lie algebra. Then $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h}^s) = \Delta(\mathfrak{g}, \mathfrak{h}^s)$ and we define the group M as

$$M = \langle m_\alpha \mid \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^s) \rangle.$$

Proposition 7.2.1 implies

$$M = \langle m_\alpha \mid \alpha \text{ simple in } \Delta(\mathfrak{g}, \mathfrak{h}^s) \rangle \cong (\mathbb{Z}_2)^n$$

for any choice of simple roots. Distinct elements of $\Delta(\mathfrak{g}, \mathfrak{h}^s)$ do not necessarily give distinct elements of M . In particular, we have the following remarkable proposition.

Proposition 7.2.2 (see [1], Lemma 5.1). *Recall $Z(G) \cong \mathbb{Z}_2$. If α is a short root in $\Delta(\mathfrak{g}, \mathfrak{h}^s)$, then m_α is equal to the nontrivial element of $Z(G)$. In particular $m_\alpha = m_\beta$ whenever α and β are short roots.*

It is important to be able to manipulate elements of M formally in our abstract setting. There are two ways we can accomplish this. The first is by simply declaring $\mathfrak{h}^a = \mathfrak{h}^s$ and then viewing M as a subgroup of H^s as above. Alternatively, we can define an abstract group having the same formal properties as M . In any case, the m_α are viewed as formal group elements attached to abstract roots with multiplication determined by Proposition 7.2.1. The following examples illustrate formal calculations in M .

Example 7.2.3. Let $n = 2$ and suppose $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$ are the simple (abstract) roots for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$. Write $\gamma = e_1 + e_2$ and $\delta = e_1$ for the remaining positive roots. Then

$$\begin{aligned}\gamma^\vee &= \alpha_1^\vee + \alpha_2^\vee \\ \delta^\vee &= \gamma^\vee + \alpha_1^\vee\end{aligned}$$

and in coordinates

$$\begin{aligned}\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.\end{aligned}$$

In particular, $m_\delta = m_\gamma m_{\alpha_1} = m_{\alpha_2} m_{\alpha_1} m_{\alpha_1} = m_{\alpha_2}$ (Proposition 7.2.2) and we have

$$M = \{1, m_{\alpha_1}, m_{\alpha_2} = m_\delta, m_{\alpha_1} m_{\alpha_2} = m_\gamma\} \cong (\mathbb{Z}_2)^2.$$

In general, let $\alpha_i = e_i - e_{i+1} \in \Delta$ for $(1 \leq i < n)$ and write α_n for the simple short root. Then any element of M can be written uniquely as an ordered product $m_{\alpha_1}^{\epsilon_1} m_{\alpha_2}^{\epsilon_2} \dots m_{\alpha_n}^{\epsilon_n}$, where $\epsilon_i \in \{0, 1\}$. In particular, elements of M are determined simply by the numbers ϵ_i and can be represented by ordered bit strings of the kind in Chapter 3. In this representation, multiplication in M becomes the familiar bitwise *exclusive or* operation.

Example 7.2.4. In the setting of Example 7.2.3, the bit strings corresponding to the elements of M are given by

$$\begin{aligned}e &\longleftrightarrow 00 \\ m_{\alpha_1} &= \longleftrightarrow 10 \\ m_{\alpha_2} = m_\delta &\longleftrightarrow 01 \\ m_\gamma = m_{\alpha_1} m_{\alpha_2} &\longleftrightarrow 11.\end{aligned}$$

Proposition 7.2.5. *In the notation above, we have the following correspondence between elements of M and bit strings*

$$\begin{aligned}m_{e_i} &\longleftrightarrow 00 \dots \underset{n}{01} \\ m_{e_i - e_j} &\longleftrightarrow 0 \dots \underset{i}{11} \dots \underset{j-1}{1} \underset{j-1}{1} 0 \dots 0 \\ m_{e_i + e_j} &\longleftrightarrow 0 \dots \underset{i}{11} \dots \underset{j-1}{1} \underset{j-1}{1} 0 \dots 1.\end{aligned}$$

Proof. The first identification follows from Proposition 7.2.2. For the second one, let $\beta = e_i - e_j$ and observe

$$\beta^\vee = \alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_{j-1}^\vee.$$

Finally, if $\gamma = e_i + e_j$ the identity

$$\gamma^\vee = \beta^\vee + 2e_j$$

verifies the third correspondence. \square

Set $\mathfrak{h}^a = \mathfrak{h}^s$ so that $M \subset H^s$ and let $\Pi \subset \Delta = \Delta(\mathfrak{g}, \mathfrak{h}^s)$ be a choice of simple roots. Our interest in M is explained by the following proposition.

Proposition 7.2.6 (see [3]). *The elements $\{m_\alpha \mid \alpha \in \Pi\}$ live in distinct connected components of H^s and*

$$M = \langle m_\alpha \mid \alpha \in \Pi \rangle \cong \pi_0(H^s).$$

In particular, the simple m_α generate the component group of H^s .

In Chapter 4, we determined

$$H^s \cong (\mathbb{R}^\times)^n$$

so that

$$\pi_0(H^s) \cong (\mathbb{Z}_2)^n.$$

Proposition 7.2.6 implies this fact as well, but it also gives explicit representatives for the connected components of H^s . A fixed, simple m_α can therefore be thought of as the ‘ -1 ’ of the corresponding \mathbb{R}^\times factor. Having explicit representatives for elements of $\pi_0(H^s)$ will be useful in the next sections.

7.3 The Group \widetilde{M}

Recall we have the following short exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \widetilde{G} \xrightarrow{\pi} G \rightarrow 1.$$

Let $\widetilde{H}^s = \pi^{-1}(H^s)$ be a split Cartan subgroup of \widetilde{G} and $\widetilde{M} = \pi^{-1}(M)$ its corresponding subgroup. For each $m_\alpha \in M$, choose once and for all an inverse image \tilde{m}_α under π and write $\pi^{-1}(m_\alpha) = \{\tilde{m}_\alpha, -\tilde{m}_\alpha\}$. Clearly we have

$$|\widetilde{M}| = 2^{n+1}.$$

Choose a set of simple roots $\Pi \subset \Delta(\mathfrak{g}, \mathfrak{h}^s)$ and fix an ordering $\alpha_1 < \alpha_2 < \dots < \alpha_n$ of elements in Π . The structure of \widetilde{M} is given by the following proposition and corollary.

Proposition 7.3.1 ([1], Lemma 4.8). *The group \widetilde{M} is generated by the elements*

$$\{\tilde{m}_\alpha \mid \alpha \in \Pi\}$$

subject to the following relations

$$\begin{aligned} \tilde{m}_\alpha^2 &= \begin{cases} -1 & \alpha \text{ is long} \\ 1 & \alpha \text{ is short} \end{cases} \\ [\tilde{m}_\alpha, \tilde{m}_\beta] &= \begin{cases} (-1)^{(\alpha, \beta^\vee)} & \alpha, \beta \text{ are both long} \\ 1 & \text{otherwise} \end{cases}. \end{aligned}$$

Corollary 7.3.2. *Every element of \widetilde{M} has a unique expression of the form*

$$\pm \tilde{m}_{\alpha_1}^{\epsilon_1} \tilde{m}_{\alpha_2}^{\epsilon_2} \dots \tilde{m}_{\alpha_n}^{\epsilon_n}$$

where $\epsilon_i \in \{0, 1\}$.

Suppose we again set $\mathfrak{h}^a = \mathfrak{h}^s$ so that $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^s)$ and $\widetilde{M} \subset \widetilde{H}^s$. Let $\Pi \subset \Delta$ be the simple roots with the usual ordering. Corollary 7.3.2 implies an element of \widetilde{M} is uniquely specified (up to sign) by the numbers ϵ_i . We can therefore represent elements of \widetilde{M} as *signed* bit strings. This is the obvious ‘cover’ of our bit string representation for elements of M .

Example 7.3.3. Fix $n > 2$. Then α_1 and α_2 are both long roots in Π with

$$(\alpha_1, \alpha_2^\vee) = -1.$$

In particular, Proposition 7.3.1 implies $[\tilde{m}_{\alpha_1}, \tilde{m}_{\alpha_2}] = -1$ and we see \widetilde{M} cannot be abelian. In terms of our signed bit string representation for $n = 3$ we have

$$\begin{aligned} 100 \cdot 010 &= 110 \\ 010 \cdot 100 &= -110. \end{aligned}$$

It will be convenient to have an explicit way of determining when two elements of \widetilde{M} (expressed as signed bit strings) commute. Suppose

$$\begin{aligned} x &= x_1 x_2 \dots x_n \\ y &= y_1 y_2 \dots y_n \end{aligned}$$

with $x_i, y_i \in \{0, 1\}$. The following sets

$$\begin{aligned} n_L(x, y) &= \left| \{i \mid x_i = y_{i-1} = 1\}_{2 \leq i \leq n-1} \right| \\ n_R(x, y) &= \left| \{i \mid x_i = y_{i+1} = 1\}_{1 \leq i \leq n-2} \right|. \end{aligned}$$

determine the noncommutativity of x and y .

Lemma 7.3.4. *Two elements $x, y \in \widetilde{M}$ commute if and only if $n_L(x, y) \equiv n_R(x, y) \pmod{2}$.*

Proof. Since \tilde{m}_{α_n} is central in \widetilde{M} (Proposition 7.3.1), we may assume $x_n = y_n = 0$. Then either $x \cdot y = y \cdot x$ or $x \cdot y = -y \cdot x$, so it remains to determine the sign of the product. Moreover, we have

$$(\alpha_i, \alpha_j^\vee) = \begin{cases} -1 & \alpha_i \text{ and } \alpha_j \text{ are long and adjacent} \\ 0 & \text{otherwise} \end{cases}$$

so that \tilde{m}_{α_i} and \tilde{m}_{α_j} commute if and only if α_i and α_j are not adjacent.

Consider first the product $x \cdot y$ and suppose \tilde{m}_{α_i} appears in the expression for x . Then \tilde{m}_{α_i} commutes with any \tilde{m}_{α_j} ($1 \leq j \leq i-2$) appearing in the expression for y and anti-commutes with the $\tilde{m}_{\alpha_{i-1}}$ term if it exists. Therefore we get a (-1) factor in the product for each \tilde{m}_{α_i} term appearing in the expression for x with $\tilde{m}_{\alpha_{i-1}}$ appears in the expression for y . The number of these factors is exactly $n_L(x, y)$ for an aggregate sign factor of

$$(-1)^{n_L(x, y)}.$$

Similarly, the product $y \cdot x$ will have an aggregate sign factor of

$$(-1)^{n_R(x, y)}$$

and these two factors are equal if and only if $n_L(x, y) \equiv n_R(x, y) \pmod{2}$. □

Proposition 7.3.5. *Fix $n > 1$. Then*

$$\left| Z(\widetilde{M}) \right| = \begin{cases} 8 & n \text{ is even} \\ 4 & n \text{ is odd} \end{cases}.$$

Proof. Proposition 7.3.1 implies $\{1, -1, \tilde{m}_{\alpha_n}, -\tilde{m}_{\alpha_n}\} \subset Z(\widetilde{M})$ for all n . If $n = 2$, the two simple elements $\{\tilde{m}_{\alpha_1}, \tilde{m}_{\alpha_2}\}$ commute and \widetilde{M} is abelian. Since $\left| \widetilde{M} \right| = 2^{2+1} = 8$, the proposition holds. If $n = 3$, Example 7.3.3 verifies there are no central elements in \widetilde{M} except the four elements listed above.

If $n \geq 4$, suppose there exists a central element $z = z_1 z_2 \dots z_{n-1} 0$. If $y = y_1 y_2 \dots y_{n-1} 0$ is any element of \widetilde{M} , lemma 7.3.4 implies

$$n_L(z, y) \equiv n_R(z, y) \pmod{2}.$$

If we let $y = 10 \dots 0$, then

$$\begin{aligned} n_R(z, y) &= 0 \\ n_L(z, y) &= z_2 \end{aligned}$$

and thus

$$z_2 = 0. \tag{7.1}$$

If we let $y = 0 \dots 010$, then

$$\begin{aligned} n_L(z, y) &= 0 \\ n_R(z, y) &= z_{n-2} \end{aligned}$$

and thus

$$z_{n-2} = 0. \tag{7.2}$$

Finally, if we let $y = 0 \dots 010 \dots 0$ have a nonzero bit in the i th position ($1 < i < n - 1$), then

$$\begin{aligned} n_L(z, y) &= z_{i+1} \\ n_R(z, y) &= z_{i-1} \end{aligned}$$

and thus

$$z_{i-1} = z_{i+1}. \tag{7.3}$$

Combining (7.1) and (7.3) gives $z_i = 0$ for i even.

Suppose first n is odd. Then $n - 2$ is odd and (7.2) and (7.3) imply $z_i = 0$ for all i . In particular, z must be trivial and we have $|Z(\widetilde{M})| = 4$. On the other hand, if n is even there is no such restriction for the odd bits and we get nontrivial central elements of the form $z = \pm 1010 \dots 10$. Therefore $|Z(\widetilde{M})| = 8$ as desired. \square

Example 7.3.6. For $n = 4$ we have

$$Z(\widetilde{M}) = \{\pm 1, \pm \tilde{m}_{\alpha_4}, \pm \tilde{m}_{\alpha_1} \tilde{m}_{\alpha_3}\} \cong (\mathbb{Z}_2)^3$$

or in bit string notation

$$Z(\widetilde{M}) = \left\{ \begin{array}{l} \pm 0000 \\ \pm 0001 \\ \pm 1010 \\ \pm 1011 \end{array} \right\}.$$

The following result is the expected analog of Proposition 7.2.6 for \widetilde{H}^s .

Proposition 7.3.7. *The elements $\{1, -1\}$ live in distinct connected components of \widetilde{H}^s . Moreover we have*

$$\widetilde{M} = \langle \tilde{m}_\alpha \mid \alpha \in \Pi \rangle \cong \pi_0(\widetilde{H}^s).$$

In particular, the split Cartan subgroup \widetilde{H}^s has double the number of connected components of H^s and their multiplicative structure is given by the group \widetilde{M} . Fix an element $\phi \in (\mathfrak{h}^s)^*$ and let $(\theta, \varepsilon, \lambda)$ be the abstract triple corresponding to the pair (\widetilde{H}^s, ϕ) (Section 6.2). Since $\theta = -1$, all roots in $\Delta(\mathfrak{g}, \mathfrak{h}^a)$ are real with respect to θ . Therefore $(\theta, \varepsilon, \lambda)$ is supportable and the pair $(\widetilde{H}^s, \phi)_\lambda$ extends to a genuine triple $(\widetilde{H}^s, \phi, \widetilde{\Gamma})_\lambda$ (Proposition 6.3.4). The number of distinct (conjugacy classes of) genuine triples extending $(\widetilde{H}^s, \phi)_\lambda$ will be denoted $\left[(\widetilde{H}^s, \phi)_\lambda \right]$.

To determine $\left[(\widetilde{H}^s, \phi)_\lambda \right]$, recall the behavior of a genuine representation $\widetilde{\Gamma}$ on $(\widetilde{H}^s)_0$ is determined by ϕ (Section 2.2). Then Proposition 7.1.3 implies each such $\widetilde{\Gamma}$ corresponds to a genuine character of $\pi_0(Z(\widetilde{H}^s))$, i.e., a character of $\pi_0(Z(\widetilde{H}^s))$ where the connected component containing -1 acts nontrivially. Since $(\widetilde{H}^s)_0$ is central in \widetilde{H}^s (Proposition 7.1.1), the group $\pi_0(Z(\widetilde{H}^s))$ appears naturally as a subgroup of $\pi_0(\widetilde{H}^s)$. Proposition 7.3.7 allows us to determine this subgroup.

In particular, Proposition 7.3.7 provides a choice of representatives for the distinct connected components of \widetilde{H}^s whose multiplicative structure is known (Proposition 7.3.1). Therefore a connected component of \widetilde{H}^s is central if and only if its corresponding representative is central in \widetilde{M} . We thus have a one-to-one correspondence between genuine representations of \widetilde{H}^s extending $(\widetilde{H}^s, \phi)_\lambda$ and genuine characters of $Z(\widetilde{M})$. We will denote the set of genuine characters of $Z(\widetilde{M})$ by $\Pi_g(Z(\widetilde{M}))$.

Proposition 7.3.8. *Let $(\tilde{H}^s, \phi)_\lambda$ be a pair with $\phi \in (\mathfrak{h}^s)^*$ and corresponding abstract triple $(\theta, \varepsilon, \lambda)$. Then*

$$\left[(\tilde{H}^s, \phi)_\lambda \right] = \left| \Pi_g(Z(\tilde{M})) \right|$$

and

$$\left| \Pi_g(Z(\tilde{M})) \right| = \left| Z(\tilde{M}) \right| / 2.$$

Proof. The first statement follows from the above argument. The second statement follows from Proposition 7.3.7 since -1 must act nontrivially for genuine characters of \tilde{H}^s . \square

Corollary 7.3.9 ([1], Example 3.13). *Let $(\tilde{H}^s, \phi)_\lambda$ be a pair and let $(\theta, \varepsilon, \lambda)$ denote its corresponding abstract triple. Then*

$$\left[(\tilde{H}^s, \phi)_\lambda \right] = \begin{cases} 4 & n \text{ is even} \\ 2 & n \text{ is odd} \end{cases} .$$

Proof. Combine Proposition 7.3.8 with Proposition 7.3.5. \square

CHAPTER 8

GENUINE TRIPLES FOR \tilde{H}

Let λ be a regular symmetric infinitesimal character and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to the pair $(\tilde{H}^s, \phi)_\lambda$. In the previous section we determined the number of genuine triples $(\tilde{H}^s, \phi, \tilde{\Gamma})_\lambda$ extending $(\tilde{H}^s, \phi)_\lambda$ (Corollary 7.3.9). We accomplished this by constructing a subgroup \tilde{M} of \tilde{H}^s that modeled the component group $\pi_0(\tilde{H}^s)$. Genuine triples $(\tilde{H}^s, \phi, \tilde{\Gamma})_\lambda$ were then in bijective correspondence with genuine representations of $Z(\tilde{M})$.

In this section we extend this program to the other (conjugacy classes of) Cartan subgroups in \tilde{G} . In particular, if \tilde{H} is a Θ -stable Cartan subgroup, we look for a subset of \tilde{H} whose elements live in distinct connected components and whose multiplicative structure we understand.

8.1 Cayley Transforms in G

We begin our analysis in the linear group G by recalling the familiar Cayley transform. This will be our main tool for transferring information about one Cartan subgroup to another. Let $\mathfrak{h}_\mathbb{R}$ be a Θ -stable Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$ with corresponding Cartan subgroup $H = TA$. Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a real root and choose a nonzero root vector $X_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{R}$. Then $\Theta X_\alpha \in \mathfrak{g}_{-\alpha} \cap \mathfrak{g}_\mathbb{R}$ and $X_\alpha + \Theta X_\alpha$ is fixed by Θ . Define the *Cayley transform* of $\mathfrak{h}_\mathbb{R}$ with respect to α as

$$(\mathfrak{h}_\mathbb{R})_\alpha = \ker(\alpha|_{\mathfrak{h}_\mathbb{R}}) \oplus \mathbb{R}(X_\alpha + \Theta X_\alpha).$$

Then $(\mathfrak{h}_\mathbb{R})_\alpha$ is a Θ -stable Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$. Let $H_\alpha = T_\alpha A_\alpha$ denote its corresponding Cartan subgroup (the Cayley transform of H with respect to α).

The Cayley transform replaces a one-dimensional subspace of $\mathfrak{h}_\mathbb{R}$ contained in the negative eigenspace of Θ with a one-dimensional subspace contained in its positive eigenspace. The Cartan subgroup H_α is thus ‘more compact’ than H . It is a fundamental fact that we obtain any Cartan subgroup of G (up to conjugacy) by starting with H^s and performing a sequence of Cayley transforms with respect to (orthogonal) real roots.

Fix a Cartan subgroup $H \subset G$ and suppose α and β are real roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. Recall the corresponding elements $m_\alpha, m_\beta \in H$ defined in Section 7.2. It will be important to understand how these elements are affected by Cayley transforms. Let $M(H)$ denote the subgroup of H generated by the m_α

$$M(H) = \langle m_\alpha \mid \alpha \text{ real in } \Delta(\mathfrak{g}, \mathfrak{h}) \rangle.$$

Since $m_\alpha = m_{-\alpha}$, we assume α and β are positive with respect to a fixed positive system in $\Delta(\mathfrak{g}, \mathfrak{h})$.

Denote the adjoint action of m_α on \mathfrak{g} by $\text{Ad}(m_\alpha)$. The following proposition describes this action for real roots.

Proposition 8.1.1 ([16], Lemma 4.3.19e). *Let $H \subset G$ be a Θ -stable Cartan subgroup and suppose α and β are distinct (positive) real roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. Let X_α be a nonzero root vector for α . Then*

$$\text{Ad}(m_\beta)(X_\alpha) = (-1)^{(\alpha, \beta^\vee)} X_\alpha.$$

Remark 8.1.2. If the root $\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ is short, m_β is the unique nontrivial central element in G by Proposition 7.2.2. In this case $\text{Ad}(m_\beta)$ should act trivially on each root space \mathfrak{g}_α . This is consistent with the proposition since $(\alpha, \beta^\vee) \in 2\mathbb{Z}$ for all roots α .

Corollary 8.1.3. *In the setting of Proposition 8.1.1, suppose β is a long root. Then $m_\beta \in H_\alpha$ if and only if α and β are orthogonal. If β is a short root, then $m_\beta \in H_\alpha$.*

Proof. Since $m_\beta \in H$, m_β centralizes $\ker(\alpha|_{\mathfrak{h}_\mathbb{R}})$ by definition. Therefore m_β will be an element of H_α if and only if $\text{Ad}(m_\beta)$ centralizes X_α . If β is long, this will happen if and only if

$$(\alpha, \beta^\vee) = (\alpha, \beta) \in 2\mathbb{Z}.$$

by Proposition 8.1.1. Since $\alpha \neq \pm\beta$ we have $(\alpha, \beta) \in \{0, 1, -1\}$ and the first result follows. If β is short the result follows from Remark 8.1.2. \square

Let

$$M(H)_\alpha = M(H) \cap H_\alpha$$

denote the subgroup of elements in $M(H)$ that are also in H_α .

Proposition 8.1.4. *Let H be a Θ -stable Cartan subgroup of G and suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a real root. Then*

$$M(H_\alpha) \subseteq M(H)_\alpha.$$

Proof. $M(H_\alpha)$ is contained in H_α by definition. Moreover we have

$$\begin{aligned} M(H_\alpha) &= \langle m_\beta \in H_\alpha \mid \beta \text{ is real in } \Delta(\mathfrak{g}, \mathfrak{h}_\alpha) \rangle \\ &= \langle m_\beta \in H \mid \beta \text{ is real in } \Delta(\mathfrak{g}, \mathfrak{h}) \text{ and } (\alpha, \beta) = 0 \rangle \\ &\subseteq M(H) \end{aligned}$$

as desired. □

Next we consider the element $m_\alpha \in M(H)$ and show this is also an element of H_α (and thus $M(H)_\alpha$). To see this, define

$$B_\alpha = \exp_G \{ \mathbb{R}(X_\alpha + \Theta X_\alpha) \}.$$

Then $B_\alpha \subset T_\alpha$ is a connected compact abelian subgroup of H_α .

Proposition 8.1.5 ([16], Lemma 8.3.13). *In the setting above,*

$$T \cap B_\alpha = \{1, m_\alpha\}.$$

In particular $m_\alpha \in (H_\alpha)_0$, the identity component of H_α .

Corollary 8.1.6. *Let $H \subset G$ be a Θ -stable Cartan subgroup and suppose α is a real root in $\Delta(\mathfrak{g}, \mathfrak{h})$. Then*

$$M(H)_\alpha = \langle M(H_\alpha), m_\alpha \rangle.$$

Proof. Propositions 8.1.4 and 8.1.5 imply

$$M(H)_\alpha \supseteq \langle M(H_\alpha), m_\alpha \rangle.$$

Conversely, Proposition 3.3.6 implies the real roots of H form a root system of type $B_m \times A_1^l$. Choose a system of positive roots in $\Delta_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h})$ for which α is simple. If α is contained in the A_1^l factor, then the other simple roots are orthogonal to α and the result easily follows as in Proposition 8.1.4. Hence we may assume $\Delta_{\mathbb{R}}(\mathfrak{g}, \mathfrak{h})$ is of type B_m .

For $m \in M(H)$, write $m = m_{\alpha_1} \cdots m_{\alpha_k}$ where the α_i are simple and suppose $m \in M(H)_\alpha$. Then m centralizes X_α (the root vector for α) and we need to show $m \in$

$\langle M(H_\alpha), m_\alpha \rangle$. If $(\alpha, \alpha_i) \in \{0, 2\}$ for all i , then each $m_{\alpha_i} \in \langle M(H_\alpha), m_\alpha \rangle$ and we are done. Otherwise α is long and there are exactly two terms (say m_{α_i} and m_{α_j}) in the expression for m with $(\alpha, \alpha_i) = (\alpha, \alpha_j) = -1$. Define a new root

$$\gamma = \alpha_i + \alpha + \alpha_j$$

and observe $(\alpha, \gamma) = 0$ so that $m_\gamma \in M(H_\alpha)$. But $m_\gamma m_\alpha = m_{\alpha_i} m_{\alpha_j}$ by Proposition 7.2.1 and thus $m_{\alpha_i} m_{\alpha_j} \in \langle M(H_\alpha), m_\alpha \rangle$. It follows $m \in \langle M(H_\alpha), m_\alpha \rangle$ and we have shown

$$M(H)_\alpha \subseteq \langle M(H_\alpha), m_\alpha \rangle$$

as desired. □

Informally, the corollary implies we can move the root α inside the parentheses as long as we add the element m_α to the resulting group.

Let H be a Θ -stable Cartan subgroup and suppose $\alpha_1, \alpha_2, \dots, \alpha_k$ are mutually orthogonal real roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. Then the *iterated* Cayley transform

$$H_{\alpha_1 \alpha_2 \dots \alpha_k} = ((H_{\alpha_1})_{\alpha_2} \dots)_{\alpha_k}$$

of H with respect to $\alpha_1, \alpha_2, \dots, \alpha_k$ is defined. Similarly we can define

$$M(H)_{\alpha_1 \alpha_2 \dots \alpha_k} = M(H) \cap H_{\alpha_1 \alpha_2 \dots \alpha_k}.$$

To simplify notation we will sometimes let $\mathfrak{c} = \alpha_1 \alpha_2 \dots \alpha_k$ denote a sequence of mutually orthogonal roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ and write

$$\begin{aligned} H_{\mathfrak{c}} &= H_{\alpha_1 \alpha_2 \dots \alpha_k} \\ M(H)_{\mathfrak{c}} &= M(H)_{\alpha_1 \alpha_2 \dots \alpha_k}. \end{aligned}$$

We conclude this section with the following extension of Corollary 8.1.6.

Proposition 8.1.7. *Suppose H is a Θ -stable Cartan subgroup of G and $\alpha_1, \alpha_2, \dots, \alpha_k$ is a sequence of mutually orthogonal real roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. Then*

$$M(H)_{\alpha_1 \alpha_2 \dots \alpha_k} = \langle M(H_{\alpha_1 \alpha_2 \dots \alpha_k}), m_{\alpha_1}, m_{\alpha_2}, \dots, m_{\alpha_k} \rangle.$$

Proof. The proof is by induction on the number of roots. The base case is Corollary 8.1.6. In general we have

$$\begin{aligned}
M(H)_{\alpha_1\alpha_2\dots\alpha_k} &= M(H) \cap H_{\alpha_1\alpha_2\dots\alpha_k} \\
&= M(H) \cap H_{\alpha_1\alpha_2\dots\alpha_{k-1}} \cap H_{\alpha_1\alpha_2\dots\alpha_k} \\
&= \langle M(H_{\alpha_1\dots\alpha_{k-1}}), m_{\alpha_1}, \dots, m_{\alpha_{k-1}} \rangle \cap H_{\alpha_1\dots\alpha_k} \\
&= \langle M(H_{\alpha_1\dots\alpha_{k-1}}) \cap H_{\alpha_1\dots\alpha_k}, m_{\alpha_1}, \dots, m_{\alpha_{k-1}} \rangle \\
&= \langle M(H_{\alpha_1\alpha_2\dots\alpha_{k-1}})_{\alpha_k}, m_{\alpha_1}, m_{\alpha_2}, \dots, m_{\alpha_{k-1}} \rangle \\
&= \langle M(H_{\alpha_1\alpha_2\dots\alpha_k}), m_{\alpha_1}, m_{\alpha_2}, \dots, m_{\alpha_k} \rangle
\end{aligned}$$

as desired. \square

8.2 Cayley Transforms in W

In this section we extend Cayley transforms to our abstract Weyl group W . To begin, let H be a Θ -stable Cartan subgroup of G and fix nonsingular $\lambda \in (\mathfrak{h}^a)^*$ and $\phi \in \mathfrak{h}^*$ so that λ and ϕ define the same infinitesimal character. The conjugation map $i_{\lambda, \phi}^{-1}$ (Section 6.1) induces an abstract involution θ of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ as usual. Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a real root and consider the Cayley transform of \mathfrak{h} by α . If ϕ' denotes the image of ϕ in $(\mathfrak{h}_\alpha)^*$, conjugation by $i_{\lambda, \phi'}^{-1}$ induces a different abstract involution θ_α of Δ . It is easy to describe how θ and θ_α are related in W .

Proposition 8.2.1 ([16], Lemma 8.6.13). *In the situation above $\theta_\alpha = s_\alpha\theta$, where s_α is the root reflection in W corresponding to (the image of) α .*

If θ is an involution and α is a real root for θ , we define the *abstract* Cayley transform of θ with respect to α to be $\theta_\alpha = s_\alpha\theta$.

Example 8.2.2. Let $n = 2$ and consider the positive abstract roots in Δ

$$\begin{aligned}
\alpha &= e_1 - e_2 \\
\beta &= e_1 + e_2 \\
\gamma &= e_1 \\
\delta &= e_2.
\end{aligned}$$

Let $\theta = -1$ so that all roots are real. In terms of our matrix representation we have

$$M_\theta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix representatives for the various Cayley transforms are then

$$\begin{aligned} M_{\theta_\alpha} &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ M_{\theta_\beta} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ M_{\theta_\gamma} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ M_{\theta_\delta} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Note the first two involutions and the last two involutions are conjugate in W . In terms of diagrams we have

$$\begin{aligned} D_{\theta_\alpha} &= (2) (1) \\ D_{\theta_\beta} &= 2 \ 1 \\ D_{\theta_\gamma} &= + \ - \\ D_{\theta_\delta} &= - \ +. \end{aligned}$$

If we instead suppose θ is of the form

$$M_\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then α is still real for θ and we have

$$M_{\theta_\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or in terms of diagrams

$$D_{\theta_\alpha} = + \ +.$$

We now describe the effect of a general abstract Cayley transform in terms of diagrams. Let $\theta \in W$ be an involution and suppose $\alpha \in \Delta$ is a real root for θ . There are several cases to consider since the relation between D_θ and D_{θ_α} depends on both θ and α .

Case I. Suppose $\alpha = e_i$ is a short root. Since α is real, our diagram is of the form

$$D_\theta : \cdots \underset{i}{-} \cdots$$

with a $-$ sign in the i th position. The Cayley transform of θ with respect to α then has diagram

$$D_{\theta_\alpha} : \cdots \underset{i}{+} \cdots.$$

Case II. Suppose $\alpha = e_i - e_j$ and θ has the form

$$D_\theta : \cdots \frac{-}{i} \cdots \frac{-}{j} \cdots .$$

Then α is real for θ and the Cayley transform of θ by α is

$$D_{\theta_\alpha} : \cdots \frac{(j)}{i} \cdots \frac{(i)}{j} \cdots .$$

Case III. Suppose $\alpha = e_i - e_j$ and θ has the form

$$D_\theta : \cdots \frac{j}{i} \cdots \frac{i}{j} \cdots .$$

Again α is real for θ and the Cayley transform of θ by α is

$$D_{\theta_\alpha} : \cdots \frac{+}{i} \cdots \frac{+}{j} \cdots .$$

Case IV. Suppose $\alpha = e_i + e_j$ and θ has the form

$$D_\theta : \cdots \frac{-}{i} \cdots \frac{-}{j} \cdots .$$

Then α is real for θ and the Cayley transform of θ by α is

$$D_{\theta_\alpha} : \cdots \frac{j}{i} \cdots \frac{i}{j} \cdots .$$

Case V. Suppose $\alpha = e_i + e_j$ and θ has the form

$$D_\theta : \cdots \frac{(j)}{i} \cdots \frac{(i)}{j} \cdots .$$

Again α is real for θ and the Cayley transform of θ by α is

$$D_{\theta_\alpha} : \cdots \frac{+}{i} \cdots \frac{+}{j} \cdots .$$

Most of these cases were verified in Example 8.2.2 for $n = 2$. The general statements can be verified by first reducing to this setting.

If $\alpha_1, \alpha_2, \dots, \alpha_k$ is a sequence of mutually orthogonal real roots for θ , the *iterated* Cayley transform of θ with respect to $\alpha_1, \alpha_2, \dots, \alpha_k$ is defined and we denote it by

$$\theta_{\alpha_1 \alpha_2 \dots \alpha_k} = ((\theta_{\alpha_1}) \dots)_{\alpha_k}.$$

Example 8.2.3. Suppose θ is of the form

$$D_\theta : \cdots \frac{-}{i} \cdots \frac{-}{j} \cdots .$$

Then the iterated Cayley transform with respect to the real roots $\alpha = e_i - e_j$ and $\beta = e_i + e_j$ is defined and gives the same result as taking $\alpha = e_i$ and $\beta = e_j$

$$D_{\theta_{\alpha\beta}} : \cdots \frac{+}{i} \cdots \frac{+}{j} \cdots .$$

The proof of the following proposition will be important in later sections.

Proposition 8.2.4. *Fix the involution $\theta^s = -1$ in W . Then up to conjugacy, every involution in W can be obtained from θ^s via an iterated Cayley transform.*

Proof. Let θ be an involution in W . Since we are only interested in obtaining θ up to conjugacy, we may assume θ has the form

$$(2) \frac{(1)}{1} \cdots \frac{(n_c^\theta)}{n_c^\theta} \frac{(n_c^\theta - 1)}{n_c^\theta} \frac{+}{n_c^\theta + 1} + \cdots + \frac{+}{n_c^\theta + n_s^\theta} \frac{-}{n - n_r^\theta + 1} - \cdots - \frac{-}{n} .$$

Label the following distinguished positive roots in Δ

$$\begin{aligned} \alpha_i &= e_i - e_{i+1} \\ \beta_i &= e_i. \end{aligned}$$

We construct our iterated Cayley transform using these roots in a two-step process.

Step I. Let $k = n_c^\theta$. If $k = 0$ we proceed to step II. If $k \neq 0$, consider the sequence of (long) simple roots $\alpha_1, \alpha_3, \dots, \alpha_{k-1}$. Since these roots are mutually orthogonal, the corresponding iterated Cayley transform of θ^s is defined. By a repeated application of Case II above we have

$$D_{\theta_{\alpha_1 \alpha_3 \dots \alpha_{k-1}}^s} : (2) \frac{(1)}{1} \cdots \frac{(k)}{k} \frac{(k-1)}{k} \frac{-}{k+1} - \cdots - \frac{-}{n} .$$

Step II. Let $s = n_s^\theta$. If $s = 0$ we are done. If $s \neq 0$ consider the sequence of (short) positive roots $\beta_{k+1}, \beta_{k+2}, \dots, \beta_{k+s}$. These roots are mutually orthogonal and are orthogonal to the roots $\alpha_1, \alpha_3, \dots, \alpha_{k-1}$ appearing in Step I. Therefore the corresponding iterated Cayley transform of $\theta_{\alpha_1 \alpha_3 \dots \alpha_{k-1}}^s$ is defined. By Case I above we have

$$D_{\theta_{\alpha_1 \dots \alpha_{k-1} \beta_{k+1} \dots \beta_{k+s}}^s} : (2) \frac{(1)}{1} \cdots \frac{(k)}{k} \frac{(k-1)}{k} \frac{+}{k+1} \cdots \frac{+}{k+s} - \cdots - \frac{-}{n}$$

and thus $\theta_{\alpha_1 \dots \alpha_{k-1} \beta_{k+1} \dots \beta_{k+s}}^s$ is the desired involution. □

8.3 Component Groups for $H_{\mathbf{c}}^s$

Recall H^s denotes a fixed split Cartan subgroup of G and $M = M(H^s)$. In this section we describe how M changes under the specific set of Cayley transforms outlined in Proposition 8.2.4. This will ultimately lead to a description of $\pi_0(H)$ for each conjugacy class of Cartan subgroups in G .

Recall the set of conjugacy classes of Cartan subgroups in G is in bijective correspondence with the set of conjugacy classes of involutions in W . Proposition 8.2.4 constructs a set of representative involutions for these conjugacy classes by applying a sequence \mathbf{c} of abstract Cayley transforms to the involution $\theta^s = -1$. Since θ^s is the abstract involution corresponding to any split Cartan subgroup, we can similarly apply this sequence to H^s . This sets up a correspondence

$$\theta_{\mathbf{c}}^s \longleftrightarrow H_{\mathbf{c}}^s$$

between representative involutions in W and representative Cartan subgroups of G .

In the notation of Proposition 8.2.4, $\mathbf{c} = \alpha_1\alpha_3 \dots \alpha_{k-1}\beta_{k+1} \dots \beta_{k+s}$ for some numbers k and s , where

$$\begin{aligned} \alpha_i &= e_i - e_{i+1} \\ \beta_i &= e_i. \end{aligned}$$

Sequences of this form occur frequently and will be referred to as *standard*. Additionally, we will write

$$\gamma_i = e_i + e_{i+1}$$

and

$$m_{\beta} = m_{\beta_i}$$

for the unique element of M corresponding to short roots in Δ . Proposition 8.1.7 determines the structure of the group $M_{\mathbf{c}}$. There are three distinct cases that we record here for convenience.

- Suppose $\mathbf{c} = \alpha_1\alpha_3 \dots \alpha_{k-1}$. Then we have

$$M_{\mathbf{c}} = \langle M(H_{\mathbf{c}}^s), m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}} \rangle.$$

- Suppose $\mathbf{c} = \beta_1 \dots \beta_s$. Then we have

$$M_{\mathbf{c}} = \langle M(H_{\mathbf{c}}^s), m_{\beta} \rangle.$$

- Suppose $\mathfrak{c} = \alpha_1 \alpha_3 \dots \alpha_{k-1} \beta_{k+1} \dots \beta_{k+s}$. Then we have

$$M_{\mathfrak{c}} = \langle M(H_{\mathfrak{c}}^s), m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}}, m_{\beta} \rangle.$$

The following proposition explains our interest in $M_{\mathfrak{c}}$.

Proposition 8.3.1. *Suppose \mathfrak{c} is any sequence of mutually orthogonal real roots in H^s and*

$$M_{\mathfrak{c}} = \langle M(H_{\mathfrak{c}}^s), m_{\delta_1}, \dots, m_{\delta_m} \rangle$$

as in Proposition 8.1.7. Then

$$\begin{aligned} \pi_0(H_{\mathfrak{c}}^s) &\cong M_{\mathfrak{c}} / \langle m_{\delta_1}, \dots, m_{\delta_m} \rangle \\ &\cong M(H_{\mathfrak{c}}^s) / (M(H_{\mathfrak{c}}^s) \cap \langle m_{\delta_1}, \dots, m_{\delta_m} \rangle). \end{aligned}$$

Proof. This follows from Proposition 7.2.6, Proposition 8.1.5, and the fact that elements in $M(H_{\mathfrak{c}}^s)$ generate $\pi_0(H_{\mathfrak{c}}^s)$ (see [3]). The last isomorphism follows from the Second Isomorphism Theorem for groups. \square

Proposition 7.2.6 implies the group $M = M(H^s)$ is a model for the component group $\pi_0(H^s)$. Proposition 8.3.1 extends this model to the Cartan subgroups $H_{\mathfrak{c}}^s$. The extension is almost perfect in the sense that $\pi_0(H_{\mathfrak{c}}^s)$ is generated by elements of $M(H_{\mathfrak{c}}^s)$. Unfortunately distinct elements of $M(H_{\mathfrak{c}}^s)$ can live in the same connected component of $H_{\mathfrak{c}}^s$. However this relationship is completely determined by the standard sequence \mathfrak{c} . In particular, we obtain a model for $\pi_0(H_{\mathfrak{c}}^s)$ in $H_{\mathfrak{c}}^s$ by choosing elements of $M(H_{\mathfrak{c}}^s)$ that are distinct modulo those elements determined by \mathfrak{c} .

Proposition 8.3.1 describes $\pi_0(H_{\mathfrak{c}}^s)$ as an appropriate subquotient of M . This is our second description of $\pi_0(H_{\mathfrak{c}}^s)$ and it will turn out to be the one most useful in determining the central components in $\pi_0(\tilde{H}_{\mathfrak{c}}^s)$. Before we turn to this however, we verify that Proposition 8.3.1 is consistent with the results of Section 4.2.

Case I. Suppose $\mathfrak{c} = \beta_1 \dots \beta_s$ and let $r = n - s$. Then we have

$$D_{\theta_{\mathfrak{c}}^s} : \begin{array}{cccc} + & \cdots & + & - \cdots - \\ 1 & & s & n \end{array}.$$

If $s = 0$, then $\theta_{\mathfrak{c}}^s = \theta^s = -1$ and by Proposition 7.2.6 we have $\pi_0(H^s) \cong M$ as desired. If $s = n$, then $\theta_{\mathfrak{c}}^s = 1$ and $\pi_0(H_{\mathfrak{c}}^s)$ is trivial. This is consistent with Proposition 8.3.1 since $M(H_{\mathfrak{c}}^s)$ is also trivial. Finally, if $0 < s < n$ we have

$$M_{\mathfrak{c}} = \langle M(H_{\mathfrak{c}}^s), m_{\beta} \rangle.$$

$M(H_{\mathfrak{c}}^s)$ has generators

$$\{m_{\alpha_{s+1}}, m_{\alpha_{s+2}}, \dots, m_{\alpha_{n-1}}, m_{\beta}\}$$

and thus $|M(H_{\mathfrak{c}}^s)| = 2^r$. Therefore Proposition 8.3.1 implies

$$|\pi_0(H_{\mathfrak{c}}^s)| = 2^{r-1}.$$

This is consistent with Case I from section 4.2.

Case II. Suppose $\mathfrak{c} = \alpha_1 \alpha_3 \dots \alpha_{k-1} \beta_{k+1} \dots \beta_n$ with $0 < k < n$. Then we have

$$D_{\theta^s_{\alpha_1 \alpha_3 \dots \alpha_{k-1} \beta_{k+1} \dots \beta_n}} : \binom{(2)}{1} \binom{(1)}{\dots} \binom{(k)}{(k-1)} \binom{(k-1)}{k} \binom{+}{k+1} \dots \binom{+}{n}$$

and

$$M_{\mathfrak{c}} = \langle M(H_{\mathfrak{c}}^s), m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}}, m_{\beta} \rangle.$$

Also $M(H_{\mathfrak{c}}^s)$ has generators

$$\{m_{\gamma_1}, m_{\gamma_3}, \dots, m_{\gamma_{k-1}}\}.$$

In Proposition 7.2.5, we saw these elements differ from the elements $m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}}$ by a factor of m_{β} . Therefore every element of $M(H_{\mathfrak{c}}^s)$ is also an element of

$$\langle m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}}, m_{\beta} \rangle$$

and Proposition 8.3.1 implies the group $\pi_0(H)$ is trivial. This is consistent with Case II from Section 4.2.

Case III. Suppose $\mathfrak{c} = \alpha_1 \alpha_3 \dots \alpha_{k-1}$ with $0 < k < n$ and let $r = n - k$. Then we have

$$D_{\theta^s_{\alpha_1 \alpha_3 \dots \alpha_{k-1}}} : \binom{(2)}{1} \binom{(1)}{\dots} \binom{(k)}{(k-1)} \binom{(k-1)}{k} \binom{-}{\dots} \binom{-}{n}$$

and

$$M_{\mathfrak{c}} = \langle M(H_{\mathfrak{c}}^s), m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}} \rangle.$$

Also $M(H_{\mathfrak{c}}^s)$ has generators

$$\{m_{\gamma_1}, m_{\gamma_3}, \dots, m_{\gamma_{k-1}}, m_{\alpha_{k+1}}, m_{\alpha_{k+2}}, \dots, m_{\alpha_{n-1}}, m_{\beta}\}$$

and therefore the quotient $M_{\mathfrak{c}} / \langle m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}} \rangle$ has generators

$$\{m_{\alpha_{k+1}}, m_{\alpha_{k+2}}, \dots, m_{\alpha_{n-1}}, m_{\beta}\}.$$

By Proposition 8.3.1 we have

$$|\pi_0(H_{\mathfrak{c}}^s)| = 2^r.$$

This is consistent with Case III of Section 4.2.

Case IV. Suppose $\mathfrak{c} = \alpha_1\alpha_3 \dots \alpha_{k-1}\beta_{k+1} \dots \beta_{k+s}$ with $0 < k < n$ and $0 < s < n - k$. Let $r = n - k - s$. Then we have

$$D_{\alpha_1\alpha_3 \dots \alpha_{k-1}\beta_{k+1} \dots \beta_{k+s}}^{\theta^s} : \binom{(2)}{1} \binom{(1)}{\dots} \binom{(k)}{(k-1)} \binom{(k-1)}{k} \binom{+}{k+1} \dots \binom{+}{k+s} \binom{-}{\dots} \binom{-}{n}.$$

The analysis for this case is exactly the same as the previous one except now m_β is trivial in the quotient. Therefore

$$|\pi_0(H_{\mathfrak{c}}^s)| = 2^{r-1}$$

and this is consistent with Case IV of Section 4.2.

Case V. Suppose $\mathfrak{c} = \alpha_1\alpha_3 \dots \alpha_{n-1}$. Note this is only possible when n is even. Then we have

$$D_{\alpha_1\alpha_3 \dots \alpha_{n-1}}^{\theta^s} : \binom{(2)}{1} \binom{(1)}{\dots} \binom{(n)}{(n-1)}$$

and

$$M_{\mathfrak{c}} = \langle M(H_{\mathfrak{c}}^s), m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{n-1}} \rangle.$$

Also $M(H_{\mathfrak{c}}^s)$ has generators

$$\{m_{\gamma_1}, m_{\gamma_3}, \dots, m_{\gamma_{n-1}}\}$$

and thus the subgroup $\langle m_{\alpha_1}, m_{\alpha_3}, \dots, m_{\alpha_{k-1}} \rangle$ has index two in $M_{\mathfrak{c}}$. By Proposition 8.3.1 we have $|\pi_0(H_{\mathfrak{c}}^s)| = 2$. This is consistent with Case V of Section 4.2.

Theorem 4.1.3 determined the order of $\pi_0(H)$ for each conjugacy class of Cartan subgroups in G . Proposition 8.3.1 determines this order as well, however it also provides a method for selecting representatives in distinct connected components of H . The key to reconciling these approaches is Proposition 7.2.1. In particular, Proposition 7.2.1 implies

$$M \cong L(\Delta^\vee)/2L(\Delta^\vee)$$

where $L(\Delta^\vee)$ denotes the coroot lattice. Let $\pi : L(\Delta^\vee) \rightarrow M$ denote the corresponding quotient map and suppose \mathfrak{c} is a standard sequence in Δ . Write $\theta = \theta_{\mathfrak{c}}^s$ and define

$$\begin{aligned} L(\Delta^\vee)_+^\theta &= \{\alpha \in L(\Delta^\vee) \mid \theta(\alpha) = \alpha\} \\ L(\Delta^\vee)_-^\theta &= \{\alpha \in L(\Delta^\vee) \mid \theta(\alpha) = -\alpha\} \\ M_+^\theta &= \pi(L(\Delta^\vee)_+^\theta) \\ M_-^\theta &= \pi(L(\Delta^\vee)_-^\theta) \\ M_\pm^\theta &= M_+^\theta \cap M_-^\theta. \end{aligned}$$

We first observe the group $M(H_{\mathfrak{c}}^s)$ is generated by elements m_α , where α is a real root for θ . In the notation above we therefore have

$$M(H_{\mathfrak{c}}^s) \cong M_-^\theta.$$

Moreover, the group $\pi_0(H_{\mathfrak{c}}^s)$ is a quotient of $M(H_{\mathfrak{c}}^s)$ by a subgroup determined from the elements in \mathfrak{c} . Suppose $\mathfrak{c} = \delta_1 \delta_2 \dots \delta_m$ so that

$$\theta = s_{\delta_m} \dots s_{\delta_2} s_{\delta_1} \theta^s.$$

Since all roots are real for θ^s and the roots δ_i are pairwise orthogonal, it follows the corresponding elements m_{δ_i} generate M_+^θ . Proposition 8.3.1 now implies

$$\pi_0(H_{\mathfrak{c}}^s) \cong M_-^\theta / M_-^\theta \cap M_+^\theta \cong M_-^\theta / M_\pm^\theta.$$

Since $L(\Delta^\vee)$ is the lattice dual to the weight lattice X , we see Proposition 8.3.1 is actually a dualized version of Theorem 4.1.3 for $\theta = \theta_{\mathfrak{c}}^s$. Since every involution in \mathcal{I} is conjugate to one of the form $\theta_{\mathfrak{c}}^s$, we easily obtain the correspondence for arbitrary $\theta \in \mathcal{I}$.

8.4 Genuine Triples for $\tilde{H}_{\mathfrak{c}}^s$

In the previous section we constructed a set of representatives for the K -conjugacy classes of Θ -stable Cartan subgroups in G . These subgroups (denoted $H_{\mathfrak{c}}^s$) were iterated Cayley transforms of H^s with respect to a standard sequence \mathfrak{c} . Writing $\tilde{H}_{\mathfrak{c}}^s = \pi^{-1}(H_{\mathfrak{c}}^s)$ gives an analogous set of Θ -stable Cartan subgroup representatives for \tilde{G} .

Let λ be a regular symmetric infinitesimal character, \mathfrak{c} a standard sequence in Δ , and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to the pair $(\tilde{H}_{\mathfrak{c}}^s, \phi)_\lambda$. Proposition 6.2.2 provides sufficient conditions for the existence of a genuine triple $(\tilde{H}_{\mathfrak{c}}^s, \phi, \tilde{\Gamma})_\lambda$ extending $(\tilde{H}_{\mathfrak{c}}^s, \phi)_\lambda$. Assuming an extension exists, we are finally in a position to determine how many such triples there are.

The approach is essentially the same as the one for \tilde{H}^s described at the end of Chapter 7. Recall the identity component $(\tilde{H}_{\mathfrak{c}}^s)_0$ is central in $\tilde{H}_{\mathfrak{c}}^s$ and the behavior of $\tilde{\Gamma}$ on $(\tilde{H}_{\mathfrak{c}}^s)_0$ is determined by ϕ . Proposition 7.1.3 reduces the problem to understanding $\pi_0(Z(\tilde{H}_{\mathfrak{c}}^s))$, which is naturally a subgroup of $\pi_0(\tilde{H}_{\mathfrak{c}}^s)$. Therefore we simply need representatives for the distinct connected components of $\tilde{H}_{\mathfrak{c}}^s$ whose multiplicative structure we understand.

Recall Proposition 8.3.1 implies

$$\pi_0(H_{\mathfrak{c}}^s) \cong M(H_{\mathfrak{c}}^s) / (M(H_{\mathfrak{c}}^s) \cap \langle m_{\delta_1}, \dots, m_{\delta_m} \rangle)$$

where the elements m_{δ_i} are determined by the sequence \mathfrak{c} . In words, representatives for the connected components of $H_{\mathfrak{c}}^s$ are given by choosing elements of $M(H_{\mathfrak{c}}^s)$ that are distinct ‘modulo \mathfrak{c} ’. The following proposition (whose proof is easy) extends this to $\tilde{H}_{\mathfrak{c}}^s$.

Proposition 8.4.1. *In the situation above*

$$\left| \pi_0(\tilde{H}_{\mathfrak{c}}^s) \right| = \begin{cases} 2 |\pi_0(H_{\mathfrak{c}}^s)|, & -1 \notin (\tilde{H}_{\mathfrak{c}}^s)_0 \\ |\pi_0(H_{\mathfrak{c}}^s)|, & -1 \in (\tilde{H}_{\mathfrak{c}}^s)_0 \end{cases}.$$

Moreover, if $m_{\delta} \in M(H_{\mathfrak{c}}^s)$ then \tilde{m}_{δ} and $-\tilde{m}_{\delta}$ are contained in distinct connected components of $\tilde{H}_{\mathfrak{c}}^s$ if and only if $-1 \notin (\tilde{H}_{\mathfrak{c}}^s)_0$.

Let $\{m_{\epsilon_1}, \dots, m_{\epsilon_k}\}$ be a set of representatives for the distinct connected components of $H_{\mathfrak{c}}^s$. Proposition 8.4.1 implies $(\pm) \{\tilde{m}_{\epsilon_1}, \dots, \tilde{m}_{\epsilon_k}\}$ is a set of representatives for the distinct connected components of $\tilde{H}_{\mathfrak{c}}^s$. Since the elements $\pm \tilde{m}_{\epsilon_i}$ are contained in \tilde{M} by definition, their multiplicative structure is given by Proposition 7.3.1.

Corollary 8.4.2. *Let λ be a regular symmetric infinitesimal character, \mathfrak{c} a standard sequence, and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to the pair $(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda}$. Let $\{m_{\epsilon_1}, \dots, m_{\epsilon_k}\}$ be a representative set for $\pi_0(H_{\mathfrak{c}}^s)$ and assume $(\theta, \varepsilon, \lambda)$ is supportable. Then the number of genuine triples $(\tilde{H}_{\mathfrak{c}}^s, \phi, \tilde{\Gamma})_{\lambda}$ extending $(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda}$ is given by*

$$\begin{aligned} \left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] &= \left| \left\{ \tilde{m}_{\epsilon_i} \mid [\tilde{m}_{\epsilon_i}, \tilde{m}_{\epsilon_j}] = 1, \text{ for all } 1 \leq j \leq k \right\} \right| \\ &= \left| \left\{ \tilde{m}_{\epsilon_i} \mid \tilde{m}_{\epsilon_i} \text{ is central in } \tilde{H}_{\mathfrak{c}}^s \right\} \right|. \end{aligned}$$

Remark 8.4.3. The statement of the corollary does not depend on the choices for the \tilde{m}_{ϵ_i} .

Proof. Suppose $-1 \in (\tilde{H}_{\mathfrak{c}}^s)_0$. Then the number of genuine triples extending $(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda}$ is equal to the number of connected components of $\tilde{H}_{\mathfrak{c}}^s$ contained in the center. Since the connected components are represented by the elements $\{\tilde{m}_{\epsilon_1}, \dots, \tilde{m}_{\epsilon_k}\}$, we simply need to determine the central \tilde{m}_{ϵ_i} and the result follows.

If $-1 \notin (\tilde{H}_{\mathfrak{c}}^s)_0$, then for each element of

$$\left\{ \tilde{m}_{\epsilon_i} \mid [\tilde{m}_{\epsilon_i}, \tilde{m}_{\epsilon_j}] = 1, \text{ for all } 1 \leq j \leq k \right\}$$

there are two corresponding central connected components in $\tilde{H}_{\mathfrak{c}}^s$ that differ by a factor if -1 . Since we are interested in *genuine* representations of $\tilde{H}_{\mathfrak{c}}^s$, the action of -1 is fixed and

we have only a single corresponding genuine character. Therefore the genuine characters are parameterized by the same set as the previous case and the result follows. \square

Theorem 8.4.4. *Let λ be a regular symmetric infinitesimal character, \mathfrak{c} a standard sequence, and suppose the abstract triple $(\theta, \varepsilon, \lambda)$ corresponding to the pair $(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda}$ is sup-portable. Then*

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] \in \{1, 2, 4\}.$$

Proof. Let $\{m_{\varepsilon_1}, \dots, m_{\varepsilon_k}\} \subset M$ be a set representatives for the connected components of $H_{\mathfrak{c}}^s$. By Corollary 8.4.2 we have

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] = \left| \left\{ \tilde{m}_{\varepsilon_i} \mid [\tilde{m}_{\varepsilon_i}, \tilde{m}_{\varepsilon_j}] = 1, \text{ for all } 1 \leq j \leq k \right\} \right|.$$

It suffices to verify the statement for the representative involutions $\theta_{\mathfrak{c}}^s$.

Case I. Suppose $\mathfrak{c} = \beta_1 \dots \beta_s$ and let $r = n - s$. Then we have

$$D_{\theta_{\mathfrak{c}}^s} : \begin{array}{cccc} + & \cdots & + & - \cdots - \\ 1 & & s & n \end{array}.$$

If $s = 0$, then $\theta_{\mathfrak{c}}^s = -1$ and the result follows from Corollary 7.3.9. If $s = n$, then $\theta_{\mathfrak{c}}^s = 1$ and thus

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] = 1.$$

Finally if $0 < s < n$, Case I of Section 8.3 implies $\pi_0(H_{\mathfrak{c}}^s)$ has generators

$$\{m_{\alpha_{s+1}}, m_{\alpha_{s+2}}, \dots, m_{\alpha_{n-1}}\}.$$

The proof of Proposition 7.3.5 then implies

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] = \begin{cases} 2 & r \text{ is even} \\ 1 & r \text{ is odd} \end{cases}$$

and the result holds.

Case II. Suppose $\mathfrak{c} = \alpha_1 \alpha_3 \dots \alpha_{k-1} \beta_{k+1} \dots \beta_n$ with $0 < k < n$. Then we have

$$D_{\theta_{\alpha_1 \alpha_3 \dots \alpha_{k-1} \beta_{k+1} \dots \beta_n}^s} : \begin{array}{cccc} (2) & (1) & \cdots & (k) & (k-1) & + & \cdots & + \\ 1 & & & k & k & k+1 & & n \end{array}.$$

Case II of Section 8.3 implies $\pi_0(H_{\mathfrak{c}}^s)$ is trivial and therefore

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] = 1$$

and the result holds.

Case III. Suppose $\mathfrak{c} = \alpha_1\alpha_3 \dots \alpha_{k-1}$ with $0 < k < n$ and let $r = n - k$. Then we have

$$D_{\theta_{\alpha_1\alpha_3\dots\alpha_{k-1}}}^s : \binom{(2)}{1} \binom{(1)}{\dots} \binom{(k)}{k} \binom{(k-1)}{k} - \dots - \binom{\dots}{n}.$$

Case III of Section 8.3 implies $\pi_0(H_{\mathfrak{c}}^s)$ has generators

$$\{m_{\alpha_{k+1}}, m_{\alpha_{k+2}}, \dots, m_{\alpha_{n-1}}, m_{\beta}\}.$$

The proof of Proposition 7.3.5 then implies

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] = \begin{cases} 4 & r \text{ is even} \\ 2 & r \text{ is odd} \end{cases}$$

and the result holds.

Case IV. Suppose $\mathfrak{c} = \alpha_1\alpha_3 \dots \alpha_{k-1}\beta_{k+1} \dots \beta_{k+s}$ with $0 < k < n$ and $0 < s < n - k$. Let $r = n - k - s$. Then we have

$$D_{\theta_{\alpha_1\alpha_3\dots\alpha_{k-1}\beta_{k+1}\dots\beta_{k+s}}}^s : \binom{(2)}{1} \binom{(1)}{\dots} \binom{(k)}{k} \binom{(k-1)}{k} + \dots + \binom{\dots}{k+s} - \dots - \binom{\dots}{n}.$$

Case IV of Section 8.3 implies $\pi_0(H_{\mathfrak{c}}^s)$ has generators

$$\{m_{\alpha_{k+s+1}}, m_{\alpha_{k+s+2}}, \dots, m_{\alpha_{n-1}}\}.$$

The proof of Proposition 7.3.5 then implies

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] = \begin{cases} 2 & r \text{ is even} \\ 1 & r \text{ is odd} \end{cases}$$

and the result holds.

Case V. Suppose $\mathfrak{c} = \alpha_1\alpha_3 \dots \alpha_{n-1}$. Note this is only possible when n is even. Then we have

$$D_{\theta_{\alpha_1\alpha_3\dots\alpha_{n-1}}}^s : \binom{(2)}{1} \binom{(1)}{\dots} \binom{(n)}{n} \binom{(n-1)}{n}.$$

Case V of Section 8.3 implies $\pi_0(H_{\mathfrak{c}}^s)$ has a single generator

$$\{m_{\beta}\}.$$

Therefore we have

$$\left[(\tilde{H}_{\mathfrak{c}}^s, \phi)_{\lambda} \right] = 2$$

and the result holds. □

Recall the indicator bits ϵ_s^θ , ϵ_r^θ , and ϵ_m^θ defined in Section 3.1. The results of Theorem 8.4.4 are summarized in the following corollary.

Corollary 8.4.5. *Let λ be a regular symmetric infinitesimal character, \mathbf{c} a standard sequence, and suppose the abstract triple $(\theta, \varepsilon, \lambda)$ corresponding to the pair $(\tilde{H}_{\mathbf{c}}^s, \phi)_\lambda$ is sup-portable. Then*

$$\left[(\tilde{H}_{\mathbf{c}}^s, \phi)_\lambda \right] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta (1-\epsilon_m^\theta)}.$$

Proof. Check this for each of the cases in Theorem 8.4.4. □

CHAPTER 9

K-ORBITS

Fix a nonsingular element $\lambda \in (\mathfrak{h}^a)^*$. We have seen it is important to understand the K -conjugacy classes of pairs $(H', \phi')_\lambda$, where H' is a Θ -stable Cartan subgroup in G and ϕ' is a nonsingular element in $(\mathfrak{h}')^*$ such that ϕ' and λ define the same infinitesimal character. The K -conjugacy classes of such pairs will be referred to as K -orbits (for λ). If we fix a set $\{H_i\}$ of Θ -stable Cartan subgroup representatives, the K -orbits for λ are parameterized by $N_K(H_i)$ -orbits on pairs of the form $(H_i, \phi)_\lambda$. The stabilizer of any such pair is $Z_K(H_i)$ and the quotient

$$W(G, H_i) = N_K(H_i)/Z_K(H_i)$$

is the real Weyl group for H_i (Chapter 5).

Fix a Θ -stable Cartan subgroup H and let Λ denote the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of a nonsingular element $\phi \in (\mathfrak{h})^*$. Then $W(G, H)$ acts freely on Λ and thus freely on pairs of the form $(H, \phi)_\lambda$. Therefore the K -orbits whose first entry is conjugate to H are parameterized by $W(G, H)$ -orbits in Λ . For this reason $W(G, H)$ -orbits in Λ will also be referred to as K -orbits for (the conjugacy class of) H . Since

$$|W(\mathfrak{g}, \mathfrak{h})| = |\Lambda|$$

the number of K -orbits for H is given by

$$|W(\mathfrak{g}, \mathfrak{h})/W(G, H)|.$$

In Chapter 6 we constructed an abstract triple $(\theta, \varepsilon, \lambda)$ corresponding to each pair $(H, \phi)_\lambda$. In this section we study the relationship between K -orbits and abstract triples.

9.1 The Cross Action on K -orbits

Let $\lambda \in (\mathfrak{h}^a)^*$ be a nonsingular element, H a Θ -stable Cartan subgroup of G , and suppose $\phi \in \mathfrak{h}^*$ and λ determine the same infinitesimal character. Let $\Lambda \subset \mathfrak{h}^*$ denote the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of ϕ . This section makes frequent use of the conjugation maps $\{i_{\lambda, \phi'} = i_{\phi'}\}_{\phi' \in \Lambda}$ (see Section 6.1), so we begin by recording some easy formal properties.

Lemma 9.1.1. *Let $w, \Theta \in W(\mathfrak{g}, \mathfrak{h})$ and $\theta \in W$. Then*

$$\begin{aligned}
i_{w\phi} &= w \cdot i_\phi \\
i_{w\phi}^{-1} &= i_\phi^{-1} \cdot w^{-1} \\
i_\phi(\theta) &= i_\phi \cdot \theta \cdot i_\phi^{-1} \\
i_\phi^{-1}(\Theta) &= i_\phi^{-1} \cdot \Theta \cdot i_\phi \\
i_{w\phi}(\theta) &= w \cdot i_\phi(\theta) \cdot w^{-1} \\
i_{w\phi}^{-1}(\Theta) &= i_\phi^{-1}(w^{-1} \cdot \Theta \cdot w).
\end{aligned}$$

Proof. The first two equalities are obvious and the second two are by definition. For the fifth equality we have

$$\begin{aligned}
i_{w\phi}(\theta) &= i_{w\phi} \cdot \theta \cdot i_{w\phi}^{-1} \\
&= w \cdot i_\phi \cdot \theta \cdot i_\phi^{-1} \cdot w^{-1} \\
&= w \cdot i_\phi(\theta) \cdot w^{-1}
\end{aligned}$$

as desired. The last equality is just the corresponding inverse statement. \square

Proposition 9.1.2. *Let $(H, \phi)_\lambda$ be a pair as above and suppose $w \in W(G, H)$. Then the abstract triples associated with $(H, \phi)_\lambda$ and $(H, w\phi)_\lambda$ are the same.*

Proof. Let Θ denote the Cartan involution and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple for $(H, \phi)_\lambda$. By Lemma 9.1.1 and Theorem 5.2.3 we have

$$\begin{aligned}
i_{w\phi}^{-1}(\Theta) &= i_\phi^{-1}(w^{-1} \cdot \Theta \cdot w) \\
&= i_\phi^{-1}(\Theta) \\
&= \theta.
\end{aligned}$$

Moreover if $\alpha \in \mathfrak{h}^*$ is a root,

$$i_{w\phi}^{-1}(\alpha) = i_\phi^{-1} \cdot w^{-1}(\alpha).$$

However, elements of $W(G, H)$ preserve the set of compact roots and thus the corresponding abstract grading is unchanged. \square

Proposition 9.1.2 implies the association of abstract triples to pairs $(H, \phi)_\lambda$ is defined on the level of K -orbits for H (viewed as $W(G, H)$ -orbits in \mathfrak{h}^*). We would like to understand the extent to which this association is unique. We begin by recalling the familiar action of the abstract Weyl group on Λ .

Definition 9.1.3. Let $w \in W$ and $\phi \in \Lambda$. Define the *cross action* of w on ϕ as

$$w \times \phi = i_\phi \cdot w^{-1} \cdot i_\phi^{-1}(\phi).$$

Alternatively we can define

$$w_\phi = i_\phi(w)$$

so that

$$w \times \phi = w_\phi^{-1}(\phi).$$

If $(H, \phi)_\lambda$ is the corresponding pair, the cross action of w on $(H, \phi)_\lambda$ will be denoted $w \times (H, \phi)_\lambda = (H, w \times \phi)_\lambda$.

Lemma 9.1.4. Let $w \in W$ and $\phi \in \Lambda$. Then

$$\begin{aligned} i_{w \times \phi} &= i_\phi \cdot w^{-1} \\ i_{w \times \phi}^{-1} &= w \cdot i_\phi^{-1}. \end{aligned}$$

Proof. It suffices to check the first equality. We have

$$\begin{aligned} i_{w \times \phi} &= i_{w_\phi^{-1}(\phi)} \\ &= w_\phi^{-1} \cdot i_\phi \\ &= (i_\phi(w))^{-1} \cdot i_\phi \\ &= (i_\phi \cdot w \cdot i_\phi^{-1})^{-1} \cdot i_\phi \\ &= i_\phi \cdot w^{-1} \cdot i_\phi^{-1} \cdot i_\phi \\ &= i_\phi \cdot w^{-1} \end{aligned}$$

as desired. □

Proposition 9.1.5. The cross action defines a left action of W on Λ . Moreover, the cross action commutes with the usual action of $W(\mathfrak{g}, \mathfrak{h})$ and thus descends to a transitive action at the level of K -orbits.

Proof. This is well known but we prove it anyway. For $w_1, w_2 \in W$ and $\phi \in \Lambda$ we have

$$\begin{aligned}
w_2 \times (w_1 \times \phi) &= i_{w_1 \times \phi} \cdot w_2^{-1} \cdot i_{w_1 \times \phi}^{-1}(w_1 \times \phi) \\
&= i_\phi \cdot w_1^{-1} \cdot w_2^{-1} \cdot w_1 \cdot i_\phi^{-1}(i_\phi \cdot w_1^{-1} \cdot i_\phi^{-1}(\phi)) \\
&= i_\phi \cdot w_1^{-1} \cdot w_2^{-1} \cdot i_\phi^{-1}(\phi) \\
&= (w_2 w_1) \times \phi.
\end{aligned}$$

This proves the first claim. For the second claim, let $w \in W$ and $\sigma \in W(\mathfrak{g}, \mathfrak{h})$. Then

$$\begin{aligned}
w \times (\sigma \phi) &= i_{\sigma \phi} \cdot w^{-1} \cdot i_{\sigma \phi}^{-1}(\sigma \phi) \\
&= \sigma \cdot i_\phi \cdot w^{-1} \cdot i_\phi^{-1} \cdot \sigma^{-1}(\sigma \phi) \\
&= \sigma \cdot i_\phi \cdot w^{-1} \cdot i_\phi^{-1}(\phi) \\
&= \sigma \cdot w \times \phi
\end{aligned}$$

as desired. □

If $\phi \in \Lambda$, we denote the K -orbit of ϕ by $[\phi]$ and the corresponding cross action by $w \times [\phi]$. We conclude this section with a generalization of Proposition 9.1.2.

Proposition 9.1.6. *Let $w \in W$ and suppose $(H, \phi)_\lambda$ is a pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$. Then the abstract triple associated to the pair $w \times (H, \phi)_\lambda$ is given by*

$$w \times (\theta, \varepsilon, \lambda) = (w \cdot \theta \cdot w^{-1}, w \times \varepsilon, \lambda)$$

where $w \times \varepsilon$ is the grading defined via

$$(w \times \varepsilon)(\alpha) = \varepsilon(w^{-1} \alpha).$$

In particular, the cross action on pairs induces an action (also called the cross action) on abstract triples.

Proof. Let Θ denote the Cartan involution. Then

$$\begin{aligned}
i_{w \times \phi}^{-1}(\Theta) &= i_{w \times \phi}^{-1} \cdot \Theta \cdot i_\phi \\
&= w \cdot i_\phi^{-1} \cdot \Theta \cdot i_\phi \cdot w^{-1} \\
&= w \cdot i_\phi^{-1}(\Theta) \cdot w^{-1} \\
&= w \cdot \theta \cdot w^{-1}.
\end{aligned}$$

So the abstract involution induced by the pair $w \times (H, \phi)_\lambda$ is w -conjugate to the involution induced by $(H, \phi)_\lambda$.

Now let $\alpha \in \mathfrak{h}^*$ be a root. By Lemma 9.1.4 we have

$$i_{w \times \phi}^{-1}(\alpha) = w \cdot i_{\phi}^{-1}(\alpha).$$

Therefore the cross action alters the abstract root correspondence by the regular action of w . Hence the new abstract grading $w \times \varepsilon$ will be the same as the abstract grading for $(H, \phi)_{\lambda}$ if we first compose ε with w^{-1} .

Finally, it is trivial to check this defines an action on the set of abstract triples. \square

9.2 The Fiber Over an Involution

We begin with our usual setup. Let λ be a nonsingular element in $(\mathfrak{h}^a)^*$, H a Θ -stable Cartan subgroup of G , and $\phi \in \mathfrak{h}^*$ such that ϕ and λ define the same infinitesimal character. Let $\Lambda \subset \mathfrak{h}^*$ be the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of ϕ and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to $(H, \phi)_{\lambda}$. We define the *fiber over θ* (denoted $\theta_{\lambda}^{\dagger}$) to be the set of K -orbits in Λ whose corresponding abstract triples begin with θ . Similarly, the *fiber over $(\theta, \varepsilon, \lambda)$* (denoted $(\theta, \varepsilon, \lambda)^{\dagger}$) is defined to be the set of K -orbits in Λ with associated abstract triple $(\theta, \varepsilon, \lambda)$.

Example 9.2.1. Let $n = 4$ and suppose $H = T$ is compact. Then $\theta = 1$ and every K -orbit for H is in $\theta_{\lambda}^{\dagger}$. In Example 5.2.5 we computed $|\theta_{\lambda}^{\dagger}| = 12$. Since there are only $\binom{4}{2} = 6$ possibilities for the principal abstract grading ε , there must exist an abstract triple whose fiber contains more than one K -orbit.

The order of the fiber over a fixed involution is easy to describe.

Proposition 9.2.2. *Let $(H, \phi)_{\lambda}$ be a pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$. Then*

$$|\theta_{\lambda}^{\dagger}| = \left| \frac{W^{\theta}}{W(G, H)} \right|$$

where $W(G, H) \subset W$ denotes the image of the real Weyl group for H under the map i_{ϕ}^{-1} .

Proof. By Proposition 9.1.5 and Proposition 9.1.6, W^{θ} acts transitively on $\theta_{\lambda}^{\dagger}$. Moreover, for $w \in W$ we have (by definition)

$$w \times [\phi] = [\phi] \iff w \in W(G, H).$$

So $W(G, H)$ is the stabilizer of this action and the result follows. \square

Proposition 9.2.2 implies the set of K -orbits for H can be viewed as a product of involutions and fibers. Numerically we have

$$\left| \frac{W(\mathfrak{g}, \mathfrak{h})}{W(G, H)} \right| = \left| \frac{W(\mathfrak{g}, \mathfrak{h})}{W^\theta} \right| \times \left| \frac{W^\theta}{W(G, H)} \right|.$$

of K -orbits for H # of involutions size of each fiber

In addition to determining $|\theta_\lambda^\dagger|$, it will be important to understand the transitive action of W^θ on θ_λ^\dagger . Since the situation is not changed by conjugation, it suffices to determine this action for a set of representative involutions in \mathcal{I} . Retain the setting of the proposition and recall from Section 3.3 and Section 5.2

$$\begin{aligned} W^\theta &\cong (W_i^\theta \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta \\ W(G, H) &\cong ((A \times W_{ic}^\theta) \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta \end{aligned}$$

where $A \cong \mathbb{Z}_2^k$ is a subgroup of W_i^θ . Therefore the action of $w \in W^\theta$ on θ_λ^\dagger is trivial if w is an element of $W_{ic}^\theta, W_{\mathbb{R}}^\theta$, or $W_{\mathbb{C}}^\theta$. Hence it remains to understand the action of W_i^θ on θ_λ^\dagger , and specifically the action for (reflections corresponding to) noncompact imaginary roots.

A noncompact imaginary root in $\Delta(\mathfrak{g}, \mathfrak{h})$ is said to be of *type I* if the corresponding reflection is not induced by an element of K (i.e., by an element in $N_K(H)$). Otherwise it is said to be of *type II*, with analogous terminology used for associated abstract roots. In particular, the cross action through an abstract noncompact imaginary root is nontrivial if and only if it is of type I (Proposition 12.5.1).

Example 9.2.3. We continue Example 9.2.1. Let $(\theta, \varepsilon, \lambda)$ be the abstract triple corresponding to the K -orbit $[\phi]$ where ε is given by the diagram

$$+ + \oplus \oplus .$$

Let $\alpha \in \Delta^+$ be a long root of the form $\alpha = e_i - e_j$. Note α is noncompact if and only if $1 \leq i \leq 2$ and $3 \leq j \leq 4$. In this case we have

$$\begin{aligned} (s_\alpha \times \varepsilon)(e_i) &= \varepsilon(s_\alpha e_i) \\ &= \varepsilon(e_j) \neq \varepsilon(e_i) \end{aligned}$$

and thus the abstract triples corresponding to $s_\alpha \times [\phi]$ and $[\phi]$ are different. Proposition 9.1.2 implies $s_\alpha \times [\phi] \neq [\phi]$ and therefore α is of type I. A similar argument holds for roots of

the form $\alpha = e_i + e_j$ and hence all long noncompact imaginary roots are of type I. However if $\alpha = e_3$ or $\alpha = e_4$ is a short noncompact imaginary root, then it is easy to check

$$s_\alpha \times \varepsilon = \varepsilon$$

and thus we can not determine (yet) the type of α .

We now outline an effective method for describing the action of W_i^θ on θ_λ^\dagger . Let $(H, \phi)_\lambda$ be a pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$ and suppose the imaginary roots $\alpha, \beta \in \Delta$ are noncompact. We have seen there exists a ‘dual’ pair (H', ϕ') whose corresponding abstract involution is $-\theta$. At the end of Section 8.3 we associated the following data to $-\theta$

$$\begin{aligned} M &= L(\Delta^\vee)/2L(\Delta^\vee) \\ L(\Delta^\vee)_+^{-\theta} &= \{\alpha \in L(\Delta^\vee) \mid -\theta(\alpha) = \alpha\} \\ L(\Delta^\vee)_-^{-\theta} &= \{\alpha \in L(\Delta^\vee) \mid -\theta(\alpha) = -\alpha\} \\ M_+^{-\theta} &= \pi(L(\Delta^\vee)_+^{-\theta}) \\ M_-^{-\theta} &= \pi(L(\Delta^\vee)_-^{-\theta}) \\ M_\pm^{-\theta} &= M_+^{-\theta} \cap M_-^{-\theta}. \end{aligned}$$

Note the roots α, β are real for $-\theta$ and recall the associated elements in $M_-^{-\theta}$ were denoted m_α, m_β .

Let $w \in W_i^\theta$ and write $w = s_{\alpha_n} \dots s_{\alpha_1}$ as a minimal length product of simple reflections in W_i^θ . Define the element

$$\begin{aligned} m_w^{[\phi]} &= \varepsilon(\alpha_1)m_{\alpha_1} + s_{\alpha_1} \times \varepsilon(\alpha_2)m_{\alpha_2} + \dots + s_{\alpha_{n-1}} \times \dots \times s_{\alpha_1} \times \varepsilon(\alpha_n)m_{\alpha_n} \\ &= \varepsilon(\alpha_1)m_{\alpha_1} + \varepsilon(s_{\alpha_1}\alpha_2)m_{\alpha_2} + \dots + \varepsilon(s_{\alpha_1} \dots s_{\alpha_{n-1}}\alpha_n)m_{\alpha_n} \end{aligned}$$

where the m_{α_i} are viewed as elements in $M_-^{-\theta}/M_\pm^{-\theta}$. Note the element $m_w^{[\phi]}$ depends on the grading ε and thus on the K -orbit $[\phi]$.

Proposition 9.2.4. *The definition above gives a well-defined map*

$$w \mapsto m_w^{[\phi]}$$

of W_i^θ into $M_-^{-\theta}/M_\pm^{-\theta}$.

Proof. The issue is the choice of reduced expression for w . Proposition 3.3.6 implies

$$W_i^\theta \cong W(B_m) \times W(A_1)^l$$

for some integers m and l and thus it suffices to consider the following three cases (see [6], Theorem 3.3.1).

Case I. Suppose $s_\alpha s_\beta = s_\beta s_\alpha$, where α, β are simple roots in Δ_i^θ . Then we have

$$\begin{aligned} m_{s_\beta s_\alpha}^{[\phi]} &= \varepsilon(\alpha)m_\alpha + \varepsilon(s_\alpha\beta)m_\beta \\ &= \varepsilon(\alpha)m_\alpha + \varepsilon(\beta)m_\beta \\ &= \varepsilon(\beta)m_\beta + \varepsilon(s_\beta\alpha)m_\alpha \\ &= m_{s_\alpha s_\beta}^{[\phi]}. \end{aligned}$$

Case II. Suppose $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$, where α, β are (long) adjacent simple roots in Δ_i^θ . Then we have

$$\begin{aligned} m_{s_\alpha s_\beta s_\alpha}^{[\phi]} &= \varepsilon(\alpha)m_\alpha + \varepsilon(s_\alpha\beta)m_\beta + \varepsilon(s_\alpha s_\beta \alpha)m_\alpha \\ &= \varepsilon(\alpha)m_\alpha + \varepsilon(\alpha + \beta)m_\beta + \varepsilon(\beta)m_\alpha \\ &= \varepsilon(\alpha + \beta)m_\alpha + \varepsilon(\alpha + \beta)m_\beta \\ &= \varepsilon(\beta)m_\beta + \varepsilon(\beta + \alpha)m_\alpha + \varepsilon(\alpha)m_\beta \\ &= \varepsilon(\beta)m_\beta + \varepsilon(s_\beta\alpha)m_\alpha + \varepsilon(s_\beta s_\alpha \beta)m_\beta \\ &= m_{s_\beta s_\alpha s_\beta}^{[\phi]}. \end{aligned}$$

Case III. Suppose $s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha$, where α, β are adjacent simple roots in Δ_i^θ with α short. Then

$$\begin{aligned} m_{s_\beta s_\alpha s_\beta s_\alpha}^{[\phi]} &= \varepsilon(\alpha)m_\alpha + \varepsilon(s_\alpha\beta)m_\beta + \varepsilon(s_\alpha s_\beta \alpha)m_\alpha + \varepsilon(s_\alpha s_\beta s_\alpha \beta)m_\beta \\ &= \varepsilon(\alpha)m_\alpha + \varepsilon(\beta)m_\beta + \varepsilon(\alpha + \beta)m_\alpha + \varepsilon(\beta)m_\beta \\ &= \varepsilon(\alpha)m_\alpha + \varepsilon(\alpha)m_\alpha + \varepsilon(\beta)m_\beta + \varepsilon(\beta)m_\beta + \varepsilon(\beta)m_\alpha \\ &= \varepsilon(\beta)m_\alpha. \end{aligned}$$

Similarly

$$\begin{aligned} m_{s_\alpha s_\beta s_\alpha s_\beta}^{[\phi]} &= \varepsilon(\beta)m_\beta + \varepsilon(s_\beta\alpha)m_\alpha + \varepsilon(s_\beta s_\alpha \beta)m_\beta + \varepsilon(s_\beta s_\alpha s_\beta \alpha)m_\alpha \\ &= \varepsilon(\beta)m_\beta + \varepsilon(\beta + \alpha)m_\alpha + \varepsilon(\beta)m_\beta + \varepsilon(\alpha)m_\alpha \\ &= \varepsilon(\beta)m_\beta + \varepsilon(\beta)m_\beta + \varepsilon(\alpha)m_\alpha + \varepsilon(\alpha)m_\alpha + \varepsilon(\beta)m_\alpha \\ &= \varepsilon(\beta)m_\alpha \end{aligned}$$

and thus

$$m_{s_\beta s_\alpha s_\beta s_\alpha}^{[\phi]} = m_{s_\alpha s_\beta s_\alpha s_\beta}^{[\phi]}.$$

□

The following proposition computes the elements $m_{s_\alpha}^{[\phi]}$, where $\alpha \in \Delta$ is an arbitrary (i.e., not necessarily simple) imaginary root. The result is as expected.

Proposition 9.2.5. *Let $[\phi]$ be a K -orbit in θ_λ^\dagger and let $\beta \in \Delta$ be an imaginary root for θ . Then*

$$m_{s_\beta}^{[\phi]} = \varepsilon(\beta)m_\beta.$$

Proof. It suffices to consider the case when $W_i^\theta \cong W(B_m)$. Denote the long simple roots in Δ_i^θ by

$$\alpha_i = e_i - e_{i+1}$$

for $1 \leq i \leq m-1$. We have the following three cases for β .

Case I. Suppose β is of the form

$$\beta = e_i - e_{j+1} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$$

for $1 \leq i \leq j$ so that

$$s_\beta = s_{\alpha_j} s_{\alpha_{j-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} s_{\alpha_{i+1}} \cdots s_{\alpha_{j-1}} s_{\alpha_j}$$

is a reduced expression for s_β . Then

$$\begin{aligned} m_{s_\beta}^{[\phi]} &= \varepsilon(\alpha_j)m_{\alpha_j} + \varepsilon(s_{\alpha_j}\alpha_{j-1})m_{\alpha_{j-1}} + \cdots + \varepsilon(s_{\alpha_j} \cdots s_{\alpha_{i+1}}\alpha_i)m_{\alpha_i} \\ &\quad + \varepsilon(s_{\alpha_j} \cdots s_{\alpha_i}\alpha_{i+1})m_{\alpha_{i+1}} + \cdots + \varepsilon(s_{\alpha_j} \cdots s_{\alpha_i} \cdots s_{\alpha_{j-1}}\alpha_j)m_{\alpha_j} \\ &= \varepsilon(\alpha_j)m_{\alpha_j} + \varepsilon(\alpha_j + \alpha_{j-1})m_{\alpha_{j-1}} + \cdots + \varepsilon(\alpha_j + \alpha_{j-1} + \cdots + \alpha_i)m_{\alpha_i} \\ &\quad + \varepsilon(\alpha_i)m_{\alpha_{i+1}} + \varepsilon(\alpha_i + \alpha_{i+1})m_{\alpha_{i+2}} + \cdots + \varepsilon(\alpha_i + \cdots + \alpha_{j-1})m_{\alpha_j} \\ &= \varepsilon(\alpha_i + \cdots + \alpha_j)m_{\alpha_i} + \cdots + \varepsilon(\alpha_i + \cdots + \alpha_j)m_{\alpha_j} \\ &= \varepsilon(\beta)m_{\alpha_i} + \cdots + \varepsilon(\beta)m_{\alpha_j} \\ &= \varepsilon(\beta)m_\beta \end{aligned}$$

as desired (the last step follows from Proposition 7.2.5).

Case II. Suppose β is of the form

$$\beta = e_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{m-1} + e_m$$

for $1 \leq i \leq m$ so that

$$s_\beta = s_{\alpha_i} s_{\alpha_{i+1}} \cdots s_{\alpha_{m-1}} s_{e_m} s_{\alpha_{m-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i}$$

is a reduced expression for s_β . Then

$$\begin{aligned} m_{s_\beta}^{[\phi]} &= \varepsilon(\alpha_i) m_{\alpha_i} + \varepsilon(s_{\alpha_i} \alpha_{i+1}) m_{\alpha_{i+1}} + \cdots + \varepsilon(s_{\alpha_i} \cdots s_{\alpha_{m-1}} e_m) m_{e_m} \\ &\quad + \varepsilon(s_{\alpha_i} \cdots s_{e_m} \alpha_{m-1}) m_{\alpha_{m-1}} + \cdots + \varepsilon(s_{\alpha_i} \cdots s_{e_m} \cdots s_{\alpha_{i+1}} \alpha_i) m_{\alpha_i} \\ &= \varepsilon(\alpha_i) m_{\alpha_i} + \varepsilon(\alpha_i + \alpha_{i+1}) m_{\alpha_{i+1}} + \cdots + \varepsilon(\alpha_i + \cdots + \alpha_{m-1} + e_m) m_{e_m} \\ &\quad + \varepsilon(\alpha_i + \cdots + \alpha_{m-1} + 2e_m) m_{\alpha_{m-1}} + \cdots + \\ &\quad + \varepsilon(\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{m-1} + 2e_m) m_{\alpha_i} \\ &= \varepsilon(2\beta) m_{\alpha_i} + \cdots + \varepsilon(2\beta) m_{\alpha_{m-1}} + \varepsilon(\beta) m_{e_m} \\ &= \varepsilon(\beta) m_{e_m} \\ &= \varepsilon(\beta) m_\beta \end{aligned}$$

as desired (the last step again follows from Proposition 7.2.5).

Case III. Suppose $\beta = e_i + e_{j+1}$. This case is handled in the same fashion as the previous cases. The reader is spared the details. □

Let $[\phi]$ and $[\psi]$ be two K -orbits for H in θ_λ^\dagger . For $w \in W_i^\theta$, the following proposition describes how $m_w^{[\phi]}$ and $m_w^{[\psi]}$ are related.

Proposition 9.2.6. *Let $[\phi]$ and $[\psi]$ be two K -orbits in θ_λ^\dagger and suppose $[\psi] = \tau \times [\phi]$. For $w \in W_i^\theta$ we have*

$$m_w^{[\psi]} = m_{w\tau}^{[\phi]} + m_\tau^{[\phi]}.$$

Proof. Let ε denote the abstract grading corresponding to $[\phi]$ and choose a reduced expression $w = s_{\alpha_n} \cdots s_{\alpha_1}$. By definition we have

$$\begin{aligned} m_w^{[\psi]} &= m_w^{\tau \times [\phi]} \\ &= \tau \times \varepsilon(\alpha_1) m_{\alpha_1} + \tau \times \varepsilon(s_{\alpha_1} \alpha_2) m_{\alpha_2} + \cdots + \tau \times \varepsilon(s_{\alpha_1} \cdots s_{\alpha_{n-1}} \alpha_n) m_{\alpha_n} \\ &= \varepsilon(\tau^{-1} \alpha_1) m_{\alpha_1} + \varepsilon(\tau^{-1} s_{\alpha_1} \alpha_2) m_{\alpha_2} + \cdots + \varepsilon(\tau^{-1} s_{\alpha_1} \cdots s_{\alpha_{n-1}} \alpha_n) m_{\alpha_n} \\ &= m_{w\tau}^{[\phi]} + m_\tau^{[\phi]} \end{aligned}$$

as desired. \square

Proposition 9.2.6 provides an iterative method for computing $m_w^{[\phi]}$.

Corollary 9.2.7. *Let $w \in W_i^\theta$ and suppose $w = s_{\alpha_n} \cdots s_{\alpha_1}$ is a reduced expression for w . Then*

$$m_w^{[\phi]} = m_{s_{\alpha_1}}^{[\phi]} + m_{s_{\alpha_2}}^{s_{\alpha_1} \times [\phi]} + \cdots + m_{s_{\alpha_n}}^{s_{\alpha_{n-1}} \times \cdots \times s_{\alpha_1} \times [\phi]}.$$

Proof. This follows easily by induction and Proposition 9.2.6. \square

Corollary 9.2.8. *Let $[\phi]$ and $[\psi]$ be two K -orbits in θ_λ^\dagger and suppose $[\psi] = \tau \times [\phi]$. Then*

$$\left| \left\{ m_w^{[\phi]} \mid w \in W_i^\theta \right\} \right| = \left| \left\{ m_w^{[\psi]} \mid w \in W_i^\theta \right\} \right|.$$

Proof. Proposition 9.2.6 implies these sets differ by translation through $m_\tau^{[\phi]}$. \square

Our interest in the elements $m_w^{[\phi]}$ is justified by the following remarkable theorem.

Theorem 9.2.9 ([9]). *Let $(H, \phi)_\lambda$ be a pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$ and let w_1 and w_2 be elements in W_i^θ . Then*

$$w_1 \times [\phi] = w_2 \times [\phi] \iff m_{w_1}^{[\phi]} = m_{w_2}^{[\phi]}.$$

In particular, a noncompact imaginary root α is of type II if and only if $m_{s_\alpha}^{[\phi]} = m_\alpha$ is trivial in the quotient $M_-^{-\theta} / M_\pm^{-\theta}$.

Remark 9.2.10. Recall elements in $M_-^{-\theta} / M_\pm^{-\theta}$ represent the connected components of the dual torus H' . Theorem 9.2.9 implies

$$w_1 \times [\phi] = w_2 \times [\phi]$$

if and only if the elements $m_{w_1}^{[\phi]}, m_{w_2}^{[\phi]} \in H'$ live in the same connected component.

It follows from Theorem 9.2.9 that elements in θ_λ^\dagger are parameterized by distinct elements of the form $m_w^{[\phi]}$, for $w \in W_i^\theta$. Formally we can write

$$\theta_\lambda^\dagger \longleftrightarrow \left\{ m_w^{[\phi]} \cdot [\phi] \mid w \in W_i^\theta \right\}$$

where $m_w^{[\phi]} \cdot [\phi]$ is a formal symbol representing the K -orbit $w \times [\phi]$ with $m_w^{[\phi]} \cdot [\phi] = m_\tau^{[\phi]} \cdot [\phi]$ if and only if $m_w^{[\phi]} = m_\tau^{[\phi]}$. The action of W_i^θ on θ_λ^\dagger is then

$$\begin{aligned} \tau \times (m_w^{[\phi]} \cdot [\phi]) &= m_\tau^{w \times [\phi]} + (m_w^{[\phi]} \cdot [\phi]) \\ &= (m_\tau^{w \times [\phi]} + m_w^{[\phi]}) \cdot [\phi] \\ &= m_{\tau w}^{[\phi]} \cdot [\phi] \end{aligned}$$

by Proposition 9.2.6.

Example 9.2.11. We are now able to finish Example 9.2.1. Recall $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to the K -orbit $[\phi]$, with $\theta = 1$ and ε given by the diagram

$$+ + \oplus \oplus .$$

Then $-\theta = -1$ so that $M_+^{-\theta}$ is trivial. Proposition 9.2.9 implies all noncompact imaginary roots (including the short roots e_3 and e_4) are of type I. Moreover $m_{e_3} = m_{e_4}$ in $M_-^{-\theta}$ and thus

$$s_{e_3} \times [\phi] = s_{e_4} \times [\phi].$$

We now give a complete description of the action of $W_i^\theta = W$ on θ_λ^\dagger . Let

$$\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= e_2 - e_3 \\ \alpha_3 &= e_3 - e_4 \\ \beta &= e_4 \end{aligned}$$

denote the simple roots for W . Table 9.1 summarizes the action of $W_i^\theta = W$ on θ_λ^\dagger .

Each row in the table represents a particular K -orbit in θ_λ^\dagger . The first column assigns each orbit a number, the second column gives the corresponding element of $M_-^{-\theta}$ (written as a bit string), and the third column describes the associated abstract grading. The last four columns give images of cross actions for the simple roots (in terms of orbit numbers from column one).

Note that each abstract grading appears exactly twice and the corresponding elements in $M_-^{-\theta}$ differ by 0001 (i.e., by a cross action through a short noncompact imaginary root). Therefore we see the order of the fiber over any abstract triple is two.

9.1. ACTION OF W_i^θ

<i>orbit</i>	$m_w^{[\phi]}$	<i>grading</i>	α_1	α_2	α_3	β
0	0000	+ + \oplus \oplus	0	1	0	2
1	0100	+ \oplus + \oplus	3	0	4	5
2	0001	+ + \oplus \oplus	2	5	2	0
3	1100	\oplus + + \oplus	1	3	6	7
4	0110	+ \oplus \oplus +	6	4	1	4
5	0101	+ \oplus + \oplus	7	2	8	1
6	1110	\oplus + \oplus +	4	9	3	6
7	1101	\oplus + + \oplus	5	7	10	3
8	0111	+ \oplus \oplus +	10	8	5	8
9	1010	\oplus \oplus + +	9	6	9	9
10	1111	\oplus + \oplus +	8	11	7	10
11	1011	\oplus \oplus + +	11	10	11	11

9.3 Fibers for G

Fix a nonsingular element $\lambda \in (\mathfrak{h}^a)^*$ and suppose $(H, \phi)_\lambda$ is a pair for G with corresponding abstract triple $(\theta, \varepsilon, \lambda)$. The previous section outlined a procedure for computing the action of W_i^θ on θ_λ^\dagger . In this section we use this procedure to describe the fibers over arbitrary involutions and abstract triples for G . As usual, the picture is unchanged by conjugation and thus it suffices to consider a representative set of involutions in \mathcal{I} . This will essentially give a complete description of the set of K -orbits for λ in G .

Given λ and θ , we can extend these parameters to an abstract triple $(\theta, \varepsilon, \lambda)$ by selecting a principal imaginary grading ε . Recall ε is specified by choosing exactly $\frac{n_s^\theta + \varepsilon_p^\theta}{2}$ of the short imaginary roots to be noncompact. In particular, the number of abstract triples beginning with θ is given by

$$\binom{n_s^\theta}{\frac{n_s^\theta + \varepsilon_p^\theta}{2}}.$$

Proposition 9.1.2 implies there will be at least $\binom{n_s^\theta}{\frac{n_s^\theta + \varepsilon_p^\theta}{2}}$ elements in θ_λ^\dagger . The following theorem completes the picture by determining the number of elements in the fiber over each abstract triple for θ .

Theorem 9.3.1. *Let $(H, \phi)_\lambda$ be a pair for G and suppose $(\theta, \varepsilon, \lambda)$ is the corresponding abstract triple. Then*

$$\left| (\theta, \varepsilon, \lambda)^\dagger \right| \in \{1, 2\}.$$

Proof. The proof is by cases for a representative set of involutions in \mathcal{I} . In each case we determine the number of elements in $(\theta, \varepsilon, \lambda)^\dagger$ as well as how the elements are related by cross action.

Case I. Let $\theta \in \mathcal{I}$ be given by the diagram

$$D_\theta : \begin{array}{c} + \\ 1 \end{array} + \cdots + \begin{array}{c} + \\ n \end{array}$$

so that $n_s^\theta = n$, $n_r^\theta = 0$, and all roots are imaginary. Suppose $w \in W_i^\theta = W$ with $w = (00 \cdots 0, \sigma)$. Then

$$w \times \varepsilon = \varepsilon$$

if and only if σ preserves the set of compact roots. Such permutations are given by elements of W_{ic}^θ and thus act trivially on K -orbits.

Suppose now $w = (\epsilon_1 \epsilon_2 \cdots \epsilon_n, 1)$ is a product of reflections in short noncompact imaginary roots. Since $m_{e_i} = m_{e_j}$ in $M_-^{-\theta}$ for all i and j , Theorem 9.2.9 implies the cross action through any two such roots is the same. Moreover, this action is nontrivial since $M_+^{-\theta} = \{0\}$. Hence

$$\left| (\theta, \varepsilon, \lambda)^\dagger \right| = 2$$

with the elements related by a cross action in any short noncompact imaginary root (this is a generalization of Example 9.2.11). We conclude

$$\left| \theta_\lambda^\dagger \right| = \begin{cases} 2 \binom{n}{\frac{n}{2}} & n \text{ is even} \\ 2 \binom{n+1}{\frac{n+1}{2}} & n \text{ is odd} \end{cases} .$$

Case I'. Let $\theta \in \mathcal{I}$ be given by the diagram

$$D_\theta : \begin{array}{c} + \\ 1 \end{array} \cdots + \begin{array}{c} + \\ n_s^\theta \end{array} - \cdots - \begin{array}{c} - \\ n \end{array}$$

with $n_r^\theta = n - n_s^\theta \neq 0$. The situation is the same as the previous case, except the elements m_{e_i} are now in $M_\pm^{-\theta}$. Hence all short noncompact imaginary roots are of type II and thus

$$\left| (\theta, \varepsilon, \lambda)^\dagger \right| = 1.$$

Therefore

$$\left| \theta_\lambda^\dagger \right| = \binom{n_s^\theta}{\frac{n_s^\theta + \epsilon_p^\theta}{2}}.$$

Case II. Let $\theta \in \mathcal{I}$ be given by the diagram

$$D_\theta : \begin{array}{ccccccc} (2) & (1) & \cdots & (k) & (k-1) & + & \cdots + \\ \underset{1}{} & & & & \underset{k}{} & \underset{k+1}{} & \underset{n}{} \end{array}$$

where $k = n_c^\theta$, $n_s^\theta = n - k \neq 0$, and $n_r^\theta = 0$. Then

$$W_i^\theta \cong W(A_1)^{k/2} \times W(B_{n-k})$$

and cross actions for elements of W_i^θ that live in the $W(B_{n-k})$ factor behave as in Case I.

Suppose now i is odd with $1 \leq i \leq k - 1$ and define the roots

$$\begin{aligned} \alpha_i &= e_i - e_{i+1} \\ \beta_i &= e_i + e_{i+1}. \end{aligned}$$

Then each root β_i is real for θ and imaginary for $-\theta$ and each root α_i is noncompact imaginary for θ and real for $-\theta$. Therefore it remains to determine the type of each α_i . By Proposition 7.2.5 we have

$$m_{\alpha_i} = m_{e_j}$$

in $M_-^{-\theta}/M_\pm^{-\theta}$, where $k + 1 \leq j \leq n$. Theorem 9.2.9 then implies the cross actions with respect to s_{α_i} and s_{e_j} are the same (assuming e_j is noncompact). The analysis in Case I shows this action is nontrivial and thus each α_i is of type I. Since the actions are the same we conclude

$$\left| (\theta, \varepsilon, \lambda)^\dagger \right| = 2.$$

These elements are related by a cross action in any of the roots α_i or in any short noncompact imaginary root e_j . We then have

$$\left| \theta_\lambda^\dagger \right| = 2 \binom{n_s^\theta}{\frac{n_s^\theta + c_p^\theta}{2}}.$$

Case III. Let $\theta \in \mathcal{I}$ be given by the diagram

$$D_\theta : \begin{array}{ccccccc} (2) & (1) & \cdots & (k) & (k-1) & - & \cdots - \\ \underset{1}{} & & & & \underset{k}{} & \underset{k+1}{} & \underset{n}{} \end{array}$$

where $k = n_c^\theta$ and $n_r^\theta = n - k \neq 0$. The principal abstract grading ε and associated abstract triple are unique for θ . By the results of Section 4.2, the corresponding dual torus is connected and Theorem 9.2.9 implies

$$\left| \theta_\lambda^\dagger \right| = \left| (\theta, \varepsilon, \lambda)^\dagger \right| = 1.$$

Given a pair $(H, \phi)_\lambda$ with corresponding abstract triple $(\theta, \varepsilon, \lambda)$, we can use Theorem 9.3.1 to determine the order of $W(G, H)$. We have

$$W(G, H) \cong ((A \times W_{ic}^\theta) \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta$$

where $A \cong \mathbb{Z}_2^k$ is the only unknown group. Let $k = \frac{n_c^\theta}{2}$. Combining Corollary 9.3.2 with Proposition 9.2.2 gives

$$\begin{aligned} |\theta_\lambda^\dagger| &= 2^{1-\epsilon_r^\theta} \binom{n_s^\theta}{\frac{n_s^\theta + \epsilon_p^\theta}{2}} = \left| \frac{W^\theta}{W(G, H)} \right| \\ &= \left| \frac{(W_i^\theta \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta}{((A \times W_{ic}^\theta) \times W_{\mathbb{R}}^\theta) \rtimes W_{\mathbb{C}}^\theta} \right| \\ &= \frac{|W_i^\theta|}{|A \times W_{ic}^\theta|} \\ &= \frac{2^k 2^{\epsilon_s^\theta} \binom{n_s^\theta}{\frac{n_s^\theta + \epsilon_p^\theta}{2}}}{|A|} \end{aligned}$$

so that

$$\begin{aligned} |A| &= \frac{2^k 2^{\epsilon_s^\theta}}{2^{1-\epsilon_r^\theta}} \\ &= 2^{k+\epsilon_s^\theta+\epsilon_r^\theta-1}. \end{aligned}$$

Comparing this to Corollary 4.2.1 leads to a verification of Proposition 5.2.4.

9.4 Fibers for \tilde{G}

Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character. Recall the set $\tilde{\mathcal{D}}_\lambda$ is parameterized by the \tilde{K} -conjugacy classes of genuine triples $(\tilde{H}, \phi, \tilde{\Gamma})$, where \tilde{H} is a Θ -stable Cartan subgroup of \tilde{G} , $\phi \in \mathfrak{h}^*$ such that ϕ and λ define the same infinitesimal character, and $\tilde{\Gamma}$ is a genuine representation of \tilde{H} . Theorem 8.4.4 reduces the problem to understanding the \tilde{K} -conjugacy classes of genuine pairs $(\tilde{H}, \phi)_\lambda$ whose corresponding abstract triples are supportable. A \tilde{K} -conjugacy class of genuine pairs $(\tilde{H}, \phi)_\lambda$ will be called a \tilde{K} -orbit for λ . We have seen the \tilde{K} -orbits for λ correspond naturally to the K -orbits for λ , and these were parameterized in Theorem 9.3.1.

Let $(\tilde{H}, \phi)_\lambda$ be a genuine pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$. We define the *genuine fiber over θ* (denoted $\tilde{\theta}_\lambda^\dagger$) to be the set of K -orbits in θ_λ^\dagger whose corresponding abstract triples are supportable. In particular $\left[(\tilde{H}, \phi)_\lambda \right] \neq 0$ if and only if $[\phi] \in \tilde{\theta}_\lambda^\dagger$. Although the order of θ_λ^\dagger can be arbitrarily large, the following theorem shows the order of $\tilde{\theta}_\lambda^\dagger$ is always small.

Theorem 9.4.1. *Let $\lambda \in (\mathfrak{h}^a)^*$ be a nonsingular symmetric infinitesimal character and suppose $(\tilde{H}, \phi)_\lambda$ is a genuine pair for λ . Let $\theta \in \mathcal{I}$ be the corresponding abstract involution and assume there exists a supportable abstract triple beginning with θ (i.e., θ is supportable).*

Then

$$|\tilde{\theta}_\lambda^\dagger| \in \{1, 2, 4\}.$$

Proof. We first determine the number of gradings ε that make the abstract triple $(\theta, \varepsilon, \lambda)$ supportable (by assumption there exists at least one). We then use Theorem 9.3.1 to determine the order of the fiber over each abstract triple. As usual we proceed by cases for the diagram of θ .

Case I. Let θ be given by the diagram

$$D_\theta : \underset{1}{+} + \cdots + \underset{n}{+}$$

so that $n_s^\theta = n$. There are two possible gradings leading to supportable abstract triples for θ if n_s^θ is even and only one if n_s^θ is odd (see Section 6.3). In either case, Theorem 9.3.1 implies the fiber over each abstract triple is two. Therefore we have

$$|\tilde{\theta}_\lambda^\dagger| = \begin{cases} 4 & n_s^\theta = n \text{ is even} \\ 2 & n_s^\theta = n \text{ is odd} \end{cases}.$$

Case I'. Let θ be given by the diagram

$$D_\theta : \underset{1}{+} \cdots + \underset{n_s^\theta}{+} - \cdots - \underset{n}{-}$$

with $n_r^\theta = n - n_s^\theta \neq 0$. Combining Case I above with Case I' of Theorem 9.3.1 gives

$$|\tilde{\theta}_\lambda^\dagger| = \begin{cases} 2 & n_s^\theta \text{ is even} \\ 1 & n_s^\theta \text{ is odd} \end{cases}.$$

Case II. Let θ be given by the diagram

$$D_\theta : \underset{1}{(2)} (1) \cdots (k) \underset{k}{(k-1)} + \underset{k+1}{+} \cdots + \underset{n}{+}$$

where $k = n_c^\theta$, $n_s^\theta = n - k \neq 0$, and $n_r^\theta = 0$. Combining Case I above with Case II of Theorem 9.3.1 gives

$$|\tilde{\theta}_\lambda^\dagger| = \begin{cases} 4 & n_s^\theta \text{ is even} \\ 2 & n_s^\theta \text{ is odd} \end{cases}.$$

Case III. Let θ be given by the diagram

$$D_\theta : \begin{array}{ccccccc} (2) & (1) & \cdots & (k) & (k-1) & - & \cdots - \\ \underset{1}{} & & & & \underset{k}{} & \underset{k+1}{} & \underset{n}{} \end{array}$$

where $k = n_c^\theta$ and $n_r^\theta = n - k \neq 0$. There is only one possible abstract triple (supportable by hypothesis) whose fiber is a singleton by Theorem 9.3.1. Therefore

$$|\tilde{\theta}_\lambda^\dagger| = 1.$$

Case IV. Let θ be given by the diagram

$$D_\theta : \begin{array}{ccccccc} (2) & (1) & \cdots & (k) & (k-1) & + & \cdots + & - & \cdots - \\ \underset{1}{} & & & & \underset{k}{} & \underset{k+1}{} & \underset{k+n_s^\theta}{} & & \underset{n}{} \end{array}$$

where $k = n_c^\theta$ and $n_r^\theta = n - k - n_s^\theta \neq 0$. Combining Case I above with Case IV of Theorem 9.3.1 gives

$$|\tilde{\theta}_\lambda^\dagger| = \begin{cases} 2 & n_s^\theta \text{ is even} \\ 1 & n_s^\theta \text{ is odd} \end{cases}.$$

Case V. Let θ be given by the diagram

$$D_\theta : \begin{array}{ccccccc} (2) & (1) & \cdots & (n) & (n-1) \\ \underset{1}{} & & & & \underset{n}{} \end{array}$$

Note this is possible only when n is even. As in Case III, there is a unique abstract triple beginning with θ . Theorem 9.3.1 implies its fiber has order two. Therefore we have

$$|\tilde{\theta}_\lambda^\dagger| = 2.$$

□

Recall the indicator bits ϵ_s^θ , ϵ_r^θ , ϵ_p^θ , and ϵ_m^θ from Section 3.1. The following corollary summarizes the results of Theorem 9.4.1.

Corollary 9.4.2. *Let $(\tilde{H}, \phi)_\lambda$ be a genuine pair with corresponding abstract involution $\theta \in \mathcal{I}$. If θ is supportable, then*

$$|\tilde{\theta}_\lambda^\dagger| = 2^{1-\epsilon_r^\theta} 2^{\epsilon_s^\theta(1-\epsilon_p^\theta)}.$$

Proof. Check this for each of the cases above. □

Fix a symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and recall we have constructed a (well-defined) abstract triple $(\theta, \varepsilon, \lambda)$ for each element in $\tilde{\mathcal{D}}_\lambda$. Let $\tilde{\mathcal{D}}_\lambda^\theta$ denote the subset of $\tilde{\mathcal{D}}_\lambda$ whose elements have abstract triples beginning with θ . We conclude this section with a description of the size of $\tilde{\mathcal{D}}_\lambda^\theta$, which also turns out to be small.

Theorem 9.4.3. *Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character and let $\theta \in \mathcal{I}$. If θ is supportable for λ , then*

$$\begin{aligned} \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} 2^{1-\epsilon_r^\theta} 2^{\epsilon_s^\theta(1-\epsilon_p^\theta)} \\ &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)}. \end{aligned}$$

In particular

$$\left| \tilde{\mathcal{D}}_\lambda^\theta \right| \in \{1, 2, 4\}.$$

Proof. Combine Corollary 8.4.5 with Corollary 9.4.2. □

Example 9.4.4. Fix $n = 4$ and let $\lambda = (3, 2, \frac{3}{2}, \frac{1}{2})$ be a symmetric infinitesimal character. We are now in a position to compute the size of the basis $\tilde{\mathcal{D}}_\lambda$ for $\mathcal{KHC}(\mathfrak{g}, \tilde{K})_\lambda^{\text{gen}}$. This is the \tilde{G} analog of Example 5.2.5.

To accomplish this, we identify the involutions in \mathcal{I} that are supportable for λ and then apply Theorem 9.4.3. Since the indicator bits ϵ_s^θ , ϵ_r^θ , ϵ_p^θ , and ϵ_m^θ depend only on the conjugacy class of θ , it makes sense to work one conjugacy class at a time. Since there are $5 + 3 + 1 = 9$ conjugacy classes of involutions by Corollary 3.3.2, we have the following nine cases.

Case I. Suppose θ is given by the diagram

$$D_\theta : + + + + .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_s^\theta &= 1 \\ \epsilon_p^\theta = \epsilon_r^\theta = \epsilon_m^\theta &= 0 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 4. \end{aligned}$$

Since $\theta = 1$, there are no other supportable involutions in \mathcal{I} conjugate to θ . Therefore there are

$$1 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 1 \times 4 = 4$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case II. Suppose θ is given by the diagram

$$D_\theta : + + + - .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_s^\theta = \epsilon_r^\theta &= 1 \\ \epsilon_p^\theta = \epsilon_m^\theta &= 1 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 1. \end{aligned}$$

There are four conjugate involutions in \mathcal{I} that are supportable for λ and therefore there are

$$4 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 4 \times 1 = 4$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case III. Suppose θ is given by the diagram

$$D_\theta : - + + - .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_s^\theta = \epsilon_r^\theta &= 1 \\ \epsilon_p^\theta = \epsilon_m^\theta &= 0 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 4. \end{aligned}$$

There are four conjugate involutions in \mathcal{I} that are supportable for λ and therefore there are

$$4 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 4 \times 4 = 16$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case IV. Suppose θ is given by the diagram

$$D_\theta : + - - - .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_s^\theta = \epsilon_r^\theta &= 1 \\ \epsilon_p^\theta = \epsilon_m^\theta &= 1 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 1. \end{aligned}$$

There are four conjugate involutions in \mathcal{I} that are supportable for λ and therefore there are

$$4 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 4 \times 1 = 4$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case V. Suppose θ is given by the diagram

$$D_\theta : - - - - .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_r^\theta &= 1 \\ \epsilon_s^\theta = \epsilon_p^\theta = \epsilon_m^\theta &= 0 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 4. \end{aligned}$$

Since $\theta = -1$, there are no other supportable involutions in \mathcal{I} conjugate to θ . Therefore there are

$$1 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 1 \times 4 = 4$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case VI. Suppose θ is given by the diagram

$$D_\theta : + 3 2 + .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_s^\theta &= 1 \\ \epsilon_p^\theta = \epsilon_r^\theta = \epsilon_m^\theta &= 0 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 4. \end{aligned}$$

There are eight conjugate involutions in \mathcal{I} that are supportable for λ and therefore there are

$$8 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 8 \times 4 = 32$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case VII. Suppose θ is given by the diagram

$$D_\theta : + 3 2 - .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_s^\theta = \epsilon_r^\theta &= 1 \\ \epsilon_p^\theta = \epsilon_m^\theta &= 1 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 1. \end{aligned}$$

There are sixteen conjugate involutions in \mathcal{I} that are supportable for λ and therefore there are

$$16 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 16 \times 1 = 16$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case VIII. Suppose θ is given by the diagram

$$D_\theta : - 3 2 - .$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_r^\theta &= 1 \\ \epsilon_s^\theta = \epsilon_p^\theta = \epsilon_m^\theta &= 0 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2 - (\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 4. \end{aligned}$$

There are eight conjugate involutions in \mathcal{I} that are supportable for λ and therefore there are

$$8 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 8 \times 4 = 32$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Case IX. Suppose θ is given by the diagram

$$D_\theta : 4 3 2 1.$$

Then θ is supportable for λ and we have

$$\begin{aligned} \epsilon_s^\theta = \epsilon_p^\theta = \epsilon_r^\theta = \epsilon_m^\theta &= 0 \\ \left| \tilde{\mathcal{D}}_\lambda^\theta \right| &= 2^{2 - (\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} = 4. \end{aligned}$$

There are eight conjugate involutions in \mathcal{I} that are supportable for λ and therefore there are

$$8 \times \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 8 \times 4 = 32$$

elements in $\tilde{\mathcal{D}}_\lambda$ for this conjugacy class.

Finally we have

$$\left| \tilde{\mathcal{D}}_\lambda \right| = 4 + 4 + 16 + 4 + 4 + 32 + 16 + 32 + 32 = 144.$$

Example 9.4.5. Table 9.2 lists (computer generated) values for $\left| \tilde{\mathcal{D}}_\lambda \right|$ at symmetric infinitesimal character λ for small values of n

9.2. VALUES OF $|\tilde{\mathcal{D}}_\lambda|$

n	1	2	3	4	5	6	7	8	9	10
	4	18	30	144	314	1598	4166	22344	66512	373988

PART III

DUALITY

CHAPTER 10

NUMERICAL DUALITY FOR NONLINEAR PARAMETERS

In Chapter 3 we defined the bit flip involution

$$\Psi : \mathcal{I} \rightarrow \mathcal{I}$$

on the set of abstract involutions in W . In addition to describing a symmetry of \mathcal{I} , Ψ determined symmetries on most of the structure theoretic information we computed for G in Chapters 3, 4, and 5. One might then expect Ψ to lead to a symmetry on the set \mathcal{D}_χ of representation theoretic parameters for G . Unfortunately Example 5.2.5 shows this is not the case. Our goal in this part is to show that, in some cases, the expected symmetry does appear in the set $\tilde{\mathcal{D}}_\chi$ of representation theoretic parameters for the nonlinear group \tilde{G} .

Before attempting to define Ψ explicitly on the level of parameters, we can use the results of the previous section to verify such a map is numerically possible. It turns out this is the case only when the rank of \tilde{G} is even.

10.1 Restriction to Even Rank

Let $\lambda \in (\mathfrak{h}^a)^*$ be a fixed symmetric infinitesimal character and suppose θ is an involution in W . Recall θ is said to be supportable for λ if there exists an imaginary grading ε such that the abstract triple $(\theta, \varepsilon, \lambda)$ satisfies the conditions of Proposition 6.2.2. In order for Ψ to induce a symmetry on $\tilde{\mathcal{D}}_\lambda$, we clearly must have θ supportable if and only if $\Psi(\theta)$ is supportable. The following example shows this is not always the case.

Example 10.1.1. Suppose $n = 3$ and $\lambda = (2, 1, \frac{1}{2})$ is a symmetric infinitesimal character. If we take $\theta \in \mathcal{I}$ such that

$$D_\theta : \quad - \quad - \quad +$$

then θ is supportable for λ . However the diagram for $\Psi(\theta) = -\theta$ is given by

$$D_{\Psi(\theta)} : \quad + \quad + \quad -$$

which is not supportable for λ .

Let $\theta \in \mathcal{I}$ and recall the indicator bits $\epsilon_s^\theta, \epsilon_r^\theta, \epsilon_p^\theta, \epsilon_m^\theta$ for θ from Section 3.1. The indicator bits for $\Psi(\theta)$ are related to the ones for θ through the following obvious lemma.

Lemma 10.1.2. *Let $\theta \in \mathcal{I}$. Then*

$$\begin{aligned} n_s^{\Psi(\theta)} &= n_r^\theta \\ n_c^{\Psi(\theta)} &= n_c^\theta \\ \epsilon_s^{\Psi(\theta)} &= \epsilon_r^\theta \\ \epsilon_p^{\Psi(\theta)} &= \epsilon_m^\theta. \end{aligned}$$

The next proposition shows the parity of n is the only obstruction to duality in supportable involutions.

Proposition 10.1.3. *Let $\lambda \in (\mathfrak{h}^a)^*$ be a fixed symmetric infinitesimal character, $\theta \in \mathcal{I}$, and suppose n is even. Then θ is supportable for λ if and only if $\Psi(\theta)$ is supportable for λ .*

Proof. Since n is even and λ is symmetric, $\frac{n}{2}$ of the coordinates for λ will be integers and $\frac{n}{2}$ will be strict half-integers. Recall θ is supportable for λ if and only if $\frac{n_s^\theta + \epsilon_p^\theta}{2}$ of the imaginary coordinates are of one type (say integers wlog) and $\frac{n_s^\theta - \epsilon_p^\theta}{2}$ are of the other type (say strict half-integers wlog). Then the number of real integral coordinates for θ is given by

$$\begin{aligned} \frac{n}{2} - \frac{n_c^\theta}{2} - \frac{n_s^\theta + \epsilon_p^\theta}{2} &= \frac{n - n_c^\theta - n_s^\theta - \epsilon_p^\theta}{2} \\ &= \frac{n_r^\theta - \epsilon_p^\theta}{2} \\ &= \frac{n_r^\theta - \epsilon_m^\theta}{2} \end{aligned}$$

(note $\epsilon_m^\theta = \epsilon_p^\theta$). Similarly the number of real strict half-integral coordinates for θ is given by

$$\begin{aligned} \frac{n}{2} - \frac{n_c^\theta}{2} - \frac{n_s^\theta - \epsilon_p^\theta}{2} &= \frac{n - n_c^\theta - n_s^\theta + \epsilon_p^\theta}{2} \\ &= \frac{n_r^\theta + \epsilon_p^\theta}{2} \\ &= \frac{n_r^\theta + \epsilon_m^\theta}{2}. \end{aligned}$$

Since $n_r^\theta = n_s^{\Psi(\theta)}$ and $\epsilon_m^\theta = \epsilon_p^{\Psi(\theta)}$ by Lemma 10.1.2, the result follows. \square

Prior to Proposition 10.1.3, none of the results in these notes depended on the the parity of n . Because of Proposition 10.1.3 and Example 10.1.1 we will often assume n is even in what follows. We begin with the following specialization of Theorem 9.4.3.

Proposition 10.1.4. *Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character, $\theta \in \mathcal{I}$, and suppose n is even. Then*

$$\left| \tilde{\mathcal{D}}_\lambda^\theta \right| \in \{1, 4\}$$

Proof. Theorem 9.4.3 implies

$$\left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 2^{2 - (\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)}.$$

Since n is even, we clearly have $\epsilon_p^\theta = \epsilon_m^\theta$. Now

$$\epsilon_p^\theta = \epsilon_m^\theta = 0 \implies \left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 2^{2-0} = 4.$$

On the other hand

$$\epsilon_p^\theta = \epsilon_m^\theta = 1 \implies \epsilon_s^\theta = \epsilon_r^\theta = 1$$

and $\left| \tilde{\mathcal{D}}_\lambda^\theta \right| = 2^{2-2} = 1$ as desired. \square

The following theorem shows numerical duality exists for nonlinear parameters in even rank.

Theorem 10.1.5. *Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character, $\theta \in \mathcal{I}$, and suppose n is even. Then*

$$\left| \tilde{\mathcal{D}}_\lambda^{\Psi(\theta)} \right| = \left| \tilde{\mathcal{D}}_\lambda^\theta \right|.$$

Proof. Proposition 10.1.3 implies θ is supportable for λ if and only if $\Psi(\theta)$ is. In particular we have

$$\left| \tilde{\mathcal{D}}_\lambda^\theta \right| = \left| \tilde{\mathcal{D}}_\lambda^{\Psi(\theta)} \right| = 0$$

if θ is not supportable for λ . Otherwise, Theorem 9.4.3 and Lemma 10.1.2 give

$$\begin{aligned} \left| \tilde{\mathcal{D}}_\lambda^{\Psi(\theta)} \right| &= 2^{2 - (\epsilon_s^{\Psi(\theta)} \epsilon_p^{\Psi(\theta)} + \epsilon_r^{\Psi(\theta)} \epsilon_m^{\Psi(\theta)})} \\ &= 2^{2 - (\epsilon_r^\theta \epsilon_m^\theta + \epsilon_s^\theta \epsilon_p^\theta)} \\ &= 2^{2 - (\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)} \\ &= \left| \tilde{\mathcal{D}}_\lambda^\theta \right| \end{aligned}$$

as desired. \square

CHAPTER 11

CENTRAL CHARACTERS

Fix a symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$. Example 10.1.1 and Proposition 10.1.3 of the previous section imply an extension of Ψ to $\tilde{\mathcal{D}}_\lambda$ is possible only when the rank of \tilde{G} is even. In this case, Proposition 10.1.4 implies $|\tilde{\mathcal{D}}_\lambda^\theta| \in \{1, 4\}$ for each supportable involution $\theta \in \mathcal{I}$. If $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$, we have $|\tilde{\mathcal{D}}_\lambda^{\Psi(\theta)}| = 1$ by Theorem 10.1.5 and the definition of Ψ is obvious. Therefore it remains to define Ψ on the sets $\tilde{\mathcal{D}}_\lambda^\theta$ for which $|\tilde{\mathcal{D}}_\lambda^\theta| = |\tilde{\mathcal{D}}_\lambda^{\Psi(\theta)}| = 4$. This definition is complicated and will occupy the next several sections.

The main idea is to find structure theoretic data that characterize the elements in $\tilde{\mathcal{D}}_\lambda^\theta$, and then use these data in extending the map Ψ . In this section we cut the problem in half by considering an action of $\tilde{\mathcal{D}}_\lambda^\theta$ on $Z(\tilde{G})$. Not surprisingly, short roots play an important role (see Proposition 7.2.2).

11.1 Properties of Short Roots

We begin by redeveloping some material from Chapter 8. Let $\mathfrak{h}_\mathbb{R}$ be a Θ -stable Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$ with root system $\Delta(\mathfrak{g}, \mathfrak{h})$. Recall $\Delta_\mathbb{R}^\Theta(\mathfrak{g}, \mathfrak{h})$ denotes the roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ that are real with respect to Θ .

Lemma 11.1.1 ([16], Lemma 4.3.7). *Suppose $\alpha \in \Delta_\mathbb{R}^\Theta(\mathfrak{g}, \mathfrak{h})$. Then there exists a root vector $X_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{R}$ with the property*

$$[\Theta X_\alpha, X_\alpha] = h_\alpha,$$

where $h_\alpha \in \mathfrak{h}$ is the coroot for α . Moreover, the vector X_α is unique up to sign.

Definition 11.1.2. For a fixed choice of X_α , let $Z_\alpha = X_\alpha + \Theta X_\alpha \in \mathfrak{h}$ and define the elements

$$\begin{aligned}
\sigma_\alpha &= \exp_G\left(\frac{\pi}{2}Z_\alpha\right) \in K \\
\tilde{\sigma}_\alpha &= \exp_{\tilde{G}}\left(\frac{\pi}{2}Z_\alpha\right) \in \tilde{K} \\
m_\alpha &= \exp_G(\pi Z_\alpha) = \exp_G(\pi i h_\alpha) \in H \\
\tilde{m}_\alpha &= \exp_{\tilde{G}}(\pi Z_\alpha) \in \tilde{H}.
\end{aligned}$$

Then $\sigma_\alpha, \tilde{\sigma}_\alpha$ are representatives of the root reflection s_α and we have $\sigma_\alpha^2 = m_\alpha$ and $\tilde{\sigma}_\alpha^2 = \tilde{m}_\alpha$ ([11], Proposition 6.52(c)).

Remark 11.1.3. It is important to note the conditions in Lemma 11.1.1 determine the vector X_α (and thus Z_α) *only up to sign*. This ambiguity potentially affects the elements of Definition 11.1.2. In the group G we have

$$\begin{aligned}
\exp_G(2\pi Z_\alpha) &= m_\alpha^2 = 1 \\
\exp_G(\pi Z_\alpha) &= \exp_G(-\pi Z_\alpha)
\end{aligned}$$

so the element m_α is well-defined. Moreover

$$\begin{aligned}
\exp_G\left(-\frac{\pi}{2}Z_\alpha\right) &= \sigma_\alpha^{-1} \\
&= \sigma_\alpha m_\alpha
\end{aligned}$$

and conjugation in σ_α and σ_α^{-1} induces the same action on H or \mathfrak{h} , but not on G or \mathfrak{g} .

In \tilde{G} , the situation is more complicated. If α is short then everything is the same as for G . If α is long

$$\begin{aligned}
\exp_{\tilde{G}}(2\pi Z_\alpha) &= \tilde{m}_\alpha^2 = -1 \\
\exp_{\tilde{G}}(\pi Z_\alpha) &= -\exp_{\tilde{G}}(-\pi Z_\alpha)
\end{aligned}$$

and the element \tilde{m}_α is determined by the root α *only up to inverse*. Moreover

$$\begin{aligned}
\exp_{\tilde{G}}\left(-\frac{\pi}{2}Z_\alpha\right) &= \tilde{\sigma}_\alpha^{-1} \\
&= -\tilde{\sigma}_\alpha \tilde{m}_\alpha
\end{aligned}$$

so conjugation in $\tilde{\sigma}_\alpha$ and $\tilde{\sigma}_\alpha^{-1}$ induces the same action on \tilde{H} if and only if \tilde{m}_α is central in \tilde{H} .

In particular, some care is required when discussing the elements $\sigma_\alpha, \tilde{\sigma}_\alpha$, and \tilde{m}_α since these elements (and their corresponding actions) are not always determined by the root

α . Depending on the context we may need to explicitly choose root vectors satisfying the condition in Lemma 11.1.1. One notable exception is when the root α is short, in which case \tilde{m}_α and the actions of $\sigma_\alpha, \tilde{\sigma}_\alpha$ are well-defined.

Although we have a choice for the root vectors X_α for each real $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, the following lemma shows certain choices can be used to induce others.

Lemma 11.1.4. *Let α, β, γ be roots in $\Delta_{\mathbb{R}}^{\ominus}(\mathfrak{g}, \mathfrak{h})$ such that*

$$s_\alpha(\beta) = \gamma.$$

Choose root vectors $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$ satisfying the conditions of Lemma 11.1.1. Then the root vector $\text{Ad}(\sigma_\alpha)(X_\beta) \in \mathfrak{g}_\gamma$ also satisfies the conditions of Lemma 11.1.1.

Proof.

$$\begin{aligned} [\Theta \text{Ad}(\sigma_\alpha)X_\beta, \text{Ad}(\sigma_\alpha)X_\beta] &= \text{Ad}(\sigma_\alpha) [\Theta X_\beta, X_\beta] \\ &= \text{Ad}(\sigma_\alpha)h_\beta \\ &= h_\gamma \end{aligned}$$

as desired. □

Let $\alpha, \beta \in \Delta_{\mathbb{R}}^{\ominus}(\mathfrak{g}, \mathfrak{h})$ be orthogonal roots and choose $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$ satisfying the conditions of Lemma 11.1.1. Lemmas 11.1.1 and 11.1.4 imply

$$\text{Ad}(\sigma_\beta)X_\alpha = \pm X_\alpha.$$

The following proposition shows the nontrivial case actually occurs.

Proposition 11.1.5. *Let $\alpha, \beta \in \Delta_{\mathbb{R}}^{\ominus}(\mathfrak{g}, \mathfrak{h})$ be short orthogonal roots. Choose $X_\beta \in \mathfrak{g}_\beta$ according to Lemma 11.1.1 and suppose $E_\alpha \in \mathfrak{g}_\alpha$ is a root vector for α . Then*

$$\text{Ad}(\sigma_\beta)E_\alpha = -E_\alpha.$$

Proof. We will calculate the adjoint action of σ_β on E_α explicitly using infinite series. We begin by calculating

$$\begin{aligned}
\operatorname{ad}\left(\frac{\pi}{2}Z_\beta\right)(E_\alpha) &= \frac{\pi}{2}([X_\beta, E_\alpha] + [\Theta X_\beta, E_\alpha]) \\
\operatorname{ad}^2\left(\frac{\pi}{2}Z_\beta\right)(E_\alpha) &= \frac{\pi^2}{4}([Z_\beta, [X_\beta, E_\alpha]] + [Z_\beta, [\Theta X_\beta, E_\alpha]]) \\
&= \frac{\pi^2}{4}([\Theta X_\beta, [X_\beta, E_\alpha]] + [X_\beta, [\Theta X_\beta, E_\alpha]]) \\
&= \frac{\pi^2}{4}(-2E_\alpha - 2E_\alpha) \\
&= -\pi^2 E_\alpha
\end{aligned}$$

where the second to last equality is given by direct calculation. Therefore we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\operatorname{ad}^k\left(\frac{\pi}{2}Z_\beta\right)E_\alpha}{k!} &= \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{(2k)!} E_\alpha \\
&\quad + \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k+1}}{(2k+1)!} ([X_\beta, E_\alpha] + [\Theta X_\beta, E_\alpha]) \\
&= \cos(\pi)E_\alpha + \frac{1}{2} \sin(\pi) ([X_\beta, E_\alpha] + [\Theta X_\beta, E_\alpha]) \\
&= -E_\alpha
\end{aligned}$$

as desired. \square

If α and β are short roots in $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$, Proposition 7.2.2 implies $\tilde{m}_\alpha = \pm \tilde{m}_\beta$. The following proposition shows we always have equality.

Proposition 11.1.6. *Let α and β be short orthogonal roots in $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$. Then $\tilde{m}_\beta = \tilde{m}_\alpha$.*

Proof. Let $\gamma = \alpha - \beta$ so that

$$\begin{aligned}
s_\gamma(\alpha) &= \alpha - (\alpha, \gamma)\gamma \\
&= \alpha - \gamma \\
&= \beta.
\end{aligned}$$

Choose a root vector $X_\alpha \in \mathfrak{g}_\alpha$ according to Lemma 11.1.1. Then

$$\begin{aligned}
\tilde{m}_\beta &= \exp_{\tilde{G}}(\pi \operatorname{Ad}(\tilde{\sigma}_\gamma)Z_\alpha) \\
&= \tilde{\sigma}_\gamma \exp_{\tilde{G}}(\pi Z_\alpha) \tilde{\sigma}_\gamma^{-1} \\
&= \tilde{\sigma}_\gamma \tilde{m}_\alpha \tilde{\sigma}_\gamma^{-1} \\
&= \tilde{m}_\alpha
\end{aligned}$$

by Proposition 7.2.2. \square

Following the notation in [16], let $\beta(m_\alpha)$ denote the scalar by which m_α acts on the root space \mathfrak{g}_β . Proposition 8.1.1 implies

$$\beta(m_\alpha) = (-1)^{(\beta, \alpha^\vee)}$$

and obviously $\beta(m_\alpha) = \beta(\tilde{m}_\alpha)$.

Lemma 11.1.7. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Θ -stable Cartan subalgebra and suppose $\alpha, \beta \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$. Choose root vectors $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$ according to Lemma 11.1.1. Then we have*

$$\begin{aligned} \tilde{m}_\alpha \tilde{\sigma}_\beta \tilde{m}_\alpha^{-1} &= \tilde{\sigma}_\beta^{\beta(\tilde{m}_\alpha)} \\ \tilde{\sigma}_\beta \tilde{m}_\alpha \tilde{\sigma}_\beta^{-1} &= \tilde{\sigma}_\beta^{(1-\beta(\tilde{m}_\alpha))} \tilde{m}_\alpha. \end{aligned}$$

Proof. The first statement is proved in exactly the same way as [16], Lemma 4.3.19(c). For the second statement we have

$$\begin{aligned} \tilde{m}_\alpha \tilde{\sigma}_\beta \tilde{m}_\alpha^{-1} &= \tilde{\sigma}_\beta^{\beta(\tilde{m}_\alpha)} \\ \tilde{m}_\alpha \tilde{\sigma}_\beta^{-1} \tilde{m}_\alpha^{-1} &= \tilde{\sigma}_\beta^{-\beta(\tilde{m}_\alpha)} \\ \tilde{\sigma}_\beta \tilde{m}_\alpha \tilde{\sigma}_\beta^{-1} &= \tilde{\sigma}_\beta \tilde{\sigma}_\beta^{-\beta(\tilde{m}_\alpha)} \tilde{m}_\alpha \\ \tilde{\sigma}_\beta \tilde{m}_\alpha \tilde{\sigma}_\beta^{-1} &= \tilde{\sigma}_\beta^{(1-\beta(\tilde{m}_\alpha))} \tilde{m}_\alpha \end{aligned}$$

as desired. □

Lemma 11.1.8. *Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be a Θ -stable Cartan subalgebra and suppose $\alpha, \beta \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ are strongly orthogonal. Then the operators $Ad(m_\alpha), Ad(m_\beta)$ commute.*

Proof. Since these operators are given by the adjoint representation, we need only check that $[Z_\alpha, Z_\beta] = 0$. However this follows easily from the fact the roots α and β are strongly orthogonal. □

We now turn our attention to noncompact imaginary roots. Recall $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ denotes the roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ that are imaginary with respect to Θ . We begin with the following analog of Lemma 11.1.1.

Lemma 11.1.9. *Suppose $\beta \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ is imaginary and noncompact. Then there exists a root vector $X_\beta \in \mathfrak{g}_\beta$ with the property*

$$[X_\beta, \overline{X_\beta}] = h_\beta,$$

where $h_\beta \in \mathfrak{h}$ is the coroot for β . Moreover, the vector X_β is unique up to sign.

Definition 11.1.10. For a fixed choice of X_β as above, let $Z_\beta = X_\beta + \overline{X_\beta} \in \mathfrak{h}$ and define the elements

$$\begin{aligned} m_\beta &= \exp_G(\pi i Z_\beta) = \exp_G(\pi i h_\beta) \in H \\ \tilde{m}_\beta &= \exp_{\tilde{G}}(-\pi i h_\beta) \in \tilde{H}. \end{aligned}$$

The reason for this definition of \tilde{m}_β will become clear in the next section (Corollary 11.2.5).

Remark 11.1.11. As before, Lemma 11.1.9 defines the vector X_β only up to sign. In the group G we have

$$\begin{aligned} \exp_G(2\pi i Z_\beta) &= m_\beta^2 = 1 \\ \exp_G(\pi i Z_\beta) &= \exp_G(-\pi i Z_\beta) \end{aligned}$$

so the element m_β is well-defined. The same is true in the group \tilde{G} if β is short. However, if β is long we have

$$\begin{aligned} \exp_{\tilde{G}}(-2\pi i h_\beta) &= \tilde{m}_\beta^2 = -1 \\ \exp_{\tilde{G}}(-\pi i h_\beta) &= -\exp_{\tilde{G}}(\pi i h_\beta). \end{aligned}$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Θ -stable Cartan subalgebra and suppose $\beta \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ is noncompact. Proposition 11.2.4 and Corollary 11.2.5 of the next section imply there exists a Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}$ and a real root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}')$ so that $\tilde{m}_\beta = \tilde{m}_\alpha$. In particular, if β is short then \tilde{m}_β is nontrivial and central in \tilde{G} . Assuming this, we have the following analog of Proposition 11.1.6.

Proposition 11.1.12. *Let $\alpha \neq \pm\beta$ be short noncompact roots in $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$. Then $\tilde{m}_\beta = \tilde{m}_\alpha$.*

Proof. Let $\gamma = \alpha - \beta$. Proposition 5.1.2 implies γ is compact and we can find an element $\tilde{\sigma}_\gamma \in \tilde{K}$ for which

$$\text{Ad}(\tilde{\sigma}_\gamma)h_\alpha = h_\beta.$$

We now proceed as in the proof of Proposition 11.1.6. □

Tracking the elements \tilde{m}_α for various roots α will be an important component in the definition of Ψ . Proposition 11.1.6 implies there is a single $\tilde{m}_\alpha \in \tilde{H}$ associated to the short roots in $\Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ and Proposition 11.1.12 gives a single element $\tilde{m}_\beta \in \tilde{H}$ associated to the short noncompact roots in $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$. Moreover, we must have $\tilde{m}_\alpha = \pm\tilde{m}_\beta$ by Proposition 7.2.2. Our goal in the next section is to determine this relationship (Corollary 11.2.9).

11.2 Cayley Transforms Revisited

In Section 8.1 we defined the Cayley transform operation for Cartan subalgebras. It turns out these transformations appear explicitly in the adjoint representation of G . Having specific realizations for Cayley transforms will be useful in what follows. To begin, let $\mathfrak{h}_{\mathbb{R}}$ be a Θ -stable Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a real root and $\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ is imaginary and noncompact. Choose nonzero root vectors $X_{\alpha} \in \mathfrak{g}_{\alpha}$ and $X_{\beta} \in \mathfrak{g}_{\beta}$ according to Lemmas 11.1.1 and 11.1.9. Let

$$\begin{aligned}\xi_{\alpha} &= \frac{\pi i}{4} (\Theta X_{\alpha} - X_{\alpha}) \\ \zeta_{\beta} &= \frac{\pi}{4} (\overline{X_{\beta}} - X_{\beta})\end{aligned}$$

and define the corresponding *Cayley transform operators*

$$\begin{aligned}\mathcal{C}_{\alpha} &= \text{Ad}(\exp_G(\xi_{\alpha})) \\ \mathcal{C}_{\beta} &= \text{Ad}(\exp_G(\zeta_{\beta})).\end{aligned}$$

Remark 11.2.1. The Cayley transform operators depend on the choices of X_{α} and X_{β} up to inverse.

The connection with our previous notion of Cayley transform is given by the following propositions (see [11], Chapter 6 for more details).

Proposition 11.2.2 ([11], 6.68). *In the notation of Section 8.1 and Lemma 11.1.1*

$$\begin{aligned}\mathcal{C}_{\alpha}(\mathfrak{h}) &= \mathfrak{h}_{\alpha} \\ \mathcal{C}_{\alpha}(h_{\alpha}) &= iZ_{\alpha} \\ \mathcal{C}_{\alpha}(Z_{\alpha}) &= ih_{\alpha}.\end{aligned}$$

Proposition 11.2.3 ([11], 6.66). *$\mathcal{C}^{\beta}(\mathfrak{h})$ is a Θ -stable Cartan subalgebra of \mathfrak{g} . In the notation of Lemma 11.1.9*

$$\begin{aligned}\mathcal{C}^{\beta}(h_{\beta}) &= Z_{\beta} \\ \mathcal{C}^{\beta}(Z_{\beta}) &= -h_{\beta}.\end{aligned}$$

We will write $\mathfrak{h}^{\beta} = \mathcal{C}^{\beta}(\mathfrak{h})$ for the Cayley transform of \mathfrak{h} with respect to β and

$$\begin{aligned}\mathcal{C}_{\alpha} &: \Delta(\mathfrak{g}, \mathfrak{h}) \rightarrow \Delta(\mathfrak{g}, \mathfrak{h}_{\alpha}) \\ \mathcal{C}^{\beta} &: \Delta(\mathfrak{g}, \mathfrak{h}) \rightarrow \Delta(\mathfrak{g}, \mathfrak{h}^{\beta})\end{aligned}$$

for the induced operations on roots.

Proposition 11.2.4 ([11], Proposition 6.69). *The root $\mathcal{C}^\beta(\beta) \in \Delta(\mathfrak{g}, \mathfrak{h}^\beta)$ is real and the root $\mathcal{C}_\alpha(\alpha) \in \Delta(\mathfrak{g}, \mathfrak{h}_\alpha)$ is imaginary and noncompact. Moreover, there exists a suitable choice of root vectors so that*

$$\mathcal{C}^{\mathcal{C}_\alpha(\alpha)} \circ \mathcal{C}_\alpha = \mathcal{C}_{\mathcal{C}^\beta(\beta)} \circ \mathcal{C}^\beta = I.$$

In particular, the Cayley transform operators defined above are essentially inverses. The following corollary explains the definition of \tilde{m}_β in the previous section.

Corollary 11.2.5. *Let $\alpha \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ be a real root and choose $X_\alpha \in \mathfrak{g}_\alpha$ according to Lemma 11.1.1. Then $\tilde{m}_\alpha = \tilde{m}_{\mathcal{C}_\alpha(\alpha)}$.*

Proof. The coroot for $\mathcal{C}_\alpha(\alpha)$ is $\mathcal{C}_\alpha(h_\alpha) = iZ_\alpha \in \mathfrak{h}_\alpha$ by Proposition 11.2.2. Therefore we have

$$\begin{aligned} \tilde{m}_{\mathcal{C}_\alpha(\alpha)} &= \exp_{\tilde{G}}(-\pi i(iZ_\alpha)) \\ &= \exp_{\tilde{G}}(\pi Z_\alpha) \\ &= \tilde{m}_\alpha \end{aligned}$$

as desired. □

Lemma 11.2.6. *Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be a Θ -stable Cartan subalgebra and suppose $\alpha, \beta \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ are strongly orthogonal. Then the operators $\{\mathcal{C}_\alpha, \mathcal{C}_\beta\}$ and $\{\mathcal{C}_\alpha, Ad(m_\beta)\}$ commute.*

Proof. The proof is the same as Lemma 11.1.8. □

Lemma 11.2.7. *Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be a Θ -stable Cartan subalgebra with $\alpha \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ a long root. Suppose there exists a short root $\beta \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ that is not orthogonal to α . Then the operators $\{\mathcal{C}_\alpha, Ad(m_\alpha)\}$ commute.*

Proof. Write $\gamma = s_\beta(\alpha)$. Then α and γ are strongly orthogonal roots in $\Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ and the operators $\{\mathcal{C}_\alpha, Ad(m_\gamma)\}$ commute by the previous lemma. However, the elements m_α and m_γ differ by a central element in \tilde{G} (Proposition 7.2.1) and therefore $Ad(m_\alpha) = Ad(m_\gamma)$ as operators on \mathfrak{g} . □

We now come to the main result of this section. Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be a Θ -stable Cartan subalgebra and suppose $\alpha, \beta \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ are short and orthogonal. Set $\gamma = \alpha - \beta$, $\delta = \alpha + \beta$, and choose a root vector X_γ for γ satisfying the conditions of Lemma 11.1.1. Write

$X_\delta = s_\beta(X_\gamma)$ (well-defined since β is short) be the induced root vector for δ according to Lemma 11.1.4 and define the corresponding Cayley transform operators \mathcal{C}_γ and \mathcal{C}_δ . Let

$$\mathcal{D} = \mathcal{C}_\delta \circ \mathcal{C}_\gamma$$

denote the composition. The roots $\mathcal{D}(\gamma)$ and $\mathcal{D}(\delta)$ in $\Delta_i^\ominus(\mathfrak{g}, \mathcal{D}(\mathfrak{h}))$ are noncompact by Proposition 11.2.4. The following proposition determines the imaginary types for $\mathcal{D}(\alpha)$ and $\mathcal{D}(\beta)$.

Proposition 11.2.8. *In the situation above, the root $\mathcal{D}(\alpha) \in \Delta_i^\ominus(\mathfrak{g}, \mathcal{D}(\mathfrak{h}))$ is noncompact and the root $\mathcal{D}(\beta) \in \Delta_i^\ominus(\mathfrak{g}, \mathcal{D}(\mathfrak{h}))$ is compact.*

Proof. Let E_β be an arbitrary root vector in \mathfrak{g}_β and set

$$E_\alpha = [\Theta X_\gamma - X_\gamma, E_\beta] = [E_\beta, X_\gamma] \in \mathfrak{g}_\alpha.$$

We begin by computing $\mathcal{C}_\gamma(E_\beta)$. We have

$$\begin{aligned} \text{ad}(\xi_\gamma)(E_\beta) &= \left(\frac{\pi i}{4}\right) E_\alpha \\ \text{ad}(\xi_\gamma)^2(E_\beta) &= \left(\frac{\pi i}{4}\right) [\xi_\alpha, [E_\beta, X_\gamma]] \\ &= \left(\frac{\pi i}{4}\right)^2 [\Theta X_\gamma - X_\gamma, [E_\beta, X_\gamma]] \\ &= \left(\frac{\pi i}{4}\right)^2 [\Theta X_\gamma, [E_\beta, X_\gamma]] \\ &= -\left(\frac{\pi i}{4}\right)^2 [E_\beta, [X_\gamma, \Theta X_\gamma]] \\ &= -\left(\frac{\pi i}{4}\right)^2 [h_\gamma, E_\beta] \\ &= \left(\frac{\pi i}{4}\right)^2 E_\beta \end{aligned}$$

and in general

$$\text{ad}(\xi_\gamma)^n(E_\beta) = \begin{cases} \left(\frac{\pi i}{4}\right)^n E_\alpha & n \text{ odd} \\ \left(\frac{\pi i}{4}\right)^n E_\beta & n \text{ even} \end{cases}.$$

Therefore

$$\begin{aligned} \mathcal{C}_\gamma(E_\beta) &= \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k}}{(2k)!} E_\beta + i \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k+1}}{(2k+1)!} E_\alpha \\ &= \cos\left(\frac{\pi}{4}\right) E_\beta + i \sin\left(\frac{\pi}{4}\right) E_\alpha. \end{aligned}$$

It remains to compute $\mathcal{C}_\delta(E_\alpha)$ and $\mathcal{C}_\delta(E_\beta)$. Set $F_\alpha = [\Theta X_\delta - X_\delta, E_\beta] = [\Theta X_\delta, E_\beta] \in \mathfrak{g}_{-\alpha}$ and $F_\beta = [\Theta X_\delta - X_\delta, E_\alpha] = [\Theta X_\delta, E_\alpha] \in \mathfrak{g}_{-\beta}$. In the same fashion as above we compute

$$\begin{aligned} \operatorname{ad}(\xi_\delta)^n(E_\alpha) &= \begin{cases} \left(\frac{\pi i}{4}\right)^n F_\beta & n \text{ odd} \\ \left(\frac{\pi i}{4}\right)^n E_\alpha & n \text{ even} \end{cases} \\ \mathcal{C}_\delta(E_\alpha) &= \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k}}{(2k)!} E_\alpha + i \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k+1}}{(2k+1)!} F_\beta \\ &= \cos\left(\frac{\pi}{4}\right) E_\alpha + i \sin\left(\frac{\pi}{4}\right) F_\beta \end{aligned}$$

and

$$\begin{aligned} \operatorname{ad}(\xi_\delta)^n(E_\beta) &= \begin{cases} \left(\frac{\pi i}{4}\right)^n F_\alpha & n \text{ odd} \\ \left(\frac{\pi i}{4}\right)^n E_\beta & n \text{ even} \end{cases} \\ \mathcal{C}_\delta(E_\beta) &= \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k}}{(2k)!} E_\beta + i \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k+1}}{(2k+1)!} F_\alpha \\ &= \cos\left(\frac{\pi}{4}\right) E_\beta + i \sin\left(\frac{\pi}{4}\right) F_\alpha. \end{aligned}$$

Combining these we have

$$\begin{aligned} \mathcal{D}(E_\beta) &= \mathcal{C}_\delta \left(\cos\left(\frac{\pi}{4}\right) E_\beta + i \sin\left(\frac{\pi}{4}\right) E_\alpha \right) \\ &= \cos\left(\frac{\pi}{4}\right) \left(\cos\left(\frac{\pi}{4}\right) E_\beta + i \sin\left(\frac{\pi}{4}\right) F_\alpha \right) \\ &\quad + i \sin\left(\frac{\pi}{4}\right) \left(\cos\left(\frac{\pi}{4}\right) E_\alpha + i \sin\left(\frac{\pi}{4}\right) F_\beta \right) \\ &= \frac{i}{2} (E_\alpha + F_\alpha) + \frac{1}{2} (E_\beta - F_\beta). \end{aligned}$$

Since $\mathcal{D}(\beta)$ is imaginary, we must have $\Theta \mathcal{D}(E_\beta) = \pm \mathcal{D}(E_\beta)$. In order to prove $\mathcal{D}(\beta)$ is compact, it suffices to prove the vector $E_\alpha + F_\alpha$ is fixed by Θ or (more specifically) $E_\alpha = \Theta F_\alpha$. To see this, we compute

$$\begin{aligned} \Theta F_\alpha &= \Theta [\Theta X_\delta, E_\beta] \\ &= [X_\delta, \Theta E_\beta] \\ &= [s_\beta X_\gamma, \Theta E_\beta] \end{aligned}$$

by our construction of X_δ . Then a simple calculation in \mathfrak{sl}_2 gives

$$\begin{aligned} [s_\beta X_\gamma, \Theta E_\beta] &= s_\beta [X_\gamma, s_\beta \Theta E_\beta] \\ &= s_\beta [X_\gamma, E_\beta] \end{aligned}$$

and now Proposition 11.1.5 implies

$$s_\beta [X_\gamma, E_\beta] = [E_\beta, X_\gamma] = E_\alpha.$$

Therefore we have $\Theta\mathcal{D}(E_\beta) = \mathcal{D}(E_\beta)$ implying the root $\mathcal{D}(\beta)$ is compact. The claim that $\mathcal{D}(\alpha)$ is noncompact follows from Proposition 5.1.2. \square

Corollary 11.2.9. *In the setting of Proposition 11.2.8, $\tilde{m}_{\mathcal{D}(\alpha)} = -\tilde{m}_\alpha$.*

Proof. The coroot for α in \mathfrak{h} is given by $h_\alpha = h_\gamma + h_\delta$. Therefore the coroot for $\mathcal{D}(\alpha)$ in $\mathcal{D}(\mathfrak{h})$ is given by

$$\begin{aligned} h_{\mathcal{D}(\alpha)} &= \mathcal{D}(h_\gamma + h_\delta) \\ &= i(Z_\gamma + Z_\delta) \\ &= i(Z_\gamma + s_\beta Z_\gamma) \end{aligned}$$

by Proposition 11.2.2. Therefore

$$\begin{aligned} \tilde{m}_{\mathcal{D}(\alpha)} &= \exp_{\tilde{G}}(-\pi i h_{\mathcal{D}(\alpha)}) \\ &= \exp_{\tilde{G}}(-\pi i^2(Z_\gamma + s_\beta Z_\gamma)) \\ &= \exp_{\tilde{G}}(\pi Z_\gamma) \exp_{\tilde{G}}(\pi s_\beta Z_\gamma) \\ &= \tilde{m}_\gamma \tilde{\sigma}_\beta \tilde{m}_\gamma \tilde{\sigma}_\beta^{-1} \\ &= \tilde{m}_\gamma \tilde{m}_\beta \tilde{m}_\gamma \\ &= -\tilde{m}_\beta \\ &= -\tilde{m}_\alpha \end{aligned}$$

as desired. Here we are using Lemma 11.1.7 and Proposition 11.1.6. Recall conjugation in β is well-defined since the root β is short. \square

11.3 Abstract Bigradings

Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character and suppose $(\tilde{H}, \phi, \tilde{\Gamma})$ is a genuine triple for λ . Recall $\tilde{\Gamma}$ is an irreducible genuine representation of \tilde{H} whose differential is compatible with ϕ (Chapter 2). Since the Cartan subgroup \tilde{H} may not be abelian, the representation $\tilde{\Gamma}$ is not necessarily one-dimensional (Proposition 7.1.3). In this section we characterize the action of $\tilde{\Gamma}$ on certain finite order elements in \tilde{H} . We begin with the following lemma.

Lemma 11.3.1 ([13], Chapter 5). *In the setting above, let $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ be a real root and choose a corresponding $\tilde{m}_\alpha \in \tilde{H}$. If α is long we have*

$$\tilde{\Gamma}(\tilde{m}_\alpha^2) = -I$$

and $\tilde{\Gamma}(\tilde{m}_\alpha)$ has eigenvalues $\pm i$ occurring with equal multiplicity. If α is short we have

$$\tilde{\Gamma}(\tilde{m}_\alpha) = \pm I.$$

The data of Lemma 11.3.1 are conveniently packaged in the following definition.

Definition 11.3.2. Following [13], we say a long root $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ satisfies the *parity condition* if it is strictly half-integral with respect to ϕ (i.e., $(\phi, \alpha^\vee) \in \mathbb{Z} + \frac{1}{2}$). We say a short root $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ satisfies the parity condition if

$$\tilde{\Gamma}(\tilde{m}_\alpha) = (-1)(-1)^{(\phi, \alpha^\vee)} \cdot I$$

(see also [16], Section 8.3 and Corollary 12.3.6). Recall the element \tilde{m}_α is well-defined for short roots and we always have $(\phi, \alpha^\vee) \in \mathbb{Z}$. Lemma 11.3.1 implies this definition makes sense and gives a well-defined map

$$\eta : \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathbb{Z}_2$$

where

$$\eta(\alpha) = \begin{cases} 0 & \alpha \text{ does not satisfy the parity condition} \\ 1 & \alpha \text{ satisfies the parity condition} \end{cases}.$$

The following proposition is the formal analog of Proposition 5.1.2 for real roots.

Proposition 11.3.3. *The map η defines a grading (Definition 5.1.1) on $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$.*

Proof. It remains to verify the properties of Definition 5.1.1. This follows easily from Proposition 11.1.6 and the fact that the parity condition for a short root α is determined by the integer (ϕ, α^\vee) . \square

In Chapter 5 we defined a grading ε on the imaginary root system $\Delta_i^{\Theta}(\mathfrak{g}, \mathfrak{h})$ for any Θ -stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Given a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$, Proposition 11.3.3 defines a grading η on the real root system $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ with the same formal properties as ε . Applying the conjugation map $i_{\lambda, \phi}^{-1}$ (Section 6.1) to $\{(\tilde{H}, \phi), \varepsilon, \eta\}$ gives an *abstract bigrading* $(\theta, \varepsilon, \eta, \lambda)$ for the abstract root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$. This extends the notion of an abstract triple from Section 6.2.

It is easy to check conjugating $(\tilde{H}, \phi, \tilde{\Gamma})$ by an element of \tilde{K} does not change the abstract bigrading. In particular, abstract bigradings are defined on the level of genuine parameters $\tilde{\mathcal{D}}_\lambda$ for \tilde{G} . Note that an abstract bigrading depends (potentially) on the full representative $(\tilde{H}, \phi, \tilde{\Gamma})$ and not just the representative pair (\tilde{H}, ϕ) . The exact nature of this dependence is determined in the next section.

11.4 Central Character

Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character. It will be convenient in what follows to fix a representative set of Cartan subgroups in \tilde{G} . To begin, let $\mathfrak{h}^s \subset \mathfrak{g}$ be a split Cartan subalgebra with corresponding Cartan subgroup $\tilde{H}^s \subset \tilde{G}$. In the usual coordinates for $\Delta(\mathfrak{g}, \mathfrak{h}^s) = \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h}^s)$ set

$$\begin{aligned}\alpha_i &= e_{2i-1} - e_{2i} \\ \beta_i &= e_{2i-1} + e_{2i} \\ \gamma &= e_n.\end{aligned}$$

With the exception of the pairs $\{\alpha_n, \gamma\}$ and $\{\beta_n, \gamma\}$, this is a strongly orthogonal set of roots. Choose root vectors $X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and $X_{\gamma} \in \mathfrak{g}_{\gamma}$ according to Lemma 11.1.1 and induce $X_{\beta_i} \in \mathfrak{g}_{\beta_i}$ via short reflection as in Proposition 11.2.8. Define the corresponding Cayley transform operators \mathcal{C}_{α_i} , \mathcal{C}_{β_i} , and \mathcal{C}_{γ} . Then (up to conjugacy) we obtain every Cartan subalgebra of \mathfrak{g} by an iterated application of these operators to \mathfrak{h}^s . We use the notation from Section 8.1 to denote the resulting Cartan subalgebras and corresponding Cartan subgroups in \tilde{G} .

Definition 11.4.1. Suppose \mathfrak{c} is a sequence of strongly orthogonal roots of the form above and let $\tilde{H}_{\mathfrak{c}}^s \subset \tilde{G}$ be the corresponding Cartan subgroup. The *central character* of a genuine triple $(\tilde{H}_{\mathfrak{c}}^s, \phi, \tilde{\Gamma})_{\lambda}$ is the genuine representation of $Z(\tilde{G})$ given by restricting $\tilde{\Gamma}$ to $Z(\tilde{G}) \subset \tilde{H}_{\mathfrak{c}}^s$.

Remark 11.4.2. The central character will turn out to be an important invariant of genuine triples. Since $\tilde{\Gamma}$ is assumed to be genuine, we automatically have $\tilde{\Gamma}(-1) = -I$. Therefore the central character of $(\tilde{H}_{\mathfrak{c}}^s, \phi, \tilde{\Gamma})_{\lambda}$ is determined by the action of $\tilde{\Gamma}$ on a single nontrivial element in $Z(\tilde{G})$. Since $|Z(\tilde{G})| = 4$, there are only two such possibilities.

Remark 11.4.3. Given a genuine triple $(\tilde{H}_{\mathfrak{c}}^s, \phi, \tilde{\Gamma})_{\lambda}$, Proposition 7.1.3 implies

$$\tilde{\Gamma}|_{Z(\tilde{H}_{\mathfrak{c}}^s)} = m\chi$$

where $m = \left| \tilde{H}_{\mathfrak{c}}^s / Z(\tilde{H}_{\mathfrak{c}}^s) \right|^{\frac{1}{2}}$ and χ is a genuine character of $Z(\tilde{H}_{\mathfrak{c}}^s)$. In particular, central characters are not always one-dimensional representations of $Z(\tilde{G})$. This makes it technically incorrect to compare central characters for distinct genuine triples unless we happen to know the dimensions of their genuine representations are equal. However, we can remedy this by comparing the associated characters $\chi|_{Z(\tilde{G})}$ from Proposition 7.1.3. Unless otherwise stated,

this convention is in effect whenever we compare central characters for different genuine triples.

Remark 11.4.4. The central character is clearly well defined on the level of genuine parameters $\tilde{\mathcal{D}}_\lambda$ for \tilde{G} .

Fix a genuine triple $(\tilde{H}_c^s, \phi, \tilde{\Gamma})_\lambda$ and let η and ε be the corresponding real and imaginary gradings. In most cases, the central character of $(\tilde{H}_c^s, \phi, \tilde{\Gamma})_\lambda$ is determined by either η or ε .

Proposition 11.4.5. *In the above setting, suppose $\alpha \in \Delta_{\mathbb{R}}^{\ominus}(\mathfrak{g}, \mathfrak{h}_c^s)$ is a short real root. Then \tilde{m}_α is well-defined and central in \tilde{G} and we have*

$$\tilde{\Gamma}(\tilde{m}_\alpha) = \begin{cases} (-1)^{1-\eta(\alpha)} \cdot I & (\phi, \alpha) \in \mathbb{Z} + \frac{1}{2} \\ (-1)^{\eta(\alpha)} \cdot I & (\phi, \alpha) \in \mathbb{Z} \end{cases}.$$

Proof. This follows from Definition 11.3.2 directly. \square

Proposition 11.4.6. *In the above setting, suppose $\alpha \in \Delta_i^{\ominus}(\mathfrak{g}, \mathfrak{h}_c^s)$ is short and noncompact. Then \tilde{m}_α is well-defined and central in \tilde{G} and we have*

$$\tilde{\Gamma}(\tilde{m}_\alpha) = \begin{cases} 1 \cdot I & (\phi, \alpha) \in \mathbb{Z} + \frac{1}{2} \\ -1 \cdot I & (\phi, \alpha) \in \mathbb{Z} \end{cases}.$$

Proof. Choose $X_\alpha \in \mathfrak{g}_\alpha$ according to Lemma 11.1.9 and recall $\tilde{m}_\alpha = \exp_{\tilde{G}}(-\pi i h_\alpha) = \exp_{\tilde{G}}(\pi i h_\alpha)$ is a central element in \tilde{G} . Then

$$\begin{aligned} \overline{i h_\alpha} &= \overline{i [X_\alpha, \overline{X_\alpha}]} \\ &= -i [\overline{X_\alpha}, X_\alpha] \\ &= i [X_\alpha, \overline{X_\alpha}] \\ &= i h_\alpha \end{aligned}$$

so that $i h_\alpha \in \mathfrak{g}_{\mathbb{R}}$. It follows $\tilde{m}_\alpha \in (\tilde{H}_c^s)_0$ (see also Proposition 8.3.1) and it suffices to calculate $d\Gamma(-\pi i h_\alpha)$. From the identity $\alpha(h_\alpha) = 2$ and the definition of $d\tilde{\Gamma}$ in Section 2.2 we have

$$\begin{aligned} \tilde{\Gamma}(\tilde{m}_\alpha) &= \tilde{\Gamma}(\exp_{\tilde{G}}(-\pi i h_\alpha)) \\ &= e^{d\Gamma(-\pi i h_\alpha)} \cdot \mathbf{I} \\ &= e^{-2\pi i \left((\phi, \alpha) + (\rho_i^\phi, \alpha) - (2\rho_{ic}^\phi, \alpha) \right)} \cdot \mathbf{I}. \end{aligned}$$

Now it is easy to check $(\rho_i^\phi, \alpha) \in \mathbb{Z} + \frac{1}{2}$ and $(2\rho_{ic}^\phi, \alpha) \in \mathbb{Z}$ so that

$$\tilde{\Gamma}(\tilde{m}_\alpha) = -e^{-2\pi i \cdot (\phi, \alpha)} \cdot \mathbf{I}$$

and the result follows. \square

Remark 11.4.7. If $\Delta(\mathfrak{g}, \mathfrak{h}_c^s)$ contains a short root that is not complex, we can use Proposition 11.4.5 or Proposition 11.4.6 to determine the central character of $(\tilde{H}_c^s, \phi, \tilde{\Gamma})_\lambda$. If this root is noncompact, the central character is determined by the genuine pair $(\tilde{H}_c^s, \phi)_\lambda$ and all genuine triples extending $(\tilde{H}_c^s, \phi)_\lambda$ have the same central character.

In Section 6.3 it was shown there are (at most) two possibilities for the imaginary grading ε (Remark 6.3.2). In particular, if $\alpha, \beta \in \Delta_i^\theta(\mathfrak{g}, \mathfrak{h})$ are short then

$$\varepsilon(\alpha) = \varepsilon(\beta) \iff (\phi, \alpha) \equiv (\phi, \beta) \pmod{\mathbb{Z}}.$$

Since the structure of Definition 11.3.2 is formally the same, a similar result holds for the real grading η as well. The following theorem describes the relationship between η and ε in terms of these possibilities.

Theorem 11.4.8. *Let $(\tilde{H}_c^s, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple with corresponding real and imaginary gradings η and ε . Suppose the root system $\Delta(\mathfrak{g}, \mathfrak{h}_c^s)$ contains both short real and short noncompact imaginary roots. Then the gradings η and ε are of the opposite kind. In particular, if $\alpha \in \Delta_{\mathbb{R}}^\ominus(\mathfrak{g}, \mathfrak{h}_c^s)$ and $\beta \in \Delta_i^\ominus(\mathfrak{g}, \mathfrak{h}_c^s)$ are short*

$$\eta(\alpha) = \varepsilon(\beta) \iff (\phi, \alpha) \not\equiv (\phi, \beta) \pmod{\mathbb{Z}}.$$

Proof. Specializing Proposition 11.4.5 to the case $\eta(\alpha) = 1$ gives

$$\tilde{\Gamma}(\tilde{m}_\alpha) = \begin{cases} 1 \cdot \mathbf{I} & (\phi, \alpha) \in \mathbb{Z} + \frac{1}{2} \\ -1 \cdot \mathbf{I} & (\phi, \alpha) \in \mathbb{Z} \end{cases}.$$

If $\varepsilon(\beta) = 1$, Proposition 11.4.6 gives

$$\tilde{\Gamma}(\tilde{m}_\beta) = \begin{cases} 1 \cdot \mathbf{I} & (\phi, \beta) \in \mathbb{Z} + \frac{1}{2} \\ -1 \cdot \mathbf{I} & (\phi, \beta) \in \mathbb{Z} \end{cases}.$$

However, Corollary 11.2.9 implies $\tilde{m}_\beta = -\tilde{m}_\alpha$ and the result follows. \square

11.5 Central Characters in Even Rank

We continue with the notation from the previous section and assume now the rank of \tilde{G} is *even*. Recall $|\tilde{\mathcal{D}}_\lambda^\theta|$ denotes the number of genuine parameters for λ whose corresponding abstract triples (or abstract bigradings) begin with θ . Since \tilde{G} has even rank, $|\tilde{\mathcal{D}}_\lambda^\theta| \in \{1, 4\}$ by Proposition 10.1.4. In the nontrivial case, the following proposition implies central characters evenly partition the set $\tilde{\mathcal{D}}_\lambda^\theta$.

Proposition 11.5.1. *Assume $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$ and suppose $v \in \tilde{\mathcal{D}}_\lambda^\theta$. Then exactly half of the elements in $\tilde{\mathcal{D}}_\lambda^\theta$ have the same central character as v .*

Proof. Let $\epsilon_s^\theta, \epsilon_p^\theta, \epsilon_r^\theta, \epsilon_m^\theta$ denote the indicator bits for θ (Definition 3.1.1) and recall Theorem 9.4.3 implies $|\tilde{\mathcal{D}}_\lambda^\theta| = 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)}$. If $\epsilon_s^\theta = 0$, Proposition 8.3.1 implies the nontrivial central elements in $Z(\tilde{G})$ are not in $(\tilde{H}_c^s)_0$ and the result clearly follows. Suppose now $\epsilon_s^\theta = 1 \Rightarrow \epsilon_p^\theta = 0$. Then the number of ‘+’ signs in the diagram for θ is even and there are two possible imaginary gradings for which the abstract triple $(\theta, \varepsilon, \lambda)$ is supportable (Remark 6.3.2). Theorems 8.4.4 and 9.3.1 imply each one of these gradings occurs twice in genuine triples for $\tilde{\mathcal{D}}_\lambda^\theta$ and the result follows by Proposition 11.4.6. \square

If $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$, there is obviously only one central character associated with the set $\tilde{\mathcal{D}}_\lambda^\theta$. The following proposition implies the other central character is associated with the image of $\tilde{\mathcal{D}}_\lambda^\theta$ under the map Ψ .

Proposition 11.5.2. *Suppose $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$ and let $(\theta, \varepsilon, \eta, \lambda)$ denote the abstract bigrading for $v \in \tilde{\mathcal{D}}_\lambda^\theta$. Then $|\tilde{\mathcal{D}}_\lambda^{\Psi(\theta)}| = 1$ and the corresponding abstract bigrading is $(-\theta, \eta, \varepsilon, \lambda)$.*

Proof. The fact that $|\tilde{\mathcal{D}}_\lambda^{\Psi(\theta)}| = 1$ is Theorem 10.1.5. Theorem 9.4.3 implies $|\tilde{\mathcal{D}}_\lambda^\theta| = 2^{2-(\epsilon_s^\theta \epsilon_p^\theta + \epsilon_r^\theta \epsilon_m^\theta)}$ so that $\epsilon_s^\theta = \epsilon_p^\theta = \epsilon_r^\theta = \epsilon_m^\theta = 1$. In particular, θ has an odd number of both real and imaginary coordinates. Since θ is supportable, we may assume (wlog) $\frac{n_s^\theta + 1}{2}$ of the imaginary coordinates are integral and these are exactly the short noncompact imaginary roots for ε (Remark 5.1.6). However, (the proof of) Proposition 10.1.3 implies $\frac{n_r^\theta + 1}{2}$ of the real coordinates are strictly half-integral. Since these must be noncompact imaginary coordinates for $\Psi(\theta)$ (Remark 5.1.6 again), the abstract bigrading of a genuine triple for $\tilde{\mathcal{D}}_\lambda^{\Psi(\theta)}$ should have an imaginary grading opposite to ε . The result now follows from Theorem 11.4.8. \square

Corollary 11.5.3. *In the setting of Proposition 11.5.2, the central characters of $\tilde{\mathcal{D}}_\lambda^\theta$ and $\tilde{\mathcal{D}}_\lambda^{\Psi(\theta)}$ are opposite.*

Proof. This follows from Proposition 11.5.2 and either Proposition 11.4.5 or Proposition 11.4.6. \square

CHAPTER 12

ALMOST CENTRAL CHARACTERS

Fix a symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and suppose the rank of \tilde{G} is even. We have seen $|\tilde{\mathcal{D}}_\lambda^\theta| \in \{1, 4\}$ for each supportable involution $\theta \in \mathcal{I}$. In the case $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$, the definition of Ψ on $\tilde{\mathcal{D}}_\lambda^\theta$ is obvious and it remains to define Ψ when $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$. Using properties of short roots, it was shown in the previous section that central characters can be used to partly distinguish the elements in $\tilde{\mathcal{D}}_\lambda^\theta$ (Proposition 11.5.1). In this section, we complete the distinguishing process (Theorem 12.6.4) and produce an extension of the map Ψ (Definition 12.7.1). Not surprisingly, our methods now rely on properties of long roots.

12.1 Integral Cross Actions in $\tilde{\mathcal{D}}_\lambda$

In Section 9.1 we defined the cross action of the abstract Weyl group $W = W(\mathfrak{g}, \mathfrak{h}^a)$ on \tilde{K} -conjugacy classes of genuine pairs. Here we describe an extension of the cross action to \tilde{K} -conjugacy classes of genuine triples. This material will be familiar to most readers and we refer to [16] or [19] for more details.

To begin, fix a Θ -stable Cartan subgroup $\tilde{H} \subset \tilde{G}$ and let $\Delta(\mathfrak{g}, \mathfrak{h})$ be its root system. For each $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, there is a corresponding *root character* $\tilde{\alpha}$ of \tilde{H} given by the adjoint action of \tilde{H} on \mathfrak{g}_α . The following lemma implies these characters behave like roots.

Lemma 12.1.1 ([16], Lemma 0.4.5). *Suppose we have*

$$\sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} n_\alpha \alpha = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} m_\alpha \alpha$$

in \mathfrak{h}^ with $n_\alpha, m_\alpha \in \mathbb{Z}$. Then*

$$\prod_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} \tilde{\alpha}^{n_\alpha} = \prod_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} \tilde{\alpha}^{m_\alpha}$$

as characters of \tilde{H} .

In particular, we can use root characters to translate between representations of $(\widetilde{H})_0$ whose differentials differ by a sum of roots. To this end we have the following lemmas whose proofs are similar.

Lemma 12.1.2. *Suppose $\phi \in \mathfrak{h}^*$ is regular and let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$. Then in the notation of Section 2.2*

$$\begin{aligned}\rho_i^{s_\alpha(\phi)} &= \rho_i^\phi - \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h})} n_\beta \beta \\ 2\rho_{ic}^{s_\alpha(\phi)} &= 2\rho_{ic}^\phi - \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h})} 2 \cdot m_\beta \beta\end{aligned}$$

with $n_\beta, m_\beta \in \mathbb{Z}$.

Lemma 12.1.3. *In the setting of Lemma 12.1.2, suppose $\alpha \in \Delta_i^\ominus(\mathfrak{g}, \mathfrak{h})$ is an imaginary root. Then*

$$\rho_i^{s_\alpha(\phi)} = \begin{cases} \rho_i^\phi - \sum_{\substack{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}) \\ \beta \text{ long}}} n_\beta \beta & \alpha \text{ long} \\ \rho_i^\phi - \sum_{\substack{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}) \\ \beta \text{ long}}} n_\beta \beta - \alpha & \alpha \text{ short} \end{cases}$$

with $n_\beta, m_\beta \in \mathbb{Z}$.

Proof. Let

$$\mathcal{S} = \{ \beta \in \Delta_i^\ominus(\mathfrak{g}, \mathfrak{h}) \mid (\beta, \phi) > 0 \text{ and } (\beta, s_\alpha(\phi)) < 0 \}$$

denote the set of imaginary roots that are positive with respect to ϕ and negative with respect to $s_\alpha\phi$. Then

$$\begin{aligned}\rho_i^{s_\alpha(\phi)} &= \rho_i^\phi - \frac{1}{2} \sum_{\beta \in \mathcal{S}} \beta + \frac{1}{2} \sum_{\beta \in \mathcal{S}} (-\beta) \\ &= \rho_i^\phi - \sum_{\beta \in \mathcal{S}} \beta.\end{aligned}$$

Suppose first that α is long. If every other root in \mathcal{S} is also long, the result clearly follows.

Otherwise there are exactly two short roots in \mathcal{S} , say γ_1 and γ_2 and we have

$$\begin{aligned}\rho_i^{s_\alpha(\phi)} &= \rho_i^\phi - \left(\sum_{\substack{\beta \in \mathcal{S} \\ \beta \text{ long}}} \beta \right) - \gamma_1 - \gamma_2 \\ &= \rho_i^\phi - \left(\sum_{\substack{\beta \in \mathcal{S} \\ \beta \text{ long}}} \beta \right) - \gamma\end{aligned}$$

where $\gamma = \gamma_1 + \gamma_2$ is a long root. If α is short, then α is the only such root in \mathcal{S} and the result follows. \square

Lemma 12.1.4. *In the setting of Lemma 12.1.2, suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a complex root. Then*

$$\rho_i^{s_\alpha(\phi)} = \begin{cases} \rho_i^\phi - \sum_{\substack{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}) \\ \beta \text{ long}}} n_\beta \beta - \epsilon \gamma & \alpha \text{ long} \\ \rho_i^\phi - \sum_{\substack{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}) \\ \beta \text{ long}}} n_\beta \beta & \alpha \text{ short} \end{cases}$$

where $\epsilon \in \{0, 1\}$ and $\gamma \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a short root.

In particular, reflecting ϕ by s_α alters the elements ρ_i^ϕ and $\rho_{ic}^\phi \in \mathfrak{h}^*$ by integral sums of roots. A similar result holds for ϕ itself if we restrict to a certain subgroup of $W(\mathfrak{g}, \mathfrak{h})$ (Lemma 12.1.6).

Definition 12.1.5 ([16], Definition 7.2.16). Suppose $\phi \in \mathfrak{h}^*$ is a regular element and let

$$\Delta(\mathfrak{g}, \mathfrak{h})(\phi) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid (\phi, \alpha^\vee) \in \mathbb{Z}\}$$

denote the set of *integral roots* for ϕ . Then $\Delta(\mathfrak{g}, \mathfrak{h})(\phi)$ is a subroot system of $\Delta(\mathfrak{g}, \mathfrak{h})$ and we denote the corresponding *integral Weyl group* by $W(\mathfrak{g}, \mathfrak{h})(\phi) \subset W(\mathfrak{g}, \mathfrak{h})$.

Lemma 12.1.6 ([16], Lemma 7.2.17). *In the above setting, w is an element of $W(\mathfrak{g}, \mathfrak{h})(\phi)$ if and only if*

$$w\phi - \phi = \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h})} n_\beta \beta \quad (n_\beta \in \mathbb{Z}).$$

In other words, $w \in W(\mathfrak{g}, \mathfrak{h})(\phi)$ if and only if $w\phi - \phi$ can be written as an integral sum of roots.

We now define the cross action of the abstract *integral Weyl group* on genuine triples for \tilde{G} .

Definition 12.1.7 ([16], Definition 8.3.1). Let $W = W(\mathfrak{g}, \mathfrak{h}^a)$ denote the abstract Weyl group and let $w \in W(\lambda) = W(\mathfrak{g}, \mathfrak{h}^a)(\lambda)$. Suppose $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for \tilde{G} and

write $w_\phi = i_\phi(w)$ (Section 6.1) for the image of w in $\Delta(\mathfrak{g}, \mathfrak{h})$. Then $w_\phi \in W(\mathfrak{g}, \mathfrak{h})(\phi)$ and we can write

$$w \times \phi - \phi = w_\phi^{-1}(\phi) - \phi = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} n_\alpha \alpha \quad (n_\alpha \in \mathbb{Z})$$

by Lemma 12.1.6. Similarly, Lemma 12.1.2 implies

$$(\rho_i^{w \times \phi} - 2\rho_{ic}^{w \times \phi}) - (\rho_i^\phi - 2\rho_{ic}^\phi) = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} m_\alpha \alpha \quad (m_\alpha \in \mathbb{Z}).$$

Then

$$\varphi = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} (n_\alpha + m_\alpha) \alpha$$

gives a well-defined character of \tilde{H}

$$\Phi = \prod_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} \tilde{\alpha}^{(n_\alpha + m_\alpha)}$$

by Lemma 12.1.1. We define the *cross action* of $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ by w via

$$\begin{aligned} w \times \Gamma &= \Gamma \cdot \Phi \\ w \times (\tilde{H}, \phi, \tilde{\Gamma})_\lambda &= (\tilde{H}, w \times \phi, w \times \tilde{\Gamma})_\lambda. \end{aligned}$$

Remark 12.1.8. Definition 12.1.7 describes the (abstract) cross action of $W(\phi)$ on genuine triples for \tilde{G} . If $\tilde{H} \subset \tilde{G}$ is a Cartan subgroup, we will occasionally need the (regular) cross action of $W(\mathfrak{g}, \mathfrak{h})(\phi)$ on genuine triples of the form $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$. The definition is obvious from Definition 12.1.7 and will be denoted the same way.

The following lemma describes a special case when the cross action of a genuine triple is easy to compute.

Lemma 12.1.9 ([16], Lemma 8.3.2). *Let $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $\alpha \in \Delta(\lambda) = \Delta(\mathfrak{g}, \mathfrak{h}^a)(\lambda)$ is a simple abstract root. Write $\beta = i_\phi(\alpha)$ for the image of α in $\Delta(\mathfrak{g}, \mathfrak{h})(\phi)$ and set $m = (\phi, \beta^\vee) \in \mathbb{Z}$. Then if α is compact imaginary*

$$s_\alpha \times (\tilde{H}, \phi, \tilde{\Gamma})_\lambda = (\tilde{H}, \phi - m\beta, \tilde{\Gamma} \cdot \tilde{\beta}^{-(m-1)})_\lambda,$$

if α is noncompact imaginary

$$s_\alpha \times (\tilde{H}, \phi, \tilde{\Gamma})_\lambda = (\tilde{H}, \phi - m\beta, \tilde{\Gamma} \cdot \tilde{\beta}^{-(m+1)})_\lambda,$$

and if α is real or complex

$$s_\alpha \times (\tilde{H}, \phi, \tilde{\Gamma})_\lambda = (\tilde{H}, \phi - m\beta, \tilde{\Gamma} \cdot \tilde{\beta}^{-m})_\lambda.$$

Proof. We have

$$\begin{aligned}
s_\beta^{-1}(\phi) &= s_\beta(\phi) \\
&= \phi - (\phi, \beta^\vee)\beta \\
&= \phi - m\beta
\end{aligned}$$

by the definition of m . Clearly β is simple in the positive root system for $\Delta(\mathfrak{g}, \mathfrak{h})$ determined by ϕ , and the reflection s_β permutes the positive roots other than β . Therefore if α is real or complex, $\rho_i^{s_\alpha \times \phi} = \rho_i^\phi$ and $2\rho_{ic}^{s_\alpha \times \phi} = 2\rho_{ic}^\phi$. In the notation of Definition 12.1.7, $\varphi = -m\beta$ and the result follows.

If α is compact we have $\rho_i^{s_\alpha \times \phi} = \rho_i^\phi - \beta$, $2\rho_{ic}^{s_\alpha \times \phi} = 2\rho_{ic}^\phi - 2\beta$, and $(\rho_i^{s_\alpha \times \phi} - 2\rho_{ic}^{s_\alpha \times \phi}) - (\rho_i^\phi - 2\rho_{ic}^\phi) = \beta$. Then $\varphi = -m\beta + \beta$ and the result follows. The case for noncompact imaginary roots is handled similarly. \square

Proposition 12.1.10 ([13], Chapter 4). *Let $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $w \in W(\lambda)$. Then $w \times (\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is also genuine triple. Moreover, the cross action descends to a well-defined action on the level of genuine parameters for \tilde{G} .*

Remark 12.1.11. For $v \in \tilde{\mathcal{D}}_\lambda$, we will write $w \times v \in \tilde{\mathcal{D}}_\lambda$ for the cross action of $W(\lambda)$ on genuine parameters for \tilde{G} .

Corollary 12.1.12. *In the setting of Proposition 12.1.10, suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$. Then $w \times (\theta, \varepsilon, \lambda)$ is supportable (Section 6.3).*

Proof. This follows from Proposition 12.1.10, but we can also prove it directly. It suffices to show $w \times (\theta, \varepsilon, \lambda)$ satisfies the conditions of Proposition 6.2.2. Proposition 9.1.6 implies

$$w \times (\theta, \varepsilon, \lambda) = (w \cdot \theta \cdot w^{-1}, w \times \varepsilon, \lambda)$$

and α is imaginary for $w \cdot \theta \cdot w^{-1}$ if and only if

$$w \cdot \theta \cdot w^{-1}(\alpha) = \alpha \iff \theta(w^{-1}\alpha) = w^{-1}\alpha.$$

In particular, we must have $w^{-1}\alpha$ imaginary for θ . However

$$\begin{aligned}
(\lambda, (w^{-1}\alpha)^\vee) &= (\lambda, w^{-1}\alpha^\vee) \\
&= (w\lambda, \alpha^\vee) \\
&= \left(\lambda + \sum_{\beta \in \Delta} n_\beta \beta, \alpha^\vee\right) \\
&\equiv (\lambda, \alpha^\vee) \pmod{\mathbb{Z}}.
\end{aligned}$$

Since $w \times \varepsilon(\alpha) = \varepsilon(w^{-1}\alpha)$, the result holds for imaginary roots. Complex roots are handled similarly. \square

Proposition 12.1.13. *Let $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $w \in W(\lambda)$. Then v and $w \times v$ have the same central character (Section 11.4).*

Proof. This follows from Definition 12.1.7 and the fact that root characters act trivially on $Z(\tilde{G})$. \square

Fix $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $\alpha \in \Delta(\lambda)$ is an abstract integral root. We conclude this section with a partial description of when $s_\alpha \times v = v$.

Proposition 12.1.14. *If α is complex for v , then $s_\alpha \times v \neq v$.*

Proof. The corresponding \tilde{K} -orbits not conjugate by Propositions 9.1.2 and 9.1.6. \square

Proposition 12.1.15 ([13], Lemma 6.14(a)). *If α is imaginary and compact for v , then $s_\alpha \times v = v$.*

Proposition 12.1.16. *If α is imaginary, noncompact, and of type I for v (Section 9.2), then $s_\alpha \times v \neq v$.*

Proof. The corresponding \tilde{K} -orbits not conjugate (by definition). \square

The situation for imaginary roots that are noncompact and of type II is discussed in Section 12.5. The situation for real roots is discussed in Section 12.3.

12.2 Extended Cross Actions

Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character. Definition 12.1.7 described the cross action of the abstract integral Weyl group $W(\lambda)$ on the set of genuine parameters $\tilde{\mathcal{D}}_\lambda$ with infinitesimal character λ . It will be important in what follows to consider an extended version of the cross action that includes the full abstract Weyl group W . In this section we briefly examine the definition from [13], Chapters 3 and 4.

To begin, recall the abstract root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ and let $\mathcal{R} = L(\Delta) \subset (\mathfrak{h}^a)^*$ be the corresponding root lattice. Consider the quotient

$$\mathcal{Q} = (\mathfrak{h}^a)^* / \mathcal{R}$$

and observe the natural action of W on $(\mathfrak{h}^a)^*$ descends to \mathcal{Q} . Denote the image of λ in \mathcal{Q} by $[\lambda]$.

Definition 12.2.1. A family of infinitesimal characters for $\lambda \in (\mathfrak{h}^a)^*$ is a collection of dominant representatives for the W -orbit of $[\lambda]$ in \mathcal{Q} .

Fix $\lambda \in (\mathfrak{h}^a)^*$ and suppose $\mathcal{F}(\lambda)$ is a family of infinitesimal characters for λ . Then every element of $\mathcal{F}(\lambda)$ is symmetric and Proposition 12.1.6 implies the stabilizer of λ in W is $W(\lambda)$. For even values of n we have

$$\begin{aligned} |\mathcal{F}(\lambda)| &= \frac{|W(B_n)|}{\left|W(B_{\frac{n}{2}})\right| \left|W(B_{\frac{n}{2}})\right|} \\ &= \frac{2^n \cdot n!}{2^{\frac{n}{2}} \cdot (\frac{n}{2})! \cdot 2^{\frac{n}{2}} \cdot (\frac{n}{2})!} \\ &= \binom{n}{\frac{n}{2}}. \end{aligned}$$

Remark 12.2.2. In [13], the authors define \mathcal{Q} using the weight lattice $\mathcal{P} = X$ instead of the root lattice \mathcal{R} . Either choice will suffice, however the root lattice is more convenient for our purposes.

The abstract Weyl group W acts via the extended cross action (Definition 12.2.3) on collections of genuine parameters whose infinitesimal characters live in families of the kind described above. In order to define this action, it is first necessary to specify how the infinitesimal characters in a family are related. To this end, suppose $\kappa \in \mathcal{F}(\lambda)$ and $w \in W$. Define the element $\mu_\kappa(w) \in \mathcal{R}$ by the requirement

$$\kappa + \mu_\kappa(w) \in w \cdot \mathcal{F}(\lambda)$$

with the convention that $\mu_\kappa(w) = w\kappa - \kappa$ (Lemma 12.1.6) for $w \in W(\kappa)$. We now have the following extension of Definition 12.1.7.

Definition 12.2.3. ([13], Definition 4.1) Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character with corresponding family $\mathcal{F}(\lambda)$. Suppose $\kappa \in \mathcal{F}(\lambda)$ and $(\tilde{H}, \phi, \tilde{\Gamma})_\kappa$ is a genuine triple for κ . If $w \in W$ set

$$\begin{aligned} w \times \phi &= \phi + i_\phi(\mu_\kappa(w^{-1})) \\ (\rho_i^{w \times \phi} - 2\rho_{ic}^{w \times \phi}) - (\rho_i^\phi - 2\rho_{ic}^\phi) &= \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} m_\alpha \alpha \quad (m_\alpha \in \mathbb{Z}). \end{aligned}$$

Then

$$\varphi = i_\phi(\mu_\kappa(w^{-1})) + \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} m_\alpha \alpha$$

determines a well-defined character Φ of \tilde{H} by Lemma 12.1.1. We define the (*extended*) *cross action* of $(\tilde{H}, \phi, \tilde{\Gamma})_\kappa$ by w to be

$$\begin{aligned} w \times \Gamma &= \Gamma \cdot \Phi \\ w \times (\tilde{H}, \phi, \tilde{\Gamma})_\kappa &= (\tilde{H}, w \times \phi, w \times \tilde{\Gamma}). \end{aligned}$$

Remark 12.2.4. As the notation suggests, Definition 12.1.7 and Definition 12.2.3 coincide whenever $w \in W(\kappa)$. This follows immediately from the convention $\mu_\kappa(w) = w\kappa - \kappa$ for w in $W(\kappa)$.

Remark 12.2.5. The infinitesimal character of $w \times (\tilde{H}, \phi, \tilde{\Gamma})_\kappa$ is an element of $\mathcal{F}(\lambda)$ that depends on κ and the element w .

Proposition 12.2.6 ([13], Chapter 4). *In the above setting, let $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $w \in W$. Then $w \times (\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is genuine triple and the extended cross action descends to a well-defined action on the level of genuine parameters for \tilde{G} .*

12.3 Principal Series

Fix a symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and recall the rank of \tilde{G} is even. Let $\mathfrak{h}^s \subset \mathfrak{g}$ be a split Cartan subalgebra with corresponding Cartan subgroup $\tilde{H}^s \subset \tilde{G}$.

Definition 12.3.1. Genuine representations of \tilde{G} whose corresponding genuine triples begin with a split Cartan subgroup are called *principal series*. We may also refer to any genuine triple of the form $(\tilde{H}^s, \phi, \tilde{\Gamma})$ as a principal series for \tilde{G} . Write $\text{ps}_\lambda = \tilde{\mathcal{D}}_\lambda^{-1}$ for the set of principal series representations of \tilde{G} with infinitesimal character λ .

In this section we study conjugation and cross actions in ps_λ . In the usual coordinates for $\Delta(\mathfrak{g}, \mathfrak{h}^s) = \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h}^s)$ set

$$\begin{aligned} \alpha_i &= e_{2i-1} - e_{2i} \\ \beta_i &= e_{2i-1} + e_{2i} \\ \gamma &= e_n \end{aligned}$$

as in Section 11.4. Choose root vectors $X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and $X_\gamma \in \mathfrak{g}_\gamma$ according to Lemma 11.1.1 and define

$$\tilde{z} = \tilde{m}_{\alpha_1} \tilde{m}_{\alpha_2} \cdots \tilde{m}_{\alpha_{n/2}}.$$

The element \tilde{z} is almost central in \tilde{G} and will play a critical role in our definition of Ψ . Lemma 7.1.1 and Proposition 7.3.5 imply

$$Z(\tilde{H}^s) = \langle \pm(\tilde{H}^s)_0, \pm\tilde{m}_\gamma, \pm\tilde{z} \rangle$$

and a genuine character of $Z(\tilde{H}^s)$ is determined by its differential and its values on \tilde{m}_γ and \tilde{z} . We have seen $\tilde{m}_\gamma \in Z(\tilde{G})$ has order two and the order of \tilde{z} is determined by

$$\begin{aligned} \tilde{z}^2 &= (\tilde{m}_{\alpha_1}\tilde{m}_{\alpha_2}\cdots\tilde{m}_{\alpha_{n/2}})^2 \\ &= \tilde{m}_{\alpha_1}^2\tilde{m}_{\alpha_2}^2\cdots\tilde{m}_{\alpha_{n/2}}^2 \\ &= (-1)^{\frac{n}{2}} \end{aligned}$$

to be either two or four. In particular, there are four possible *genuine* characters of $Z(\tilde{H}^s)$ with a fixed differential (Corollary 7.3.9).

Suppose $(\tilde{H}^s, \phi, \tilde{\Gamma})_\lambda$ is a principal series for \tilde{G} with infinitesimal character λ . Proposition 7.1.3 implies the isomorphism class of $\tilde{\Gamma}$ is determined by its restriction to $Z(\tilde{H}^s)$ and this restriction is given by

$$\tilde{\Gamma}|_{Z(\tilde{H}^s)} = m\chi$$

where χ is a genuine character for $Z(\tilde{H}^s)$. Therefore we may treat $\tilde{\Gamma}$ as either a genuine representation of \tilde{H}^s or a genuine character of $Z(\tilde{H}^s)$. Since the differential of $\tilde{\Gamma}$ is fixed by ϕ (Section 2.2), there are four distinct possibilities for $\tilde{\Gamma}$ and $|\text{ps}_\lambda| = 4$.

We begin with conjugation. For $\delta \in \Delta(\mathfrak{g}, \mathfrak{h}^s)$, recall $\delta(\tilde{m}_{\alpha_i})$ denotes the scalar by which \tilde{m}_{α_i} acts on the root space \mathfrak{g}_δ and Proposition 8.1.1 implies

$$\delta(\tilde{m}_{\alpha_i}) = (-1)^{(\delta, \alpha_i^\vee)}.$$

Proposition 12.3.2. *Let $(\tilde{H}^s, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $\delta \in \Delta(\mathfrak{g}, \mathfrak{h}^s)$. Choose $X_\delta \in \mathfrak{g}_\delta$ according to Lemma 11.1.1 and write $\tilde{\sigma}_\delta$ for the corresponding root reflection in \tilde{K} . Then $\tilde{\sigma}_\delta$ acts of $\tilde{\Gamma}$ by conjugation and we have*

$$(\tilde{\sigma}_\delta \cdot \tilde{\Gamma})(\tilde{z}) = \begin{cases} \tilde{\Gamma}(\tilde{z}) & \delta \text{ is long} \\ \tilde{\Gamma}(\tilde{m}_\gamma)\tilde{\Gamma}(\tilde{z}) & \delta \text{ is short} \end{cases}.$$

Proof. According to Lemma 11.1.7

$$\begin{aligned}
\tilde{\sigma}_\delta \tilde{z} \tilde{\sigma}_\delta^{-1} &= \tilde{\sigma}_\delta \tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_{n/2}} \tilde{\sigma}_\delta^{-1} \\
&= \tilde{\sigma}_\delta \tilde{m}_{\alpha_1} \tilde{\sigma}_\delta^{-1} \cdots \tilde{\sigma}_\delta \tilde{m}_{\alpha_{n/2}} \tilde{\sigma}_\delta^{-1} \\
&= \tilde{\sigma}_\delta^{(1-\delta(\tilde{m}_{\alpha_1}))} \tilde{m}_{\alpha_1} \cdots \tilde{\sigma}_\delta^{(1-\delta(\tilde{m}_{\alpha_{n/2}}))} \tilde{m}_{\alpha_{n/2}}.
\end{aligned}$$

Suppose first that δ is long. If $\delta = \alpha_j$ or $\delta = \beta_j$ for some j , we have $\delta(\tilde{m}_{\alpha_i}) = 1$ for all i by Proposition 8.1.1 and the result follows. Otherwise, $\delta(\tilde{m}_{\alpha_i}) = -1$ for exactly two values of i , say $i = j$ and $i = k$. By Proposition 7.3.1 we have

$$\begin{aligned}
\tilde{\sigma}_\delta \tilde{z} \tilde{\sigma}_\delta^{-1} &= \tilde{m}_{\alpha_1} \cdots \tilde{m}_\delta \tilde{m}_{\alpha_j} \cdots \tilde{m}_\delta \tilde{m}_{\alpha_k} \cdots \tilde{m}_{\alpha_{n/2}} \\
&= -\tilde{m}_\delta^2 \tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j} \cdots \tilde{m}_{\alpha_k} \cdots \tilde{m}_{\alpha_{n/2}} \\
&= -\tilde{m}_\delta^2 \tilde{z} \\
&= \tilde{z}
\end{aligned}$$

as desired.

If δ is short, $\delta(\tilde{m}_{\alpha_i}) = -1$ for exactly one value of i , say $i = j$. Now Propositions 7.3.1 and 7.2.2 imply

$$\begin{aligned}
\tilde{\sigma}_\delta \tilde{z} \tilde{\sigma}_\delta^{-1} &= \tilde{m}_{\alpha_1} \cdots \tilde{m}_\delta \tilde{m}_{\alpha_j} \cdots \tilde{m}_{\alpha_{n/2}} \\
&= \tilde{m}_\delta \tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j} \cdots \tilde{m}_{\alpha_{n/2}} \\
&= \tilde{m}_\delta \tilde{z} \\
&= \tilde{m}_\gamma \tilde{z}
\end{aligned}$$

as desired. □

Remark 12.3.3. Let $v \in \text{ps}_\lambda$ and suppose $(\tilde{H}^s, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple representing v . Proposition 12.3.2 implies v is *not* determined simply by its central character and the value of $\tilde{\Gamma}$ on \tilde{z} . This unfortunate fact accounts for most of the misery of this (and the previous) section.

The following proposition leads to a similar result for cross actions.

Proposition 12.3.4. *If $\delta \in \Delta(\mathfrak{g}, \mathfrak{h}^s)$, then in the notation above*

$$\delta(\tilde{z}) = \begin{cases} 1 & \delta \text{ is long} \\ -1 & \delta \text{ is short} \end{cases}.$$

Proof. Treating δ as a character of \tilde{H}^s we have

$$\begin{aligned}\delta(\tilde{z}) &= \delta(\tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_{n/2}}) \\ &= \delta(\tilde{m}_{\alpha_1}) \cdots \delta(\tilde{m}_{\alpha_{n/2}}) \\ &= (-1)^{(\delta, \alpha_1^\vee)} \cdots (-1)^{(\delta, \alpha_{n/2}^\vee)}\end{aligned}$$

and we proceed as in the proof of Proposition 12.3.2. \square

Corollary 12.3.5. *Let $(\tilde{H}^s, \phi, \tilde{\Gamma})_\lambda$ be a principal series and suppose $\delta \in \Delta(\lambda)$ is a long abstract root. Write $\bar{\delta} = i_\phi(\delta)$ for the image of δ in $\Delta(\mathfrak{g}, \mathfrak{h}^s)(\phi)$. Then in the notation of Proposition 12.3.2 we have*

$$(\tilde{\sigma}_{\bar{\delta}} \cdot \tilde{\Gamma})(\tilde{z}) = \tilde{\Gamma}(\tilde{z}) = (s_\delta \times \tilde{\Gamma})(\tilde{z}).$$

Proof. The first equality is Proposition 12.3.2. The second equality follows from Proposition 12.3.4 and the fact that $\rho_i^{s_\delta \times \phi} = \rho_{ic}^{s_\delta \times \phi} = \rho_i^\phi = \rho_{ic}^\phi = 0$ for a split Cartan subalgebra. \square

Roughly speaking, Corollary 12.3.5 implies the operations of cross action and conjugation are equal for long (integral) roots. The situation for short roots is more complicated. Let $(\tilde{H}^s, \phi, \tilde{\Gamma})_\lambda$ be a principal series and suppose $\delta \in \Delta(\lambda)$ is a short abstract root. Write $\bar{\delta} = i_\phi(\delta)$ for the image of δ in $\Delta(\mathfrak{g}, \mathfrak{h}^s)(\phi)$. By Proposition 12.3.2 and Proposition 12.3.4 we have

$$\begin{aligned}(\tilde{\sigma}_{\bar{\delta}} \cdot \tilde{\Gamma})(\tilde{z}) &= \tilde{\Gamma}(\tilde{m}_\delta) \tilde{\Gamma}(\tilde{z}) \\ (s_\delta \times \tilde{\Gamma})(\tilde{z}) &= \delta(\tilde{z})^{(\phi, \delta^\vee)} \tilde{\Gamma}(\tilde{z}) \\ &= (-1)^{(\phi, \delta^\vee)} \tilde{\Gamma}(\tilde{z})\end{aligned}$$

so that

$$(\tilde{\sigma}_{\bar{\delta}} \cdot \tilde{\Gamma})(\tilde{z}) = (s_\delta \times \tilde{\Gamma})(\tilde{z}) \iff \tilde{\Gamma}(\tilde{m}_\delta) = (-1)^{(\phi, \delta^\vee)}.$$

We have now completed the proof of the following corollary.

Corollary 12.3.6. *Let $v \in ps_\lambda$ be a principal series. If $\delta \in \Delta(\lambda)$ is an abstract integral root, then*

$$s_\delta \times v = v$$

if and only if δ does not satisfy the parity condition for v (Definition 11.3.2).

In the next section we extend Corollary 12.3.6 to a special collection of Cartan subgroups of \tilde{G} (see Corollary 12.4.10). In general, we have the following proposition.

Proposition 12.3.7 ([13], Lemma 6.14(f)). *Let $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $\alpha \in \Delta_{\mathbb{R}}^\theta(\lambda)$ is a real abstract integral root. If α does not satisfy the parity condition for v , then $s_\alpha \times v = v$.*

12.4 Even Parity Cartan Subgroups

Fix a symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and recall the rank of \tilde{G} is even. Let $\tilde{H} \subset \tilde{G}$ be a Θ -stable Cartan subgroup and suppose $\phi \in \mathfrak{h}^*$ is a regular element. Write $\theta = i_\phi^{-1}(\Theta)$ for the corresponding abstract involution and let $\epsilon_s^\theta, \epsilon_p^\theta, \epsilon_r^\theta, \epsilon_m^\theta$ be the indicator bits for θ (Definition 3.1.1). Note the indicator bits depend only on the W -conjugacy class of θ and therefore only on the \tilde{K} -conjugacy class of \tilde{H} (Chapter 3).

Since the rank of \tilde{G} is even, we have $\epsilon_p^\theta = \epsilon_m^\theta$ so that $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$ if and only if $\epsilon_p^\theta = \epsilon_m^\theta = 0$ (Proposition 10.1.4). A Cartan subgroup for which $\epsilon_p^\theta = \epsilon_m^\theta = 0$ is said to be of *even parity*. In particular, an abstract involution corresponds to an even parity Cartan subgroup if and only if its diagram has an even number of both ‘+’ and ‘-’ signs.

Lemma 12.4.1. *Let $\tilde{H} \subset \tilde{G}$ be a Θ -stable Cartan subgroup and suppose $\alpha \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ is a real root. Then \tilde{H} and \tilde{H}_α (Section 8.1) have the same parity if and only if α is long.*

Proof. This is verified in Section 8.2, Cases I–V. □

Fix a split Cartan subgroup $\tilde{H}^s \subset \tilde{G}$ and recall the Cayley transform operators \mathcal{C}_{α_i} and \mathcal{C}_{β_i} from Section 11.4. The following proposition characterizes even parity Cartan subgroups in terms of these operators.

Proposition 12.4.2. *Up to conjugacy, every even parity Cartan subgroup of \tilde{G} can be obtained from \tilde{H}^s through an iterative application of the operators \mathcal{C}_{α_i} and \mathcal{C}_{β_i} .*

Proof. This is easily verified on the level of involutions from the results of Section 8.2. □

Proposition 12.4.2 implies we can associate a (nonunique) sequence in $\{\alpha_i, \beta_i\}$ to each conjugacy class of even parity Cartan subgroups in \tilde{G} . Since the corresponding operators \mathcal{C}_{α_i} and \mathcal{C}_{β_i} commute (Lemma 11.2.6), the ordering of the roots is unimportant. To eliminate the ambiguity, we say a sequence in $\{\alpha_i, \beta_i\}$ is *standard* if it is of the form

$$\mathfrak{c}_j^k = \beta_k \cdots \beta_2 \beta_1 \alpha_j \cdots \alpha_2 \alpha_1$$

with $0 \leq k \leq j \leq \frac{n}{2}$. Then there is a unique standard sequence associated to each conjugacy class of even parity Cartan subgroups in \tilde{G} . Clearly there are

$$1 + 2 + \cdots + \left(\frac{n}{2} + 1\right) = \binom{\frac{n}{2} + 2}{2}$$

possible standard sequences (including the empty sequence) and thus $\binom{\frac{n}{2} + 2}{2}$ even parity Cartan subgroups of \tilde{G} (up to conjugacy). If \mathfrak{c}_j^k is a standard sequence we will write

$$\tilde{H}_{\mathfrak{c}_j^k}^s = \mathcal{C}_j^k(\tilde{H}^s) = \mathcal{C}_{\beta_k} \cdots \mathcal{C}_{\beta_2} \mathcal{C}_{\beta_1} \mathcal{C}_{\alpha_j} \cdots \mathcal{C}_{\alpha_2} \mathcal{C}_{\alpha_1}(\tilde{H}^s)$$

for the corresponding even parity Cartan subgroup.

In the previous section, we determined the structure of the set ps_λ by tracking a single element that was central in \tilde{H}^s but not central in \tilde{G} . The following propositions describe an extension of these methods to all even parity Cartan subgroups.

Proposition 12.4.3. *Let \mathfrak{c}_j^k be a standard sequence and suppose $1 \leq i \leq \frac{n}{2}$. Then $\tilde{m}_{\alpha_i} \in \tilde{H}_{\mathfrak{c}_j^k}^s$. In particular, the product*

$$\tilde{z} = \tilde{m}_{\alpha_1} \tilde{m}_{\alpha_2} \cdots \tilde{m}_{\alpha_{n/2}} \in \tilde{H}^s$$

(Section 12.3) is an element of $\tilde{H}_{\mathfrak{c}_j^k}^s$.

Proof. This follows from Corollary 8.1.3 and Proposition 8.1.5. □

Proposition 12.4.4. *Let \mathfrak{c}_j^k be a nonempty standard sequence and write \tilde{m}_γ for the unique element corresponding to any short real root in \tilde{H}^s (Proposition 11.1.6). Then the center of $\tilde{H}_{\mathfrak{c}_j^k}^s$ is given by*

$$Z(\tilde{H}_{\mathfrak{c}_j^k}^s) = \begin{cases} \left\langle (\tilde{H}_{\mathfrak{c}_j^k}^s)_0, \tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{n/2}}, \tilde{m}_\gamma \right\rangle & j < \frac{n}{2}, k = 0 \\ \left\langle (\tilde{H}_{\mathfrak{c}_j^k}^s)_0, \tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{n/2}} \right\rangle & j < \frac{n}{2}, k > 0 \\ \left\langle (\tilde{H}_{\mathfrak{c}_j^k}^s)_0, \tilde{m}_\gamma \right\rangle & j = \frac{n}{2}, k = 0 \\ (\tilde{H}_{\mathfrak{c}_j^k}^s)_0 & j = \frac{n}{2}, k > 0 \end{cases}$$

In particular, $\tilde{z} \in Z(\tilde{H}_{\mathfrak{c}_j^k}^s)$.

Proof. Since \mathfrak{c}_j^k is nonempty, $-1 \in (\tilde{H}_{\mathfrak{c}_j^k}^s)_0$. The first result follows from Proposition 7.3.5 and Proposition 8.3.1. The details are left to the reader. The last statement follows by Proposition 8.1.5. □

Corollary 12.4.5. *In the setting of Proposition 12.4.4, let \mathfrak{c}_j^k be a standard sequence with $j < \frac{n}{2}$. Suppose the genuine triples $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma}_1)_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma}_2)_\lambda$ are not \tilde{K} -conjugate and have the same central character. Then $\tilde{\Gamma}_2(\tilde{z}) = -\tilde{\Gamma}_1(\tilde{z})$.*

Proof. We have

$$\tilde{\Gamma}_1(\tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j}) = \tilde{\Gamma}_2(\tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j})$$

since $\tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j} \in (\tilde{H}_{\mathfrak{c}_j^k}^s)_0$ by Proposition 8.1.5. However, any genuine character extending $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda$ is determined by its restriction to $Z(\tilde{H}_{\mathfrak{c}_j^k}^s)$. Since $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma}_1)_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma}_2)_\lambda$ have the same central characters we must have

$$\tilde{\Gamma}_1(\tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{n/2}}) \neq \tilde{\Gamma}_2(\tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{n/2}})$$

and the result follows. \square

Proposition 12.4.6. *Fix a root $\delta \in \Delta(\mathfrak{g}, \mathfrak{h}^s)$ and suppose \mathfrak{c}_j^k is a standard sequence. Write $\bar{\delta} = \mathcal{C}_j^k(\delta)$ for the image of δ in $\Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^s)$. Then*

$$\bar{\delta}(\tilde{z}) = \begin{cases} 1 & \bar{\delta} \text{ is long} \\ -1 & \bar{\delta} \text{ is short} \end{cases}$$

(compare with Proposition 12.3.4).

Proof. Proposition 12.3.4 implies we need only check $\bar{\delta}(\tilde{m}_{\alpha_i}) = \delta(\tilde{m}_{\alpha_i})$ for $1 \leq i \leq \frac{n}{2}$. Recall the value of $\bar{\delta}(\tilde{m}_{\alpha_i})$ is defined by the equation

$$\text{Ad}(\tilde{m}_{\alpha_i})X_{\bar{\delta}} = \bar{\delta}(\tilde{m}_{\alpha_i})X_{\bar{\delta}}.$$

Lemmas 11.2.6 and 11.2.7 give

$$\begin{aligned} \text{Ad}(\tilde{m}_{\alpha_i})X_{\bar{\delta}} &= \text{Ad}(\tilde{m}_{\alpha_i})\mathcal{C}_j^k(X_\delta) \\ &= \mathcal{C}_j^k(\text{Ad}(\tilde{m}_{\alpha_i})X_\delta) \\ &= \mathcal{C}_j^k(\delta(\tilde{m}_{\alpha_i})X_\delta) \\ &= \delta(\tilde{m}_{\alpha_i})\mathcal{C}_j^k(X_\delta) \\ &= \delta(\tilde{m}_{\alpha_i})X_{\bar{\delta}} \end{aligned}$$

so that $\bar{\delta}(\tilde{m}_{\alpha_i}) = \delta(\tilde{m}_{\alpha_i})$ as desired. \square

Corollary 12.4.7. *Let $(\tilde{H}_{\mathfrak{c}_j^s}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})(\phi)$ is an imaginary integral root. If $m = (\phi, \alpha^\vee)$, then*

$$s_\alpha \times \tilde{\Gamma}(\tilde{z}) = \begin{cases} \tilde{\Gamma}(\tilde{z}) & \alpha \text{ long} \\ (-1)^{m+1} \tilde{\Gamma}(\tilde{z}) & \alpha \text{ short} \end{cases}$$

(see Remark 12.1.8 and compare with Proposition 12.1.9).

Proof. In the notation of Definition 12.1.7, Lemma 12.1.3 implies

$$\varphi = \begin{cases} -m\alpha + \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})}} n_\beta \beta + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})} 2 \cdot m_\beta \beta & \alpha \text{ long} \\ -m\alpha - \alpha + \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})}} n_\beta \beta + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})} 2 \cdot m_\beta \beta & \alpha \text{ short} \end{cases}$$

and the result follows by Proposition 12.4.6. \square

Corollary 12.4.8. *Let $(\tilde{H}_{\mathfrak{c}_j^s}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $\alpha \in \Delta_{\mathbb{C}}^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})(\phi)$ is a complex integral root. If $m = (\phi, \alpha^\vee)$, then in the notation of Proposition 12.1.4*

$$s_\alpha \times \tilde{\Gamma}(\tilde{z}) = \begin{cases} (-1)^\epsilon \tilde{\Gamma}(\tilde{z}) & \alpha \text{ long} \\ (-1)^m \tilde{\Gamma}(\tilde{z}) & \alpha \text{ short} \end{cases}.$$

Proof. In the notation of Definition 12.1.7 and Lemma 12.1.4 we have

$$\varphi = \begin{cases} -m\alpha - \epsilon\gamma + \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})}} n_\beta \beta + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})} 2 \cdot m_\beta \beta & \alpha \text{ long} \\ -m\alpha + \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})}} n_\beta \beta + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})} 2 \cdot m_\beta \beta & \alpha \text{ short} \end{cases}$$

and the result follows by Proposition 12.4.6. \square

We will occasionally need the following generalization of Corollary 12.4.7.

Corollary 12.4.9. *Let $\tilde{H}_{\mathfrak{c}_j^s} \subset \tilde{G}$ be an even parity Cartan subgroup with $j = \frac{n}{2}$ and let $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s})$ be an imaginary root (not necessarily integral). Suppose there exist genuine triples of the form $(\tilde{H}_{\mathfrak{c}_j^s}, \phi, \tilde{\Gamma}_1)_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^s}, s_\alpha \cdot \phi, \tilde{\Gamma}_2)_\lambda$. If $m = (\phi, \alpha^\vee)$ then*

$$\tilde{\Gamma}_2(\tilde{z}) = \begin{cases} (-1)^{2m} \tilde{\Gamma}_1(\tilde{z}) & \alpha \text{ long} \\ (-1)^{m+1} \tilde{\Gamma}_1(\tilde{z}) & \alpha \text{ short} \end{cases}.$$

Proof. Since \tilde{z} is an element of $(\tilde{H}_{\mathfrak{c}_j^s})_0$ (Proposition 12.4.4), it suffices to consider differentials. Write $\bar{\alpha}_i = \mathcal{C}_j^k(\alpha_i)$ for the image of the root α_i in $\Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s}^s)$ (Section 12.3) and let $h_{\bar{\alpha}_i}$ denote the corresponding coroot. Suppose first that α is long. Lemma 12.1.3 gives

$$\begin{aligned} d\tilde{\Gamma}_2 &= s_\alpha(\phi) + \rho_i^{s_\alpha(\phi)} - 2\rho_{\text{ic}}^{s_\alpha(\phi)} \\ &= \phi - m\alpha + \rho_1^\phi - \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s}^s)}} n_\beta \beta - (2\rho_{\text{ic}}^\phi - \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s}^s)} 2 \cdot m_\beta \beta) \\ &= d\tilde{\Gamma}_1 - m\alpha - \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s}^s)}} n_\beta \beta + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^s}^s)} 2 \cdot m_\beta \beta \end{aligned}$$

and Proposition 12.4.6 implies nontrivial changes come only from the $-m\alpha$ term. From the definition of $\tilde{m}_{\bar{\alpha}_i}$ (Definition 11.1.10) we have

$$\begin{aligned} \tilde{\Gamma}_2(\tilde{z}) &= \tilde{\Gamma}_1(\tilde{z}) e^{-m\alpha(\pi i h_{\bar{\alpha}_1})} e^{-m\alpha(\pi i h_{\bar{\alpha}_2})} \dots e^{-m\alpha(\pi i h_{\bar{\alpha}_k})} \\ &= \tilde{\Gamma}_1(\tilde{z}) e^{-m\pi i(\alpha, \bar{\alpha}_1^\vee)} e^{-m\pi i(\alpha, \bar{\alpha}_2^\vee)} \dots e^{-m\pi i(\alpha, \bar{\alpha}_k^\vee)} \end{aligned}$$

with $k = \frac{n}{2}$. Now if $\alpha = \bar{\alpha}_i$, then $(\alpha, \bar{\alpha}_i^\vee) = 2$ and $(\alpha, \bar{\alpha}_j^\vee) = 0$ for $j \neq i$. Therefore

$$\tilde{\Gamma}_2(\tilde{z}) = \tilde{\Gamma}_1(\tilde{z}) e^{-2m\pi i}$$

and the result follows. Otherwise there are exactly two numbers, say i and j , for which $(\alpha, \bar{\alpha}_i^\vee) = (\alpha, \bar{\alpha}_j^\vee) = -1$ so that

$$\begin{aligned} \tilde{\Gamma}_2(\tilde{z}) &= \tilde{\Gamma}_1(\tilde{z}) e^{m\pi i} e^{m\pi i} \\ &= \tilde{\Gamma}_1(\tilde{z}) e^{2m\pi i} \end{aligned}$$

and the result follows. The case for α short is similar. \square

Corollary 12.4.10. *Let $v \in \tilde{\mathcal{D}}_\lambda$. If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)(\lambda)$ is a real integral root for v , then $s_\alpha \times v = v$ if and only if α does not satisfy the parity condition.*

Proof. The proof is the same as for Corollary 12.3.6 using Proposition 12.4.6. \square

12.5 Cayley Transforms in $\tilde{\mathcal{D}}_\lambda$

In Section 12.1, we extended the cross action of the (integral) abstract Weyl group to $\tilde{\mathcal{D}}_\lambda$. This operation produced new elements in $\tilde{\mathcal{D}}_\lambda$ whose corresponding Cartan subgroups were conjugate. In this section we extend the Cayley transform operation of Section 11.2

to $\tilde{\mathcal{D}}_\lambda$. Not surprisingly, this operation produces new elements whose corresponding Cartan subgroups are not conjugate. This material is well known and we refer the reader to [14] and [16] for more details.

To begin, let $\tilde{H} = \tilde{T}\tilde{A}$ be a Θ -stable Cartan subgroup of \tilde{G} and fix a noncompact imaginary root $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$. Choose a corresponding Cayley transform operator \mathcal{C}^α (Section 11.2) and write

$$\begin{aligned} \mathcal{C}^\alpha(\tilde{H}) &= \tilde{H}^\alpha = \tilde{T}^\alpha \tilde{A}^\alpha \\ \tilde{T}_1^\alpha &= \tilde{T} \cap \tilde{T}^\alpha \\ \tilde{H}_1^\alpha &= \tilde{T}_1^\alpha \tilde{A}^\alpha. \end{aligned}$$

The following proposition describes the relationship between \tilde{T} and \tilde{T}^α .

Proposition 12.5.1 ([16], Lemma 8.3.5). *If α is of type I (Section 9.2) then $\tilde{T}_1^\alpha = \tilde{T}^\alpha$. If α is of type II, s_α has a representative in $\tilde{T}^\alpha \setminus \tilde{T}_1^\alpha$ and*

$$\left| \tilde{T}^\alpha / \tilde{T}_1^\alpha \right| = 2.$$

In particular, $\tilde{H}^\alpha = \tilde{H}_1^\alpha$ if and only if α is of type I.

Definition 12.5.2 ([14], before Theorem 4.4). Fix a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ and write $\phi^\alpha = \mathcal{C}^\alpha(\phi)$ for the image of ϕ in \mathfrak{h}^α . Let $\tilde{\Gamma}_1^\alpha$ be the irreducible representation of \tilde{H}_1^α satisfying

$$\begin{aligned} \tilde{\Gamma}_1^\alpha|_{\tilde{T}_1^\alpha} &= \tilde{\Gamma}|_{\tilde{T}_1^\alpha} \\ \tilde{\Gamma}_1^\alpha|_{\tilde{A}^\alpha} &= \exp_{\tilde{G}}(\phi^\alpha|_{\mathfrak{a}_{\mathbb{R}}^\alpha}). \end{aligned}$$

According to Proposition 12.5.1, we define an irreducible representation of \tilde{H}^α via

$$\tilde{\Gamma}^\alpha = \begin{cases} \tilde{\Gamma}_1^\alpha & \alpha \text{ type I} \\ \text{Ind}_{\tilde{H}_1^\alpha}^{\tilde{H}^\alpha} \tilde{\Gamma}_1^\alpha & \alpha \text{ type II} \end{cases}.$$

Proposition 12.5.3 ([14], before Theorem 4.4). *In the setting of Definition 12.5.2, suppose α is of type II. Then $\tilde{\Gamma}^\alpha = \text{Ind}_{\tilde{H}_1^\alpha}^{\tilde{H}^\alpha} \tilde{\Gamma}_1^\alpha$ is reducible if and only if the element $s_\alpha \in \tilde{T}^\alpha$ centralizes \tilde{T}_1^α . In this case we write*

$$\tilde{\Gamma}^\alpha = \tilde{\Gamma}_+^\alpha \oplus \tilde{\Gamma}_-^\alpha$$

with $\tilde{\Gamma}_\pm^\alpha$ irreducible. In particular, $\tilde{\Gamma}^\alpha$ is always reducible if \tilde{T}^α is abelian.

We now define the Cayley transform of a genuine triple through a simple noncompact imaginary root.

Definition 12.5.4 ([14]). Let $\Upsilon = (\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ and suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is noncompact and simple for ϕ . Define the *Cayley transform of Υ by α* to be

$$\mathcal{C}^\alpha(\Upsilon) = \begin{cases} \{(\tilde{H}^\alpha, \phi^\alpha, \tilde{\Gamma}^\alpha)_\lambda\} & \tilde{\Gamma}^\alpha \text{ irreducible} \\ \{(\tilde{H}^\alpha, \phi^\alpha, \tilde{\Gamma}_+^\alpha)_\lambda, (\tilde{H}^\alpha, \phi^\alpha, \tilde{\Gamma}_-^\alpha)_\lambda\} & \text{otherwise} \end{cases}.$$

In particular, $\mathcal{C}^\alpha(\Upsilon)$ is double valued if and only if α is of type II and $\tilde{\Gamma}^\alpha$ is reducible (Definition 12.5.2). If $\beta \in \Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ is an abstract noncompact simple root for Υ , we define the *abstract Cayley transform of Υ by β* to be $\mathcal{C}^{i_\phi(\beta)}(\Upsilon)$.

Proposition 12.5.5 ([13], Lemma 6.14(g)). *In the setting of Definition 12.5.4, suppose α is long. Then $\mathcal{C}^\alpha(\Upsilon)$ is single valued and we have*

$$\dim(\tilde{\Gamma}^\alpha) = m \cdot \dim(\tilde{\Gamma})$$

where

$$m = \begin{cases} 1 & \alpha \text{ type I} \\ 2 & \alpha \text{ type II} \end{cases}.$$

Proof. The fact that $\mathcal{C}^\alpha(\Upsilon)$ is single valued is proven in [13]. The statement about dimensions is an easy consequence of Proposition 12.5.1 and Definition 12.5.2. \square

The following proposition implies the Cayley transform is well-behaved and descends to the level of $\tilde{\mathcal{D}}_\lambda$.

Proposition 12.5.6 ([14]). *The elements appearing in the definition of $\mathcal{C}^\alpha(\Upsilon)$ are genuine triples for \tilde{G} . Moreover they have the same infinitesimal character as Υ and are well defined up to \tilde{K} -conjugacy.*

Remark 12.5.7. For $v \in \tilde{\mathcal{D}}_\lambda$, denote the Cayley transform on the level of genuine parameters for \tilde{G} by $\mathcal{C}^\alpha(v) \in \tilde{\mathcal{D}}_\lambda$. If $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple representing v , we will write

$$\dim(v) = \dim(\tilde{\Gamma}).$$

Proposition 12.5.8 ([13], Proposition 6.12). *Let $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $\alpha \in \Delta_i^\theta(\mathfrak{g}, \mathfrak{h}^a)(\lambda)$ is an abstract integral noncompact root that is of type II for v . Then $s_\alpha \times v = v$ if and only*

$$\dim(\mathcal{C}^\alpha(v)) = \dim(v).$$

Corollary 12.5.9. *In the setting of Proposition 12.5.8, suppose v has a representative genuine triple of the form $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma})_\lambda$. Then $s_\alpha \times v \neq v$.*

Proof. The existence of α implies we must have $j < \frac{n}{2}$ and $k > 0$ (Theorem 9.2.9). Then Proposition 7.1.3, Proposition 12.4.4, and Cases I and IV of Section 4.2 imply

$$\dim(v) = \left(\frac{2^{n_r^\theta - 1}}{2} \right)^{\frac{1}{2}} = \left(2^{n_r^\theta - 2} \right)^{\frac{1}{2}}.$$

Similarly we have

$$\begin{aligned} \dim(\mathcal{C}^\alpha(v)) &= \left(2^{n_r^\theta} \right)^{\frac{1}{2}} \\ \frac{\dim(\mathcal{C}^\alpha(v))}{\dim(v)} &= \left(\frac{2^{n_r^\theta}}{2^{n_r^\theta - 2}} \right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2 \end{aligned}$$

and the result follows from Proposition 12.5.8. \square

We will also need an inverse version of Definition 12.5.4. This is most easily stated in terms of abstract roots. We refer the reader to [13] or [14] for more details.

Definition 12.5.10. Let $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $\alpha \in \Delta_{\mathbb{R}}^\theta(\mathfrak{g}, \mathfrak{h}^a)$ is a simple abstract root that is real for v . Define the *inverse Cayley transform of v by α* to be

$$\mathcal{C}_\alpha(v) = \left\{ v' \in \tilde{\mathcal{D}}_\lambda \mid \mathcal{C}^\alpha(v') = v \right\}.$$

More explicit definitions of inverse Cayley transforms appear in [14] and [16]. The following proposition gives an important characterization of when the inverse Cayley transform is nontrivial.

Proposition 12.5.11 ([13]). *In the setting of Definition 12.5.10, $\mathcal{C}_\alpha(v)$ is nonempty if and only if α satisfies the parity condition (Definition 11.3.2).*

12.6 The Map \wp

Fix a symmetric infinitesimal character $\lambda \in (\mathfrak{h}^a)^*$ and recall the rank of \tilde{G} is even. Let \mathfrak{c}_j^k be a nonempty standard sequence and recall the element $\tilde{z} = \tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_{n/2}} \in \tilde{H}_{\mathfrak{c}_j^k}^s$ (Proposition 12.4.3). Suppose $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda$ is a genuine pair for which the corresponding abstract triple $(\theta, \varepsilon, \lambda)$ is supportable (Section 6.3). In particular, $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$. In this section we use the element \tilde{z} , along with central character (Section 11.4), to completely distinguish the elements in $\tilde{\mathcal{D}}_\lambda^\theta$.

Definition 12.6.1. Given an abstract involution $\theta \in \mathcal{I}$, let n_{cr}^θ denote *half the number of abstract coordinates that are both interchanged and negated by θ* . In terms of diagrams, n_{cr}^θ is exactly half the number of parentheses appearing in D_θ (Section 3.2).

Definition 12.6.2. The *principal series map*

$$\wp : \tilde{\mathcal{D}}_\lambda^\theta \rightarrow \mathfrak{ps}_\lambda$$

is defined as follows. For $v \in \tilde{\mathcal{D}}_\lambda^\theta$, choose a genuine triple $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma}_1)_\lambda$ beginning with $\tilde{H}_{\mathfrak{c}_j^k}^s$ and representing v . Write $\tilde{\Gamma}_1^z$ for the unique genuine character of $Z(\tilde{H}_{\mathfrak{c}_j^k}^s)$ corresponding to $\tilde{\Gamma}_1$ (Proposition 7.1.3). Then there is a unique genuine character $\tilde{\Gamma}_2^z$ of $Z(\tilde{H}^s)$ with differential $(\mathcal{C}_j^k)^{-1}(\phi)$ and

$$\begin{aligned} \tilde{\Gamma}_2^z(\tilde{m}_\gamma) &= \tilde{\Gamma}_1^z(\tilde{m}_\gamma) \\ \tilde{\Gamma}_2^z(\tilde{z}) &= \tilde{\Gamma}_1^z(\tilde{z})\tilde{\Gamma}_1^z(\tilde{m}_\gamma)^{n_{cr}^\theta}. \end{aligned}$$

Here \tilde{m}_γ denotes the nontrivial central element of \tilde{G} corresponding to the short roots in $\Delta(\mathfrak{g}, \mathfrak{h}^s)$ (Proposition 11.1.6). Let $\tilde{\Gamma}_2$ be a genuine representation of \tilde{H}^s corresponding to $\tilde{\Gamma}_2^z$ and define

$$\begin{aligned} \wp(\tilde{\Gamma}_1) &= \tilde{\Gamma}_2 \\ \wp(\Upsilon) &= (\tilde{H}^s, (\mathcal{C}_j^k)^{-1}(\phi), \tilde{\Gamma}_2)_\lambda \\ \wp(v) &= [(\tilde{H}^s, (\mathcal{C}_j^k)^{-1}(\phi), \tilde{\Gamma}_2)_\lambda] \in \mathfrak{ps}_\lambda. \end{aligned}$$

In other words, $\wp(v)$ is defined to be the principal series represented by the genuine triple $(\tilde{H}^s, (\mathcal{C}_j^k)^{-1}(\phi), \tilde{\Gamma}_2)_\lambda$. In particular, v and $\wp(v)$ have the same central characters (Remark 11.4.3). The strange definition of $\tilde{\Gamma}_2^z(\tilde{z})$ gives the map \wp nicer properties (see Theorem 12.7.3).

Proposition 12.6.3. *The map \wp is well defined.*

Proof. The issue is the choice of genuine triple $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma}_1)_\lambda$ representing v . It suffices to show $\wp(\Upsilon)$ is \tilde{K} -conjugate to $\wp(w \cdot \Upsilon)$ with

$$w \cdot \Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^s, w \cdot \phi, w \cdot \tilde{\Gamma}_1)_\lambda$$

and $w \in N_{\tilde{K}}(\tilde{H}_{\mathfrak{c}_j^k}^s)/Z_{\tilde{K}}(\tilde{H}_{\mathfrak{c}_j^k}^s) = W(\tilde{G}, \tilde{H}_{\mathfrak{c}_j^k}^s)$ (Section 5.2). Note $w \cdot \tilde{\Gamma}_1$ is defined only up to \tilde{K} -conjugacy, however the corresponding character of $Z(\tilde{H}_{\mathfrak{c}_j^k}^s)$ is well defined. Since it is

the character that matters for Definition 12.6.2, this level of precision is sufficient for our purposes.

We begin by observing the differentials of $\wp(\Upsilon)$ and $\wp(w \cdot \Upsilon)$ will always be conjugate in $(\mathfrak{h}^s)^*$ by an element of \tilde{K} . Moreover, any genuine representation of \tilde{H}^s with a fixed differential is determined by its values on \tilde{m}_γ and \tilde{z} . Since conjugation cannot change \tilde{m}_γ , we simply need to understand the effect of conjugation on \tilde{z} . This is done for principal series in Proposition 12.3.2. Therefore it remains to understand the effect of conjugation in $\tilde{H}_{\mathfrak{c}_j^k}^s$. Conjugating by w does not change the value of n_{cr}^θ , so we may effectively ignore the $\tilde{\Gamma}_1^z(\tilde{m}_\gamma)^{n_{cr}^\theta}$ term in the definition of $\tilde{\Gamma}_2^z(\tilde{z})$.

Proposition 5.2.3 implies

$$W(\tilde{G}, \tilde{H}_{\mathfrak{c}_j^k}^s) \cong W_{\mathbb{C}}^\theta \times ((A \times W_{ic}^\theta) \times W_{\mathbb{R}}^\theta).$$

Our proof is by cases based on the element w .

Case I. Suppose $w \in W_{\mathbb{R}}^\theta$. Then conjugation commutes with $(\mathcal{C}_j^k)^{-1}$ and $\wp(\Upsilon)$ is clearly conjugate to $\wp(w \cdot \Upsilon)$.

Case II. Suppose $w \in W_{ic}^\theta$. We may assume $w = s_\alpha$ for $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^s)(\phi)$ imaginary and compact. Write $\bar{\alpha} = (\mathcal{C}_j^k)^{-1}(\alpha)$ for the image of α in $\Delta(\mathfrak{g}, \mathfrak{h}^s)$. If α is long, Proposition 12.3.2 implies $(s_{\bar{\alpha}} \cdot \wp(\tilde{\Gamma}_1))(\tilde{z}) = \wp(\tilde{\Gamma}_1)(\tilde{z})$. In particular, $\wp(\Upsilon)$ is conjugate to $\wp(s_\alpha \cdot \Upsilon)$ if and only if $\tilde{\Gamma}_1(\tilde{z}) = s_\alpha \cdot \tilde{\Gamma}_1(\tilde{z})$ and this follows from Proposition 12.1.15 and Corollary 12.4.7.

If α is short, set $m = (\phi, \alpha^\vee) \in \mathbb{Z}$. Proposition 12.3.2 implies

$$(s_{\bar{\alpha}} \cdot \wp(\tilde{\Gamma}_1))(\tilde{z}) = \wp(\tilde{\Gamma}_1)(\tilde{m}_{\bar{\alpha}}) \wp(\tilde{\Gamma}_1)(\tilde{z})$$

and Corollary 12.4.7 gives

$$s_\alpha \cdot \tilde{\Gamma}_1(\tilde{z}) = (-1)^{(m+1)} \tilde{\Gamma}_1(\tilde{z}).$$

Therefore $\wp(\Upsilon)$ is conjugate to $\wp(s_\alpha \cdot \Upsilon)$ if and only if

$$\wp(\tilde{\Gamma}_1)(\tilde{m}_{\bar{\alpha}}) = (-1)^{(m+1)} \cdot \mathbf{I}$$

or if and only if $\bar{\alpha}$ satisfies the parity condition for $\wp(\Upsilon)$. Since α is assumed to compact, this follows from Theorem 11.4.8.

Case III. Suppose $w \in A$. Then

$$w \in W_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j}^{s_k}) \cong W(B_{n_s^\Theta}) \times W(A_1)^k$$

(Proposition 3.3.6) and w can be expressed as a product of orthogonal reflections in noncompact imaginary roots ([17], Corollary 5.14).

If $j < \frac{n}{2}$, the group A is generated by reflections in noncompact imaginary roots of type II (Section 9.2). Suppose first that α is long and let $w = s_\alpha$. Then α is an element of the $(A_1)^k$ factor of $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j}^{s_k})$ and we must have $\alpha = \mathcal{C}_j^k(\alpha_i)$ for some i . In particular, there exists a Cartan subgroup $\tilde{H}_\mathfrak{c}^s = \mathcal{C}^\alpha(\tilde{H}_{\mathfrak{c}_j}^s)$ (Section 11.2) containing \tilde{z} and a representative of s_α (Proposition 12.5.1). Since \tilde{z} is central in $\tilde{H}_\mathfrak{c}^s$, \tilde{z} commutes with s_α and we have $\tilde{\Gamma}_1(\tilde{z}) = s_\alpha \cdot \tilde{\Gamma}_1(\tilde{z})$. The result now follows as in Case II.

Suppose α is short and of type II. Corollary 12.4.7 gives

$$s_\alpha \times \tilde{\Gamma}_1(\tilde{z}) = (-1)^{(m+1)} \tilde{\Gamma}_1(\tilde{z})$$

and Corollary 12.5.9 implies $s_\alpha \times v \neq v$. Therefore we must have

$$s_\alpha \cdot \tilde{\Gamma}_1(\tilde{z}) = (-1)^{(m)} \tilde{\Gamma}_1(\tilde{z})$$

by Corollary 12.4.5. In particular, $\wp(\Upsilon)$ and $\wp(s_\alpha \cdot \Upsilon)$ are conjugate if and only if $\bar{\alpha}$ does not satisfy the parity condition for $\wp(\Upsilon)$. Since α is assumed to be noncompact, this follows from Theorem 11.4.8.

Finally let $j = \frac{n}{2}$ and $w = s_\epsilon s_\delta$, where δ and ϵ are orthogonal imaginary roots in $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j}^{s_k})$ that are noncompact and of type I. Set $m_1 = (\phi, \delta^\vee)$, $m_2 = (\phi, \epsilon^\vee)$, and suppose first that δ and ϵ are long. Then $m_1, m_2 \in \mathbb{Z} + \frac{1}{2}$ and Corollary 12.4.9 implies

$$\begin{aligned} s_\delta \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1) \cdot \tilde{\Gamma}_1(\tilde{z}) \\ s_\epsilon \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1) \cdot \tilde{\Gamma}_1(\tilde{z}) \\ s_\epsilon s_\delta \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1)^2 \cdot \tilde{\Gamma}_1(\tilde{z}) = \tilde{\Gamma}_1(\tilde{z}). \end{aligned}$$

as desired. If δ is long and ϵ is short, then $m_1 \in \mathbb{Z} + \frac{1}{2}$ and $m_2 \in \mathbb{Z}$. Corollary 12.4.9 implies

$$\begin{aligned} s_\epsilon \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1)^{m_2+1} \cdot \tilde{\Gamma}_1(\tilde{z}) \\ s_\epsilon s_\delta \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1)^{m_2} \cdot \tilde{\Gamma}_1(\tilde{z}). \end{aligned}$$

This is the desired result since ϵ is noncompact. Finally if both δ and ϵ are short we have

$$\begin{aligned} s_\delta \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1)^{m_1+1} \cdot \tilde{\Gamma}_1(\tilde{z}) \\ s_\epsilon \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1)^{m_2+1} \cdot \tilde{\Gamma}_1(\tilde{z}) \\ s_\epsilon s_\delta \cdot \tilde{\Gamma}_1(\tilde{z}) &= (-1)^{m_1+m_2} \cdot \tilde{\Gamma}_1(\tilde{z}) \\ &= \tilde{\Gamma}_1(\tilde{z}) \end{aligned}$$

since $m_1 + m_2 \in 2\mathbb{Z}$. The result follows from Proposition 12.3.4.

Case IV. Suppose $w \in W_{\mathbb{C}}^\theta$. This case is handled in the same fashion as the previous cases. The reader is spared the details. \square

Theorem 12.6.4. *In the setting of Definition 12.6.2, the map*

$$\wp : \tilde{\mathcal{D}}_\lambda^\theta \rightarrow ps_\lambda$$

is a bijection.

Proof. Proposition 11.5.1 implies \wp is, at worst, 2 to 1. In particular, if $\omega, v \in \tilde{\mathcal{D}}_\lambda^\theta$ have the same central character, it remains to show $\wp(\omega) \neq \wp(v)$. Recall θ is a supportable abstract involution for the genuine pair $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda$ and let $\epsilon_s^\theta, \epsilon_r^\theta, \epsilon_p^\theta = \epsilon_m^\theta = 0$ be the corresponding indicator bits (Definition 3.1.1). The proof is by cases for the standard sequence \mathfrak{c}_j^k .

Case I. Suppose $j < \frac{n}{2}$ and $k = 0$ so that $\epsilon_s^\theta = 0$ and $\epsilon_r^\theta = 1$. Corollary 8.4.5 implies

$$\left[(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda \right] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} = 4.$$

Therefore each element of $\tilde{\mathcal{D}}_\lambda^\theta$ has a representative beginning with $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda$ and the result follows from Corollary 12.4.5.

Case II. Suppose $j < \frac{n}{2}$ and $k > 0$ so that $\epsilon_s^\theta = \epsilon_r^\theta = 1$. Corollary 8.4.5 implies

$$\left[(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda \right] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} = 2$$

and there are two \tilde{K} -orbits in the genuine fiber $\tilde{\theta}_\lambda^\dagger$ of θ (Section 9.4). Since $\tilde{m}_\gamma \in (\tilde{H}_{\mathfrak{c}_j^k}^s)_0$ by Proposition 12.4.4, both genuine triples extending $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda$ have the same central character and the result follows by Corollary 12.4.5.

Case III. Suppose $j = \frac{n}{2}$ and $k > 0$ so that $\epsilon_s^\theta = 1$ and $\epsilon_r^\theta = 0$. Corollary 8.4.5 implies

$$\left[(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda \right] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} = 1.$$

Then $\tilde{m}_\gamma \in (\tilde{H}_{\mathfrak{c}_j^k}^s)_0 = \tilde{H}_{\mathfrak{c}_j^k}^s$ by Proposition 12.4.4 and there are two \tilde{K} -orbits in $\tilde{\theta}_\lambda^\dagger$ with a fixed abstract grading and central character (Proposition 11.4.6). Theorem 9.3.1 implies these orbits differ by cross action in any short noncompact root. Let $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple extending $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda$ and let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)$ be a short noncompact root. Set $m = (\lambda, \alpha^\vee)$ and write $\bar{\alpha} = i_\phi(\alpha)$ for the image of α in $\Delta(\mathfrak{g}, \mathfrak{h}^s)$. Corollary 12.4.7 and Proposition 12.3.2 imply

$$\begin{aligned} s_\alpha \times \tilde{\Gamma}(\tilde{z}) &= (-1)^{m+1} \tilde{\Gamma}(\tilde{z}) \\ s_{\bar{\alpha}} \cdot \wp(\tilde{\Gamma})(\tilde{z}) &= \wp(\tilde{\Gamma})(\tilde{m}_{\bar{\alpha}}) \wp(\tilde{\Gamma})(\tilde{z}). \end{aligned}$$

In particular, $\wp(\Upsilon) \neq \wp(s_\alpha \times \Upsilon)$ if and only if α does not satisfy the parity condition (Definition 11.3.2) for $\wp(\Upsilon)$. Since α is noncompact, this follows by Theorem 11.4.8.

Case IV. Suppose $j = \frac{n}{2}$ and $k = 0$ so that $\epsilon_s^\theta = \epsilon_r^\theta = 0$. Corollary 8.4.5 implies

$$\left[(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda \right] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} = 2.$$

Then $\tilde{z} \in (\tilde{H}_{\mathfrak{c}_j^k}^s)_0$, $\tilde{m}_\gamma \notin (\tilde{H}_{\mathfrak{c}_j^k}^s)_0$ (Proposition 12.4.4) and the \tilde{K} -orbits in $\tilde{\theta}_\lambda^\dagger$ each have two genuine triples with opposite central characters extending them. Let $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple extending $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi)_\lambda$ and suppose α is a long imaginary root in $\Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^s)$. Then α is noncompact of type I (Theorem 9.2.9) and thus $(\tilde{H}_{\mathfrak{c}_j^k}^s, \phi, \tilde{\Gamma})_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^k}^s, s_\alpha \cdot \phi, s_\alpha \cdot \tilde{\Gamma})_\lambda$ are *not* conjugate in \tilde{K} . If $m = (\phi, \alpha^\vee)$, then $m \in \mathbb{Z} + \frac{1}{2}$ and Corollary 12.4.9 implies

$$\begin{aligned} s_\alpha \cdot \tilde{\Gamma}(\tilde{z}) &= (-1)^{2m} \cdot \tilde{\Gamma}(\tilde{z}) \\ &= (-1) \cdot \tilde{\Gamma}(\tilde{z}). \end{aligned}$$

Since α is long, $\wp(\Upsilon)$ and $\wp(s_\alpha \cdot \Upsilon)$ are not conjugate in \tilde{K} by Proposition 12.3.2. \square

In particular, the map \wp associates a unique principal series to each element in $\tilde{\mathcal{D}}_\lambda^\theta$ (whenever $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$). This completes the process of distinguishing the elements of $\tilde{\mathcal{D}}_\lambda^\theta$ for even parity Cartan subgroups and will play a critical role in the next section.

12.7 Extension of Ψ

Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character and recall the rank of \tilde{G} is even. We are finally in a position to extend the bit flip involution (Definition 3.2.4)

$$\Psi : \mathcal{I} \rightarrow \mathcal{I}$$

to the level of genuine parameters for \tilde{G} . The extension requires much of the structure theory from the previous sections including central characters (Definition 11.4.1) and the map \wp (Definition 12.6.2).

Fix a split Cartan subgroup $\tilde{H}^s \subset \tilde{G}$ and recall ps_λ denotes the set of principal series for λ (Section 12.3). Choose an involution

$$\tau : \text{ps}_\lambda \rightarrow \text{ps}_\lambda$$

that *switches central character*. In other words, for $v \in \text{ps}_\lambda$ we require $\tau(v)$ and v to have opposite central characters. Since $|\text{ps}_\lambda| = 4$, Proposition 11.5.1 implies there are exactly two choices for τ .

Definition 12.7.1. Let θ be a supportable abstract involution for λ (Section 6.3) and let $v \in \tilde{\mathcal{D}}_\lambda^\theta$. If $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$, $\Psi(v)$ is defined to be the unique element in $\tilde{\mathcal{D}}_\lambda^{-\theta}$ (Theorem 10.1.5). If $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$, define

$$\Psi(v) = \omega \in \tilde{\mathcal{D}}_\lambda^{-\theta}$$

where ω is the unique element in $\tilde{\mathcal{D}}_\lambda^{-\theta}$ with $\tau(\wp(v)) = \wp(\omega)$.

The construction in Definition 12.7.1 immediately leads to the following extension of Proposition 11.5.2.

Proposition 12.7.2. *Let $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $(\theta, \varepsilon, \eta, \lambda)$ is the corresponding abstract bigrading (Section 11.3). Then $\Psi(v)$ has abstract bigrading $(-\theta, \eta, \varepsilon, \lambda)$.*

Proof. If $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$, this was proven in Proposition 11.5.2. If $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$, we observe v and $\Psi(v)$ have opposite central characters and the result follows from Theorem 11.4.8. \square

We now prove two important properties of the map Ψ .

Theorem 12.7.3. *Let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)(\lambda)$ be an abstract integral root. If $v \in \tilde{\mathcal{D}}_\lambda$, then*

$$\Psi(s_\alpha \times v) = s_\alpha \times \Psi(v).$$

Proof. Let $(\theta, \varepsilon, \eta, \lambda)$ be the abstract bigrading for v . We have

$$-(s_\alpha \theta s_\alpha^{-1}) = s_\alpha (-\theta) s_\alpha^{-1}$$

and the result holds on the level of involutions (see Proposition 9.1.6). If $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$, the elements v and $\Psi(v)$ are determined by their involutions and we are done.

If $|\tilde{\mathcal{D}}_\lambda^\theta| = 4$, we need to show $\tau(\wp(s_\alpha \times v)) = \wp(s_\alpha \times \Psi(v))$. Since $\tau(\wp(v)) = \wp(\Psi(v))$ by definition, it suffices to show

$$\wp(s_\alpha \times v) = \wp(v) \iff \wp(s_\alpha \times \Psi(v)) = \wp(\Psi(v)).$$

The proof is by cases (for a change) based on the type of α for v .

Case I. Suppose α is imaginary and compact for v . Then α is real for $\Psi(v)$ and does not satisfy the parity condition (Proposition 12.7.2). Therefore,

$$\begin{aligned} s_\alpha \times v &= v \\ s_\alpha \times \Psi(v) &= \Psi(v) \end{aligned}$$

by Propositions 12.1.15 and 12.3.7 and we have

$$\begin{aligned} \wp(s_\alpha \times v) &= \wp(v) \\ \wp(s_\alpha \times \Psi(v)) &= \wp(\Psi(v)) \end{aligned}$$

by Proposition 12.6.3.

Case II. Suppose α is imaginary and noncompact for v . Proposition 6.2.2 implies α is short and Proposition 12.7.2 implies α is real for $\Psi(v)$ and satisfies the parity condition. Then

$$\begin{aligned} s_\alpha \times v &\neq v \\ s_\alpha \times \Psi(v) &\neq \Psi(v) \end{aligned}$$

by Corollaries 12.5.9 and 12.4.10. Therefore

$$\begin{aligned} \wp(s_\alpha \times v) &\neq \wp(v) \\ \wp(s_\alpha \times \Psi(v)) &\neq \wp(\Psi(v)) \end{aligned}$$

by Theorem 12.6.4.

Case III. Suppose α is short and complex for v . Choose a genuine triple $(\tilde{H}_{\mathfrak{c}_j}^s, \phi, \tilde{\Gamma})_\lambda$ representing v and recall the element n_{cr}^θ from Definition 12.6.1. It is easy to check

$$n_{cr}^{s_\alpha \times \theta} = n_{cr}^\theta \pm 1.$$

If $m = (\lambda, \alpha^\vee)$, then Proposition 12.4.8 implies

$$\begin{aligned} \wp(s_\alpha \times v) = \wp(v) &\iff \\ (-1)^m \tilde{\Gamma}(\tilde{z}) \tilde{\Gamma}(\tilde{m}_\gamma)^{n_{cr}^\theta + 1} = \tilde{\Gamma}(\tilde{z}) \tilde{\Gamma}(\tilde{m}_\gamma)^{n_{cr}^\theta} \tilde{\Gamma}(\tilde{m}_\gamma) &\iff \\ (-1)^m = 1. & \end{aligned}$$

By the same argument we also have $\wp(s_\alpha \times \Psi(v)) = \wp(\Psi(v)) \iff (-1)^m = 1$ and the result follows.

Case IV. Suppose α is long and complex for v . This is handled in the same fashion as Case III and is left to the reader. □

Theorem 12.7.4. *Let $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)$ is an abstract simple root that is imaginary and noncompact. Then α is real for $\Psi(v)$ and we have*

$$\Psi(\mathcal{C}^\alpha(v)) = \mathcal{C}_\alpha(\Psi(v)).$$

Note this is an equality of sets (Definitions 12.5.4 and 12.5.10).

Proof. Let $(\theta, \varepsilon, \eta, \lambda)$ be the abstract bigrading for v . Since $(-\theta, \eta, \varepsilon, \lambda)$ is the abstract bigrading for $\Psi(v)$ (Proposition 12.7.2), α is real for $\Psi(v)$ and satisfies the parity condition.

Case I. Suppose $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$ and α is long. Then $\mathcal{C}^\alpha(v)$ must be the unique element of $\tilde{\mathcal{D}}_\lambda^{s_\alpha \theta}$ (Proposition 8.2.1). Similarly, $\mathcal{C}_\alpha(\Psi(v))$ is the unique element of $\tilde{\mathcal{D}}_\lambda^{s_\alpha(-\theta)} = \tilde{\mathcal{D}}_\lambda^{-s_\alpha(\theta)}$ and the result follows.

Case II. Suppose $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$ and α is short. Using Theorem 9.2.9, it is easy to check α must be of type II. Since $|\tilde{\mathcal{D}}_\lambda^\theta| = 1$, we must have $s_\alpha \times v = v$ and Proposition 12.5.8 implies

$$\mathcal{C}^\alpha(v) = \{\omega_+, \omega_-\} \subset \tilde{\mathcal{D}}_\lambda^{s_\alpha(\theta)}$$

is double valued. In particular, $\Psi(\omega_+)$ and $\Psi(\omega_-)$ are exactly the elements in $\tilde{\mathcal{D}}_\lambda^{s_\alpha(-\theta)}$ with the same central character as $\Psi(v)$ (Proposition 11.5.1). Therefore it suffices to show $\mathcal{C}_\alpha(\Psi(v))$ is double valued and this follows immediately from Corollary 12.5.9.

Case III. Suppose $|\widetilde{\mathcal{D}}_\lambda^\theta| = 4$ and α is short. Then both $\mathcal{C}^\alpha(v)$ and $\mathcal{C}_\alpha(\Psi(v))$ are single valued and the result follows for the same reasons as in Case I.

Case IV. Suppose $|\widetilde{\mathcal{D}}_\lambda^\theta| = 4$ and α is long. Choose a representative $\Upsilon = (\widetilde{H}_{\mathfrak{c}_j^k}^s, \phi, \widetilde{\Gamma})_\lambda$ for v with $\widetilde{H}_{\mathfrak{c}_j^k}^s \subset \widetilde{G}$ an even parity Cartan subgroup. Proposition 12.5.5 implies $\mathcal{C}^\alpha(v)$ is single-valued and we have

$$\begin{aligned} W_i^\theta &\cong W(B_{n_\theta}) \times W(A_1)^k \\ W(G, \widetilde{H}_{\mathfrak{c}_j^k}^s) &\cong ((A \times W_{ic}^\theta) \times W_{\mathbb{R}}^\theta) \times W_{\mathbb{C}}^\theta \end{aligned}$$

from Propositions 3.3.6 and 5.2.3. Suppose first that α is an element of the A_1^k factor of Δ_i^θ . Then $j > k$ and conjugation in $W_{\mathbb{C}}^\theta$ allows us to choose the representative Υ such that

$$\bar{\alpha} = i_\phi(\alpha) = \mathcal{C}_j^k(\pm\alpha_j).$$

In particular, $\mathcal{C}^{\bar{\alpha}}(\widetilde{H}_{\mathfrak{c}_j^k}^s) = \widetilde{H}_{\mathfrak{c}_{j-1}^k}^s$ is an even parity Cartan subgroup of \widetilde{G} . Since α is long, Definition 12.5.2 directly implies

$$\wp(\Upsilon) = \wp(\mathcal{C}^{\bar{\alpha}}(\Upsilon))$$

so that

$$\wp(v) = \wp(\mathcal{C}^\alpha(v)).$$

A similar argument holds for $\Psi(v)$ and ultimately gives

$$\wp(\Psi(v)) = \wp(\mathcal{C}_\alpha(\Psi(v)))$$

and the result follows.

Now suppose α is an element of the $W(B_{n_\theta})$ factor of Δ_i^θ . Then $k > 0$ and conjugation in W_{ic}^θ allows us to choose the representative Υ such that

$$\bar{\alpha} = i_\phi(\alpha) = \mathcal{C}_j^k(\pm\alpha_k).$$

In particular, $\mathcal{C}^{\bar{\alpha}}(\widetilde{H}_{\mathfrak{c}_j^k}^s) = \widetilde{H}_{\mathfrak{c}_j^{k-1}}^s$ is an even parity Cartan subgroup of \widetilde{G} and we proceed as above.

□

We will need an extension of Theorem 12.7.3 to the full abstract Weyl group W . The precise statement requires a bit of setup. Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character and suppose $\mathcal{F}(\lambda)$ is a family for λ (Definition 12.2.1). For each $\kappa \in \mathcal{F}(\lambda)$ we define a map $\wp_\kappa : \tilde{\mathcal{D}}_\kappa \rightarrow \mathfrak{ps}_\kappa$ as in Definition 12.6.2 and let

$$\wp' = \prod_{\kappa \in \mathcal{F}(\lambda)} \wp_\kappa : \prod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_\kappa \rightarrow \prod_{\kappa \in \mathcal{F}(\lambda)} \mathfrak{ps}_\kappa.$$

Identifying \mathfrak{ps}_κ with \mathfrak{ps}_λ in the obvious way gives a map

$$\wp : \prod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_\kappa \rightarrow \mathfrak{ps}_\lambda.$$

Fix an involution $\tau : \mathfrak{ps}_\lambda \rightarrow \mathfrak{ps}_\lambda$ as above and define maps $\Psi_\kappa : \tilde{\mathcal{D}}_\kappa \rightarrow \tilde{\mathcal{D}}_\kappa$ for $\kappa \in \mathcal{F}(\lambda)$ (Definition 12.7.1). Finally set

$$\Psi = \prod_{\kappa \in \mathcal{F}(\lambda)} \Psi_\kappa : \prod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_\kappa \rightarrow \prod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_\kappa.$$

Theorem 12.7.5. *In the notation above, let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)$ be a long abstract root and suppose $s_\alpha \notin W(\lambda)$. If $\kappa \in \mathcal{F}(\lambda)$ and $v \in \tilde{\mathcal{D}}_\kappa$ then*

$$\Psi(s_\alpha \times v) = s_\alpha \times \Psi(v).$$

Proof. Let $(\theta, \varepsilon, \eta, \kappa)$ be the abstract bigrading for v . On the level of involutions, the result follows as in Theorem 12.7.3. If $|\tilde{\mathcal{D}}_\kappa^\theta| = 1$, the elements v and $\Psi(v)$ are determined by their involutions and we are done.

If $|\tilde{\mathcal{D}}_\kappa^\theta| = 4$ we again need to show

$$\wp(s_\alpha \times v) = \wp(v) \iff \wp(s_\alpha \times \Psi(v)) = \wp(\Psi(v))$$

and Proposition 6.2.2 implies it suffices to check this for complex and noncompact imaginary roots. Combined with Remark 12.2.5, the details are as in Theorem 12.7.3 and are left to the reader. \square

12.8 Character Multiplicity Duality

Let $\lambda \in (\mathfrak{h}^a)^*$ be a symmetric infinitesimal character. We begin with one final definition.

Definition 12.8.1 ([13], Definition 6.7). Let θ be an involution in \mathcal{I} and recall Δ^+ denotes the set of abstract roots that are positive for λ . The *length* of θ is defined to be

$$\ell(\theta) = \frac{1}{2} |\{\alpha \in \Delta^+ \mid \theta(\alpha) \notin \Delta^+\}| + \frac{1}{2} \dim(\theta_{-1})$$

where θ_{-1} is the negative eigenspace for θ . If $v \in \tilde{\mathcal{D}}_\lambda$ and $(\theta, \varepsilon, \lambda)$ is the corresponding abstract triple, we define $\ell(v) = \ell(\theta)$.

Proposition 12.8.2. *In the setting of Definition 12.8.1, $\ell(v) \in \mathbb{N}^+$.*

Proof. First suppose $v \in \text{ps}_\lambda$ so that $\theta = -1$ and recall $n > 0$ denotes the rank of \tilde{G} . From Section 3.1 we have

$$\begin{aligned} \ell(v) &= \frac{1}{2}n^2 + \frac{1}{2}n \\ &= \frac{n(n+1)}{2} = \binom{n+1}{2} \end{aligned}$$

and the result follows. We now proceed as in [16], Lemma 8.6.13. \square

Suppose now the rank of \tilde{G} is even and fix a map $\Psi : \tilde{\mathcal{D}}_\lambda \rightarrow \tilde{\mathcal{D}}_\lambda$ as in Definition 12.7.1.

Proposition 12.8.3. *Let $v \in \tilde{\mathcal{D}}_\lambda$ and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple for v . Then $\ell(\Psi(v)) = \binom{n+1}{2} - \ell(v)$.*

Proof. Set

$$\begin{aligned} m_1 &= |\{\alpha \in \Delta^+ \mid \theta(\alpha) \notin \Delta^+\}| \\ m_2 &= \dim(\theta_{-1}). \end{aligned}$$

Clearly we have

$$\begin{aligned} \ell(\Psi(v)) = \ell(\Psi(\theta)) &= \ell(-\theta) \\ &= \frac{n^2 - m_1}{2} + \frac{n - m_2}{2} \\ &= \frac{n(n+1)}{2} - \frac{m_1 + m_2}{2} \\ &= \binom{n+1}{2} - \ell(v) \end{aligned}$$

as desired. \square

Fix a family $\mathcal{F}(\lambda)$ of infinitesimal characters for λ and suppose $\tilde{\Gamma}$ is a genuine central character of $Z(\tilde{G})$. Let $\mathcal{B} = \{\gamma_1, \dots, \gamma_r\} \subset \tilde{\mathcal{D}}_\lambda$ be the collection of genuine parameters in $\tilde{\mathcal{D}}_\lambda$ with central character $\tilde{\Gamma}$. If $\delta_i = \Psi(\gamma_i)$, write $\mathcal{B}' = \{\delta_1, \dots, \delta_r\} = \tilde{\mathcal{D}}_\lambda \setminus \mathcal{B}$ and extend the map Ψ (and thus the sets \mathcal{B} and \mathcal{B}') as in Theorem 12.7.5. Let \mathcal{M} (respectively \mathcal{M}') denote the free $\mathbb{Z}[q, q^{-1}]$ module with basis \mathcal{B} (respectively \mathcal{B}'). As in [13], we view \mathcal{M} and \mathcal{M}' as Hecke modules for the extended action of the Hecke algebra $\mathcal{H}(W)$ ([13], Definition 9.4). The integer matrix M (Section 2.4) for \mathcal{B} (respectively \mathcal{B}') is then determined from

the combinatorics of \mathcal{M} (respectively \mathcal{M}'). The interested reader is referred to [19] for a reasonably concise account of this process.

For our purposes, only the following formalism is important. Define the dual $\mathbb{Z}[q, q^{-1}]$ module

$$\mathcal{M}^* = \text{Hom}_{\mathbb{Z}[q, q^{-1}]}(\mathcal{M}, \mathbb{Z}[q, q^{-1}])$$

and extend \mathcal{M}^* to an $\mathcal{H}(W)$ -module as in [13], Theorem 11.1. Write $\check{\mathcal{B}} = \{\check{\gamma}_1, \dots, \check{\gamma}_r\}$ for the dual basis of \mathcal{M}^* and define the $\mathbb{Z}[q, q^{-1}]$ -linear map

$$\begin{aligned} \Phi : \mathcal{M}^* &\longrightarrow \mathcal{M} \\ \check{\gamma}_i &\longmapsto (-1)^{\ell(\gamma_i)} \delta_i. \end{aligned}$$

Theorem 12.8.4. *In the above setting, Φ is an isomorphism of $\mathcal{H}(W)$ -modules.*

Proof. It suffices to check the equivariance of the operators in $\mathcal{H}(W)$ corresponding to simple roots. Depending on the root type and length, there are many cases to consider. For integral roots, the details are as in [17], Proposition 13.10. For strictly half integral roots, the details are as in [13], Theorem 11.1. In each case the result is a formal consequence of Theorems 12.7.3, 12.7.4, and 12.7.5. \square

Theorem 12.8.5. *Let λ be a symmetric infinitesimal character and suppose the rank of \tilde{G} is even. Fix a genuine central character $\tilde{\Gamma}$ of $Z(\tilde{G})$ and let $\mathcal{B} = \{\gamma_1, \dots, \gamma_r\} \subset \tilde{\mathcal{D}}_\lambda$ be the collection of genuine parameters in $\tilde{\mathcal{D}}_\lambda$ with central character $\tilde{\Gamma}$. If $\delta_i = \Psi(\gamma_i)$ and $\mathcal{B}' = \{\delta_1, \dots, \delta_r\} = \tilde{\mathcal{D}}_\lambda \setminus \mathcal{B}$, then*

$$M(\gamma_i, \overline{\gamma_j}) = (-1)^{\ell(\gamma_j) - \ell(\gamma_i)} m(\overline{\delta_j}, \delta_i) \quad (12.1)$$

(Section 2.4).

Proof. This follows immediately from Theorem 12.8.4 and Lemma 13.7 of [17]. \square

Example 12.8.6. We verify Theorem 12.8.5 in the case $n = 2$ and $\lambda = (\frac{3}{2}, 1)$. Let α (respectively β) denote the unique short (respectively long) abstract simple root in Δ^+ and recall Example 9.4.5 implies $|\tilde{\mathcal{D}}_\lambda| = 18$. Write $\mathcal{B} = \{\gamma_0, \dots, \gamma_8\} \subset \tilde{\mathcal{D}}_\lambda$ for the collection of parameters in $\tilde{\mathcal{D}}_\lambda$ with trivial genuine central character. The structure of \mathcal{B} is given by Table 12.1.

Each row in the table corresponds to the element $\gamma_i \in \mathcal{B}$ listed in the first column. The second column gives the length of γ_i and the third column gives the image of the (integral)

cross action for s_α . The final two columns give the images of the Cayley transforms (when defined) for the simple roots β and α respectively. If $\delta_i = \Psi(\gamma_i)$ then the structure of $\mathcal{B}' = \{\delta_8, \dots, \delta_0\}$ is given by Table 12.2.

Using the methods of [13] one verifies the matrix M for \mathcal{B} is given by

$$M = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 \\ & 1 & 0 & -1 & -1 & 1 & 1 & -1 & 0 \\ & & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & -1 & 0 & 0 \\ & & & & 1 & -1 & -1 & 0 & 0 \\ & & & & & 1 & 0 & -1 & 0 \\ & & & & & & 1 & 0 & -1 \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{pmatrix}$$

with respect to the ordering above. Similarly, the matrix m for \mathcal{B}' is given by

$$m = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \\ & & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \\ & & & 1 & 1 & 0 & 1 & 1 & 1 & \\ & & & & 1 & 0 & 0 & 1 & 1 & \\ & & & & & 1 & 0 & 1 & 0 & \\ & & & & & & 1 & 0 & 1 & \\ & & & & & & & 1 & 0 & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{pmatrix}$$

with respect to the opposite order. Theorem 12.8.5 implies the matrix M equals the antitranspose (i.e., reflection about the opposite diagonal) of the matrix m up to sign. The reader is invited to verify this for the above matrices.

12.1. STRUCTURE OF \mathcal{B}

\mathcal{B}	length	$s_\alpha \times \gamma_i$	$\beta : \mathcal{C}(\gamma_i)$	$\alpha : \mathcal{C}(\gamma_i)$
γ_0	0	γ_1	γ_2	γ_4
γ_1	0	γ_0	γ_3	γ_4
γ_2	1	γ_5	γ_0	*
γ_3	1	γ_6	γ_1	*
γ_4	1	γ_4	*	$\{\gamma_1, \gamma_0\}$
γ_5	2	γ_2	γ_7	*
γ_6	2	γ_3	γ_8	*
γ_7	3	γ_7	γ_5	*
γ_8	3	γ_8	γ_6	*

12.2. STRUCTURE OF \mathcal{B}'

\mathcal{B}'	length	$s_\alpha \times \delta_i$	$\beta : \mathcal{C}(\gamma_i)$	$\alpha : \mathcal{C}(\gamma_i)$
δ_8	0	δ_8	δ_6	*
δ_7	0	δ_7	δ_5	*
δ_6	1	δ_3	δ_8	*
δ_5	1	δ_2	δ_7	*
δ_4	2	δ_4	*	$\{\gamma_1, \gamma_0\}$
δ_3	2	δ_6	δ_1	*
δ_2	2	δ_5	δ_0	*
δ_1	3	δ_0	δ_3	δ_4
δ_0	3	δ_1	δ_2	δ_4

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