Weak existence of a solution to a differential equation driven by a very rough fBm*

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Abstract

We prove that if $f : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous, then for every $H \in (0\,, 1/4]$ there exists a probability space on which we can construct a fractional Brownian motion X with Hurst parameter H, together with a process Y that: (i) is Hölder-continuous with Hölder exponent γ for any $\gamma \in (0\,, H)$; and (ii) solves the differential equation $\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}X_t$. More significantly, we describe the law of the stochastic process Y in terms of the solution to a non-linear stochastic partial differential equation.

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1 Introduction

Let us choose and fix some $T > 0$ throughout, and consider the differential equation

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}X_t \qquad (0 < t \leqslant T), \tag{DE$_0$}$$

that is driven by a given, possibly-random, signal $X := \{X_t\}_{t\in[0,T]}$ and is subject to some given initial value $Y_0 \in \mathbf{R}$ which we hold fixed throughout. The sink/source function $f : \mathbf{R} \to \mathbf{R}$ is also fixed throughout, and is assumed to be Lipschitz continuous, globally, on all of $\mathbf{R}$.

It is well known—and not difficult to verify from first principles—that when the signal X is a Lipschitz-continuous function, then:

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(i) The differential equation (DE_0) has a solution Y that is itself Lipschitz continuous;

(ii) The Radon–Nikodým derivative $\mathrm{d}Y_t/\mathrm{d}X_t$ exists, is continuous, and solves $\mathrm{d}Y_t/\mathrm{d}X_t = f(Y_t)$ for every $0 < t \leqslant T$; and

(iii) The solution to (DE_0) is unique.

Therefore, the Lebesgue differentiation theorem implies that we can recast (DE_0) equally well as the solution to the following: As $\varepsilon \downarrow 0$,

$$\frac{Y_{t+\varepsilon} - Y_t}{X_{t+\varepsilon} - X_t} = f(Y_t) + o(1), \tag{DE}$$

for almost every $t \in [0\,, T]$.[1] Note that (DE) always has an "elementary" solution, even when X is assumed only to be continuous. Namely, if y is a solution to the ODE, $y' = f(y)$, and we set $Y_t = y(X_t)$, then $Y_{t+\varepsilon} - Y_t = f(Y_t)(X_{t+\varepsilon} - X_t) + o(|X_{t+\varepsilon} - X_t|)$. Also note that if Y is a solution to (DE) and ξ is a process that is smoother than X in the sense that $\xi_{t+\varepsilon} - \xi_t = o(|X_{t+\varepsilon} - X_t|)$, then $Y + \xi$ is also a solution to (DE).

Differential equations such as (DE_0) and/or (DE) arise naturally also when X is Hölder continuous with some positive index $\gamma < 1$. One of the best-studied such examples is when X is Brownian motion on the time interval $[0\,, T]$. In that case, it is very well known that X is Hölder continuous with index γ for any $\gamma < 1/2$. It is also very well known that (DE_0) and/or (DE) has infinitely-many strong solutions [36], and that there is a unique pathwise solution provided that we specify what we mean by the stochastic integral $\int_0^t f(Y_s)\,\mathrm{d}X_s$ [consider the integrals of Itô and Stratonovich, for instance].

This view of stochastic differential equations plays an important role in the pathbreaking work [26, 25] of T. Lyons who invented his *theory of rough paths* in order to solve (DE_0) when X is rougher than Lipschitz continuous. Our reduction of (DE) to (DE_0) is motivated strongly by Gubinelli's theory of *controlled rough paths* [17], which we have learned from a recent paper of Hairer [18]. In the present context, Gubinelli's theory of controlled rough paths basically states that if we could prove *a priori* that the $o(1)$ term in (DE) has enough structure, then there is a unique solution to (DE), and hence (DE_0).

Lyons' theory builds on older ideas of Fox [12] and Chen [4], respectively in algebraic differentiation and integration theory, in order to construct, for a large family of functions X, "rough-path integrals" $\int_0^t f(Y_s)\,\mathrm{d}X_s$ that are defined uniquely provided that a certain number of "multiple stochastic integrals" of X are pre specified. Armed with a specified definition of the stochastic integral $\int_0^t f(Y_s)\,\mathrm{d}Y_s$, one can then try to solve the differential equation (DE) and/or (DE_0) pathwise [that is ω-by-ω]. To date, this program has been particularly successful when X is Hölder continuous with index $\gamma \in [1/3\,, 1]$: When $\gamma \in$

[1] To be completely careful, we might have to define $0 \div 0 := 0$ in the cases that X has intervals of constancy. But with probability one, this will be a moot issue for the examples that we will be considering soon.

$(1/2\,,1]$ one uses Young's theory of integration; $\gamma = 1/2$ is covered in essence by martingale theory; and Errami and Russo [10] and Chapter 5 of Lyons and Qian [24] both discuss the more difficult case $\gamma \in [1/3\,,1/2)$. There is also mounting evidence that one can extend this strategy to cover values of $\gamma \in [1/4\,,1]$—see [1, 2, 3, 6, 7, 16, 31]—and possibly even $\gamma \in (0\,,1/4)$—see the two recent papers by Unterberger [35] and Nualart and Tindel [29].

As far as we know, very little is known about the probabilistic structure of the solution when $\gamma < 1/2$ [when the solution is in fact known to exist]. Our goal is to say something about the probabilistic structure of *a* solution for a concrete, but highly interesting, family of choices for X in (DE).

A standard fractional Brownian motion [fBm] with Hurst parameter $H \in (0,1)$—abbreviated fBm(H)—is a continuous, mean-zero Gaussian process $X := \{X_t\}_{t \geqslant 0}$ with $X_0 = 0$ a.s. and

$$\mathrm{E}\left(|X_t - X_s|^2\right) = |t-s|^{2H} \qquad (s,t \geqslant 0). \tag{1.1}$$

Note that fBm($1/2$) is a standard Brownian motion. We refer to any constant multiple of a standard fractional Brownian motion, somewhat more generally, as fractional Brownian motion [fBm].

Here, we study the differential equation (DE) in the special case that X is fBm(H) with

$$0 < H \leqslant \tfrac{1}{4}. \tag{1.2}$$

It is well known that (1.1) implies that X is Hölder continuous with index γ for every $\gamma < H$, up to a modification.[2] Since $H \in (0\,,1/4]$, we are precisely in the regime where not a great deal is known about (DE).

In analogy with the classical literature on stochastic differential equations [36] the following theorem establishes the "weak existence" of a solution to (DE), provided that we interpret the little-o term in (DE$_0$), somewhat generously, as "little-o in probability." Our theorem says some things about the law of the solution as well.

Theorem 1.1. *Let $g : \mathbf{R} \to \mathbf{R}$ be Lipschitz continuous uniformly on all of $\mathbf{R}$. Choose and fix $H \in (0\,,1/4]$. Then there exists a probability space $(\Omega\,,\mathcal{F},\mathrm{P})$ on which we can construct a fractional Brownian motion X, with Hurst parameter H, together with a stochastic process $Y \in \cap_{\gamma\in(0,H)} C^\gamma([0\,,T])$ such that*

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in(0,T]} \mathrm{P}\left\{ \left| \frac{Y_{t+\varepsilon} - Y_t}{X_{t+\varepsilon} - X_t} - g(Y_t) \right| > \delta \right\} = 0 \qquad \text{for all } \delta > 0. \tag{1.3}$$

Moreover, $Y := \{Y_t\}_{t\in[0,T]}$ has the same law as $\{\kappa_H u_t(0)\}_{t\in[0,T]}$, where

$$\kappa_H := \left(\frac{(1-2H)\Gamma(1-2H)}{2\pi H} \right)^{1/2}, \tag{1.4}$$

[2] In other words, $X \in \cap_{\gamma\in(0,H)} C^\gamma([0\,,T])$ a.s., where $C^\gamma([0\,,T])$ denotes as usual the collection of all continuous functions $f : [0\,,T] \to \mathbf{R}$ such that $|f(t) - f(s)| \leqslant \mathrm{const} \cdot |t-s|^\gamma$ uniformly for all $s,t \in [0\,,T]$.

and u denotes the mild solution to the nonlinear stochastic partial differential equation,

$$\frac{\partial}{\partial t}u_t(x) = \frac{1}{2}(\Delta_{\alpha/2}u_t)(x) + \frac{1}{2^{(1-2H)/2}\cdot\kappa_H^2}\,g(\kappa_H u_t(x))\dot{W}_t(x), \tag{1.5}$$

on $(t\,,x)\in(0\,,T]\times\mathbf{R}$*, subject to* $u_0(x)\equiv Y_0$ *for all* $x\in\mathbf{R}$*, where* $\dot{W}$ *denotes a space-time white noise.*

The preceding can be extended to all of $H\in(0\,,1/2)$ by replacing, in (3.1) below, the space-time white noise $\dot{W}_t(x)$ by a generalized Gaussian random field $\psi_t(x)$ whose covariance measure is described by

$$\mathrm{Cov}(\psi_t(x)\,,\psi_s(y)) = \frac{\delta_0(t-s)}{|x-y|^\theta}, \tag{1.6}$$

for a suitable choice of $\theta\in(0\,,1)$. We will not pursue this matter further here since we do not know how to address the more immediately-pressing question of uniqueness in Theorem 1.1. Namely, we do not know a good answer to the following: "*What are [necessarily global] non-trivial conditions that ensure that our solution Y is unique in law*"?

Throughout this paper, A_q denotes a finite constant that depends critically only on a [possibly vector-valued] parameter q of interest. We will not keep track of parameter dependencies for the parameters that are held fixed throughout; they include α and H of (2.16) below, as well as the functions g [see Theorem 1.1] and f [see (5.1) below].

The value of A_q might change from line to line, and sometimes even within the line.

In the absence of interesting parameter dependencies, we write a generic "const" in place of "A."

We prefer to write $\|\cdots\|_k$ in place of $\|\cdot\|_{L^k(\Omega)}$, where $k\in[1\,,\infty)$ can be an arbitary real number. That is, for every random variable Y, we set

$$\|Y\|_k := \left\{\mathrm{E}\left(|Y|^k\right)\right\}^{1/k}. \tag{1.7}$$

On a few occasions we might write Lip_φ for the optimal Lipschitz constant of a function $\varphi:\mathbf{R}\to\mathbf{R}$; that is,

$$\mathrm{Lip}_\varphi := \sup_{-\infty<x<y<\infty}\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|. \tag{1.8}$$

2 Some Gaussian random fields

In this section we recall a decomposition theorem of Lei and Nualart [23] which will play an important role in this paper; see Mueller and Wu [27] for a related set of ideas. We also work out an example that showcases further the Lei–Nualart theorem.

2.1 fBm and bi-fBm

Suppose that $H \in (0\,,1)$ and $K \in (0\,,1]$ are fixed numbers.[3] A standard bifractional Brownian motion, abbreviated as bi-fBm$(H\,,K)$, is a continuous mean-zero Gaussian process $B^{H,K} := \{B_t^{H,K}\}_{t\geqslant 0}$ with $B_0^{H,K} := 0$ a.s. and covariance function

$$\operatorname{Cov}\left(B_t^{H,K},\, B_{t'}^{H,K}\right) = 2^{-K}\left(\left[t^{2H} + (t')^{2H}\right]^K - |t-t'|^{2HK}\right), \tag{2.1}$$

for all $t', t \geqslant 0$. Note that $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0\,,1)$. More generally, any constant multiple of a standard bifractional Brownian motion will be referred as bifractional Brownian motion.

Bifractional Brownian motion was invented by Houdré and Villa [19] as a concrete example (besides fractional Brownian motion) of a family of processes that yield natural "quasi–helices" in the sense of Kahane [21] and/or "screw lines" of classical Hilbert-space theory [28, 32]. Some sample path properties of bi-fBm$(H\,,K)$ have been studied by Russo and Tudor [30], Tudor and Xiao [34] and Lei and Nualart [23]. In particular, the following decomposition theorem is due to Lei and Nualart [23, Proposition 1].

Proposition 2.1. *Let $B^{H,K}$ be a* bi-fBm$(H\,,K)$. *There exists a fractional Brownian motion B^{HK} with Hurst parameter HK and a stochastic process ξ such that $B^{H,K}$ and ξ are independent and, outside a single* P*-null set,*

$$B_t^{H,K} = 2^{(1-K)/2} B_t^{HK} + \xi_t \qquad \text{for all } t \geqslant 0. \tag{2.2}$$

Moreover, the process ξ is a centered Gaussian process, with sample functions that are infinitely differentiable on $(0\,,\infty)$ and absolutely continuous on $[0\,,\infty)$.

In fact, it is shown in [23, eq.'s (4) and (5)] that we can write

$$\xi_t = \left(\frac{K}{2^K\Gamma(1-K)}\right)^{1/2} \int_0^\infty \frac{1-\exp(-st^{2H})}{s^{(1+K)/2}}\, \mathrm{d}W_s, \tag{2.3}$$

where W is a standard Brownian motion that is independent of $B^{H,K}$.

2.2 The linear heat equation

Let $\widehat{\ }$ denote the Fourier transform, normalized so that for every rapidly-decreasing function $\varphi : \mathbf{R} \to \mathbf{R}$,

$$\widehat{\varphi}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi x} \varphi(x)\, \mathrm{d}x \qquad (\xi \in \mathbf{R}). \tag{2.4}$$

Let $\Delta_{\alpha/2} := -(-\Delta)^{\alpha/2}$ denote the fractional Laplace operator, which is usually defined by the property that $(\Delta_{\alpha/2}\varphi)\widehat{\ }(\xi) = -|\xi|^\alpha \widehat{\varphi}(\xi)$; see Jacob [20, Vol. II].

[3] Although we are primarily interested in $H \in (0\,,1/4]$, we study the more general case $H \in (0\,,1)$ in this section.

Consider the linear stochastic PDE

$$\frac{\partial}{\partial t}v_t(x) = \frac{1}{2}(\Delta_{\alpha/2}v_t)(x) + \dot{W}_t(x), \tag{2.5}$$

where $v_0(x) \equiv 0$ and $\dot{W}_t(x)$ denotes space-time white noise; that is,

$$\dot{W}_t(x) = \frac{\partial^2 W_t(x)}{\partial t \partial x}, \tag{2.6}$$

in the sense of generalized random fields [15, Chapter 2, §2.4], for a space-time Brownian sheet W.

According to the theory of Dalang [8], the condition

$$1 < \alpha \leqslant 2 \tag{2.7}$$

is necessary and sufficient in order for (2.5) to have a solution v that is a random function. Lei and Nualart [23] have shown that—in the case that $\alpha = 2$—the process $t \mapsto v_t(x)$ is a suitable bi-fBm for every fixed x. In this section we apply the reasoning of [23] to the present setting in order to show that the same can be said about the solution to (2.5) for every possible choice of $\alpha \in (1\,, 2]$.

Let $p_t(x)$ denote the fundamental solution to the fractional heat operator $(\partial/\partial t) - \frac{1}{2}\Delta_{\alpha/2}$; that is, the function $(t\,; x\,, y) \mapsto p_t(y-x)$ is the transition probability function for a symmetric stable-α Lévy process, normalized as follows (see Jacob [20, Vol. III]):

$$\hat{p}_t(\xi) = \exp\left(-t|\xi|^\alpha/2\right) \qquad (t \geqslant 0,\ \xi \in \mathbf{R}). \tag{2.8}$$

The Plancherel theorem implies the following: For all $t > 0$,

$$\|p_t\|^2_{L^2(\mathbf{R})} = \frac{1}{2\pi}\|\widehat{p}_t\|^2_{L^2(\mathbf{R})} = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-t\xi^\alpha}\,\mathrm{d}\xi = \frac{\Gamma(1/\alpha)}{\alpha\pi t^{1/\alpha}}. \tag{2.9}$$

Let us mention also the following variation: By the symmetry of the heat kernel, $\|p_t\|^2_{L^2(\mathbf{R})} = (p_t * p_t)(0) = p_{2t}(0)$. Therefore, the inversion theorem shows that

$$p_t(0) = \sup_{x\in\mathbf{R}} p_t(x) = \frac{2^{1/\alpha}\Gamma(1/\alpha)}{\alpha\pi t^{1/\alpha}} \qquad (t > 0). \tag{2.10}$$

Now we can return to the linear stochastic heat equation (2.5), and write its solution v, in mild form, as follows:

$$v_t(x) = \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,W(\mathrm{d}s\,\mathrm{d}y). \tag{2.11}$$

It is well known [37, Chapter 3] that v is a continuous, centered Gaussian random field. Therefore, we combine (2.8), and (2.9), using Parseval's identity, in order to see that

$$\begin{aligned}\mathrm{Cov}\left(v_t(x)\,,v_{t'}(x)\right) &= \int_0^{t\wedge t'} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ p_{t-s}(y)p_{t'-s}(y) \\ &= \frac{1}{2\pi}\int_0^{t\wedge t'} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}\xi\ \widehat{p}_{t-s}(\xi)\widehat{p}_{t'-s}(\xi) \qquad (2.12) \\ &= \frac{\Gamma(1/\alpha)}{\pi\alpha}\int_0^{t\wedge t'} \left(\frac{t+t'-2s}{2}\right)^{-1/\alpha} \mathrm{d}s.\end{aligned}$$

We use the substitution $r = (t+t'-2s)/2$ and note that $(t+t')/2 - (t\wedge t') = |t-t'|/2$ in order to conclude that

$$\mathrm{Cov}\left(v_t(x)\,,v_{t'}(x)\right) = c_\alpha^2 2^{(1-\alpha)/\alpha}\left(|t'+t|^{(\alpha-1)/\alpha} - |t'-t|^{(\alpha-1)/\alpha}\right), \tag{2.13}$$

where

$$c_\alpha := \left(\frac{\Gamma(1/\alpha)}{\pi(\alpha-1)}\right)^{1/2}. \tag{2.14}$$

That is, we have verified the following:

Proposition 2.2. *For every fixed $x \in \mathbf{R}$, the stochastic process $t \mapsto c_\alpha^{-1}v_t(x)$ is a* bi-fBm$(1/2\,,(\alpha-1)/\alpha)$, *where c_α is defined in* (2.14). *Therefore, Proposition 2.1 allows us to write*

$$v_t(x) = c_\alpha 2^{1/(2\alpha)}X_t + R_t \qquad (t \geqslant 0), \tag{2.15}$$

where $\{X_t\}_{t\geqslant 0}$ is fBm$((\alpha-1)/(2\alpha))$ *and $\{R_t\}_{t\geqslant 0}$ is a centered Gaussian process that is:*

(i) Independent of $v_\bullet(x)$;

(ii) Absolutely continuous on $[0\,,\infty)$, a.s.; and

(iii) Infinitely differentiable on $(0\,,\infty)$, a.s.

Remark 2.3. From now on, we choose α and H according to the following relation:

$$\alpha := \frac{1}{1-2H} \quad \text{equivalently} \quad H := \frac{\alpha-1}{2\alpha}, \tag{2.16}$$

so that Dalang's condition (2.7) is equivalent to the restriction that $H \in (0\,,1/4]$. Propositions 2.1 and 2.2 together show that $t \mapsto v_t(x)$ is a smooth perturbation of a [non-standard] fractional Brownian motion. In particular, we may compare (1.4) and (2.14) in order to conclude that

$$\kappa_H = c_\alpha, \tag{2.17}$$

thanks to our convention (2.16). □

Remark 2.4. According to (2.3) the process R_t of Proposition 2.2 can be written as

$$R_t = \text{const} \cdot \int_0^\infty \frac{1 - \exp(-st)}{s^{H+(1/2)}} \, \mathrm{d}W_s. \tag{2.18}$$

This is a Gaussian process that is C^∞ away from $t = 0$, and its derivatives are obtained by differentiating under the [Wiener] integral. In particular, the first derivative of R, away from $t = 0$, is

$$R'_t = \text{const} \cdot \int_0^\infty \frac{\exp(-st)}{s^{H-(1/2)}} \, \mathrm{d}W_s \qquad (t > 0). \tag{2.19}$$

Consequently, $\{R'_q\}_{q>0}$ defines a centered Gaussian process, and Wiener's isometry shows that $\mathrm{E}(|R'_q|^2) = \text{const} \cdot q^{2H-2}$ for all $q > 0$. Therefore,

$$\begin{aligned} \|R_{t+\varepsilon} - R_t\|_k = A_k \|R_{t+\varepsilon} - R_t\|_2 &\leqslant A_k \int_t^{t+\varepsilon} \|R'_q\|_2 \, \mathrm{d}q \\ = A_k \int_t^{t+\varepsilon} q^{H-1} \, \mathrm{d}q &\leqslant A_k \, t^{H-1} \varepsilon, \end{aligned} \tag{2.20}$$

uniformly over all $t > 0$ and $\varepsilon \in (0\,,1)$. □

3 The non-linear heat equation

In this section we consider the non-linear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2} (\Delta_{\alpha/2} u_t)(x) + f(c_\alpha u_t(x)) \dot{W}_t(x) \tag{3.1}$$

on $(t\,,x) \in (0\,,T] \times \mathbf{R}$, subject to $u_0(x) \equiv Y_0$ for all $x \in \mathbf{R}$, where c_α was defined in (2.14) and $f : \mathbf{R} \to \mathbf{R}$ is a globally Lipschitz-continuous function.

As is customary [37, Chapter 3], we interpret (3.1) as the non-linear random evolution equation,

$$u_t(x) = Y_0 + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y - x) f(c_\alpha u_s(y)) \, W(\mathrm{d}s \, \mathrm{d}y). \tag{3.2}$$

Dalang's condition (2.7) implies that the evolution equation (3.2) has an a.s.-unique random-field solution u. Moreover, (2.7) is necessary and sufficient for the existence of a random-field solution when f is a constant; see [8]. We will need the following technical estimates.

Lemma 3.1. *For all $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that:*

$$\begin{aligned} &\mathrm{E}\left(|u_t(x)|^k\right) \leqslant A_{k,T}; \qquad \text{and} \\ &\mathrm{E}\left(\left|u_t(x) - u_{t'}(x')\right|^k\right) \leqslant A_{k,T} \left(|x - x'|^{(\alpha-1)k/2} + |t - t'|^{(\alpha-1)k/(2\alpha)}\right); \end{aligned} \tag{3.3}$$

uniformly for all $t, t' \in [0\,,T]$ and $x, x' \in \mathbf{R}$.

This is well known: The first moment bound can be found explicitly in Dalang [8], and the second can be found in the appendix of Foondun and Khoshnevisan [11]. The second can also be shown to follow from the moments estimates of [8] and some harmonic analysis.

Lemma 3.1 and the Kolmogorov continuity theorem [9, Theorem 4.3, p. 10] together imply that u is continuous up to a modification. Moreover, (2.16) and Kolmogorov's continuity theorem imply that for every $x \in \mathbf{R}$,

$$u_\bullet(x) \in \bigcap_{\gamma\in(0,H)} C^\gamma([0\,,T]). \tag{3.4}$$

4 An approximation theorem

The following is the main technical contribution of this paper. Recall that v denotes the solution to the linear stochastic heat equation (2.5), and has the integral representation (2.11).

Theorem 4.1. *For every $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0\,,1)$, $x \in \mathbf{R}$, and $t \in [0\,,T]$,*

$$\mathrm{E}\left(|u_{t+\varepsilon}(x) - u_t(x) - f(c_\alpha u_t(x)) \cdot \{v_{t+\varepsilon}(x) - v_t(x)\}|^k\right) \leqslant A_{k,T}\, \varepsilon^{\mathcal{G}_H k}, \tag{4.1}$$

where

$$\mathcal{G}_H := \frac{2H}{1+H}. \tag{4.2}$$

Remark 4.2. Since $0 < H \leqslant 1/4$, it follows that

$$\frac{8}{5} \leqslant \frac{\mathcal{G}_H}{H} < 2. \tag{4.3}$$

We do not know whether the fraction $8/5 = 1.6$ is a meaningful quantity or a byproduct of the particulars of our method. For us the relevant matter is that (4.3) is a good enough estimate to ensure that $\mathcal{G}_H/H > 1$; the strict inequality will play an important role in the sequel. □

Theorem 4.1 is in essence an analysis of the temporal increments of $u_\bullet(x)$. Thanks to (3.2), we can write those increments as

$$u_{t+\varepsilon}(x) - u_t(x) := \mathscr{J}_1 + \mathscr{J}_2, \tag{4.4}$$

where

$$\begin{aligned} \mathscr{J}_1 &:= \int_{(0,t)\times\mathbf{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, f(c_\alpha u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y);\\ \mathscr{J}_2 &:= \int_{(t,t+\varepsilon)\times\mathbf{R}} p_{t+\varepsilon-s}(y-x) f(c_\alpha u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{4.5}$$

Our proof of Theorem 4.1 proceeds by analyzing $\mathscr{J}_1$ and $\mathscr{J}_2$ separately. Let us begin with the latter quantity, as it is easier to estimate than the former term.

4.1 Estimation of $\mathscr{J}_2$

Define

$$\widetilde{\mathscr{J}_2} := f(c_\alpha u_t(x)) \cdot \int_{(t,t+\varepsilon)\times\mathbf{R}} p_{t+\varepsilon-s}(y-x)\, W(\mathrm{d}s\,\mathrm{d}y). \tag{4.6}$$

Proposition 4.3. *For every $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that for all $\varepsilon \in (0\,,1)$,*

$$\sup_{x\in\mathbf{R}}\sup_{t\in[0,T]} \mathrm{E}\left(\left|\mathscr{J}_2 - \widetilde{\mathscr{J}_2}\right|^k\right) \leqslant A_{k,T}\,\varepsilon^{2Hk}. \tag{4.7}$$

We split the proof in 2 parts: First we show that $\mathscr{J}_2 \approx \mathscr{J}_2'$ in $L^k(\Omega)$, where

$$\mathscr{J}_2' := \int_{(t,t+\varepsilon)\times\mathbf{R}} p_{t+\varepsilon-s}(y-x) f(c_\alpha u_s(x))\, W(\mathrm{d}s\,\mathrm{d}y). \tag{4.8}$$

After that we will verify that $\mathscr{J}_2' \approx \widetilde{\mathscr{J}_2}$ in $L^k(\Omega)$. Proposition 4.3 follows immediately from Lemmas 4.4 and 4.5 below and Minkowski's inequality. Therefore, we will state and prove only those two lemmas.

Lemma 4.4. *For all $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0\,,1)$,*

$$\sup_{x\in\mathbf{R}}\sup_{t\in[0,T]} \mathrm{E}\left(\left|\mathscr{J}_2 - \mathscr{J}_2'\right|^k\right) \leqslant A_{k,T}\varepsilon^{2Hk}. \tag{4.9}$$

Proof. The proof will use a particular form of the Burkholder–Davis–Gundy (BDG) inequality [5, Lemma 2.3]. Since we will make repeated use of this inequality throughout, let us recall it first.

For every $t \geqslant 0$, let $\mathscr{F}_t^0$ denote the sigma-algebra generated by every Wiener integral of the form $\int_{(0,t)\times\mathbf{R}} \varphi_s(y)\, W(\mathrm{d}s\,\mathrm{d}y)$ as φ ranges over all elements of $L^2(\mathbf{R}_+ \times \mathbf{R})$. We complete every such sigma-algebra, and make the filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ right continuous in order to obtain the "Brownian filtration" $\mathscr{F}$ that corresponds to the white noise $\dot{W}$.

Let $\Phi := \{\Phi_t(x)\}_{t\geqslant 0, x\in\mathbf{R}}$ be a predictable random field with respect to $\mathscr{F}$. Then, for every real number $k \in [2\,,\infty)$, we have the following BDG inequality:

$$\left\| \int_{(0,t)\times\mathbf{R}} \Phi_s(y)\, W(\mathrm{d}s\,\mathrm{d}y) \right\|_k^2 \leqslant 4k \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; \|\Phi_s(y)\|_k^2. \tag{4.10}$$

The BDG inequality (4.10) and eq. (3.2) together imply that

$$\begin{aligned}
&\|\mathscr{J}_2 - \mathscr{J}_2'\|_{L^k(\Omega)}^2 \\
&\qquad \leqslant 4k \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \;\; [p_{t+\varepsilon-s}(y-x)]^2 \, \|f(c_\alpha u_s(y)) - f(c_\alpha u_s(x))\|_k^2 \\
&\qquad \leqslant 4kc_\alpha^2 \mathrm{Lip}_f^2 \cdot \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \;\; [p_{t+\varepsilon-s}(y-x)]^2 \, \|u_s(y) - u_s(x)\|_k^2 \qquad (4.11) \\
&\qquad \leqslant A_{k,T} \int_0^{\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \;\; [p_s(y)]^2 \left(|y|^{\alpha-1} \wedge 1\right).
\end{aligned}$$

The last inequality uses both moment inequalities of Lemmas 3.1. Furthermore, measurability issues do not arise, since the solution to (3.2) is continuous in the time variable t and adapted to the Brownian filtration $\mathscr{F}$.

In order to proceed from here, we need to recall two basic facts about the transition functions of stable processes: First of all,

$$p_s(y) = s^{-1/\alpha} p_1\left(|y|/s^{1/\alpha}\right) \qquad \text{for all } s>0 \text{ and } y \in \mathbf{R}. \tag{4.12}$$

This fact is a consequence of scaling and symmetry; see (2.8). We also need to know the fact that $p_1(z) \leqslant \mathrm{const} \cdot (1+|z|)^{-(1+\alpha)}$ for all $z \in \mathbf{R}$ [22, Proposition 3.3.1, p. 380], whence

$$p_s(y) \leqslant \mathrm{const} \times \begin{cases} s^{-1/\alpha} & \text{if } |y| \leqslant s^{1/\alpha}, \\ s|y|^{-(1+\alpha)} & \text{if } |y| > s^{1/\alpha}. \end{cases} \tag{4.13}$$

Consequently,

$$\begin{aligned}
&\int_0^{\varepsilon} \mathrm{d}s \int_0^1 \mathrm{d}y \;\; [p_s(y)]^2 \left(y^{\alpha-1} \wedge 1\right) \\
&\qquad \leqslant \mathrm{const} \cdot \left(\int_0^{\varepsilon} s^{-2/\alpha} \, \mathrm{d}s \int_0^{s^{1/\alpha}} y^{\alpha-1} \, \mathrm{d}y + \int_0^{\varepsilon} s^2 \, \mathrm{d}s \int_{s^{1/\alpha}}^1 y^{-3-\alpha} \, \mathrm{d}y \right) \qquad (4.14) \\
&\qquad \leqslant \mathrm{const} \cdot \varepsilon^{2(\alpha-1)/\alpha}.
\end{aligned}$$

We obtain the following estimate by similar means:

$$\begin{aligned}
\int_0^{\varepsilon} \mathrm{d}s \int_1^{\infty} \mathrm{d}y \;\; [p_s(y)]^2 \left(y^{\alpha-1} \wedge 1\right) &\leqslant \mathrm{const} \cdot \int_0^{\varepsilon} s^2 \, \mathrm{d}s \int_1^{\infty} y^{-2-2\alpha} \, \mathrm{d}y \qquad (4.15) \\
&= \mathrm{const} \cdot \varepsilon^3 \\
&\leqslant \mathrm{const} \cdot \varepsilon^{2(\alpha-1)/\alpha},
\end{aligned}$$

uniformly for all $\varepsilon \in (0\,,1)$. Since $p_s(y) = p_s(-y)$ for all $s > 0$ and $y \in \mathbf{R}$, the preceding two displays and (4.11) together imply that

$$\|\mathscr{J}_2 - \mathscr{J}_2'\|_{L^k(\Omega)}^2 \leqslant \mathrm{const} \cdot \varepsilon^{2(\alpha-1)/\alpha}. \tag{4.16}$$

We may conclude the lemma from this inequality, using our convention about α and H; see (2.16). □

In light of Lemma 4.4, Proposition 4.3 follows at once from

Lemma 4.5. *For all $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0\,,1)$,*

$$\sup_{x\in\mathbf{R}}\sup_{t\in[0,T]} \mathrm{E}\left(\left|\mathscr{J}_2' - \widetilde{\mathscr{J}_2}\right|^k\right) \leqslant A_{k,T}\,\varepsilon^{2Hk}. \tag{4.17}$$

Proof. We apply the BDG inequality (4.10), as we did in the derivation of (4.11), in order to see that

$$\begin{aligned}
&\left\| \mathscr{J}_2' - \widetilde{\mathscr{J}_2}\right\|_{L^k(\Omega)}^2 \\
&\quad\leqslant 4kc_\alpha^2 \mathrm{Lip}_f^2 \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \ [p_{t+\varepsilon-s}(y)]^2 \left\|u_s(x) - u_t(x)\right\|_{L^k(\Omega)}^2 \\
&\quad\leqslant A_{k,T} \int_t^{t+\varepsilon} \left\|p_{t+\varepsilon-s}\right\|_{L^2(\mathbf{R})}^2 |s-t|^{(\alpha-1)/\alpha}\,\mathrm{d}s.
\end{aligned} \tag{4.18}$$

Therefore, (2.9) and a change of variables together show us that the preceding quantity is bounded above by

$$A_{k,T}\int_0^\varepsilon s^{(\alpha-1)/\alpha}(\varepsilon - s)^{-1/\alpha}\,\mathrm{d}s = A_{k,T}\varepsilon^{2(\alpha-1)/\alpha}. \tag{4.19}$$

The lemma follows from this and our convention (2.16) about the relation between α and H. □

4.2 Estimation of $\mathscr{J}_1$ and proof of Theorem 4.1

Now we turn our attention to the more interesting term $\mathscr{J}_1$ in the decomposition (4.5). The following localization argument paves the way for a successful analysis of $\mathscr{J}_1$: $p_t(x)\,\mathrm{d}x \approx \delta_0(\mathrm{d}x)$ when $t \approx 0$; therefore one might imagine that there is a small regime of values of $s \in (0\,,t)$ such that $p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)$ is highly localized [big within the regime, and significantly smaller outside that regime]. Thus, we choose and fix a parameter $a \in (0\,,1)$—whose optimal value will be made explicit later on in (4.42)—and write

$$\mathscr{J}_1 = \mathscr{J}_{1,a} + \mathscr{J}_{1,a}', \tag{4.20}$$

where

$$\begin{aligned}
\mathscr{J}_{1,a} &:= \int_{(0,t-\varepsilon^a)\times\mathbf{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, f(c_\alpha u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y), \\
\mathscr{J}_{1,a}' &:= \int_{(t-\varepsilon^a,t)\times\mathbf{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, f(c_\alpha u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y).
\end{aligned} \tag{4.21}$$

We will prove that the quantity $\mathscr{J}_{1,a}$ is small as long as we choose $a \in (0\,,1)$ carefully; that is, $\mathscr{J}_1 \approx \mathscr{J}'_{1,a}$ for a good choice of a. And because $s \in (t-\varepsilon^a, t)$ is approximately t, then we might expect that $f(u_s(y))) \approx f(u_t(y))$ [for that correctly-chosen a], and hence $\mathscr{J}_1 \approx \mathscr{J}''_{1,a}$, where

$$\mathscr{J}''_{1,a} := \int_{(t-\varepsilon^a,t)\times\mathbf{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, f(c_\alpha u_t(y))\, W(\mathrm{d}s\,\mathrm{d}y). \tag{4.22}$$

Finally, we might notice that $p_{t+\varepsilon-s}$ and p_{t-s} both act as point masses when $s \in (t-\varepsilon^a, t)$, and therefore we might imagine that $\mathscr{J}_1 \approx \mathscr{J}''_{1,a} \approx \widetilde{\mathscr{J}}_{1,a}$, where

$$\widetilde{\mathscr{J}}_{1,a} := f(c_\alpha u_t(x)) \cdot \int_{(t-\varepsilon^a,t)\times\mathbf{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, W(\mathrm{d}s\,\mathrm{d}y). \tag{4.23}$$

All of this turns out to be true; it remains to find the correct choice[s] for the parameter a so that the errors in the mentioned approximations remain sufficiently small for our later needs. Recall the parameter $\mathcal{G}_H$ from (4.2). Before we continue, let us first document the end result of this forthcoming effort. We will prove it subsequently.

Proposition 4.6. *For every $T>0$ and $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0\,,1)$, $x \in \mathbf{R}$, and $t \in [0\,,T]$,*

$$\mathrm{E}\left(\left|\mathscr{J}_1 - f(c_\alpha u_t(x)) \cdot \int_{(0,t)\times\mathbf{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, W(\mathrm{d}s\,\mathrm{d}y)\right|^k\right) \leqslant A_{k,T}\, \varepsilon^{\mathcal{G}_H k}. \tag{4.24}$$

Thanks to (4.4) and Minkowski's inequality, Theorem 4.1 follows easily from Propositions 4.3 and 4.6. It remains to prove Proposition 4.6.

We begin with a sequence of lemmas that make precise the various formal appeals to "≈" in the preceding discussion. As a first step in this direction, let us dispense with the "small" term $\mathscr{J}_{1,a}$.

Lemma 4.7. *For all $k \in [2\,,\infty)$ and $a \in (0\,,1)$ there exists a finite constant $A_{a,k,T}$ such that uniformly for all $\varepsilon \in (0\,,1)$,*

$$\sup_{x\in\mathbf{R}} \sup_{t\in[0,T]} \mathrm{E}\left(|\mathscr{J}_{1,a}|^k\right) \leqslant A_{a,k,T}\, \varepsilon^{[1-a(1-H)]k}. \tag{4.25}$$

Proof. We can modify the argument that led to (4.11), using the BDG inequality (4.10), in order to yield

$$\begin{aligned}
&\|\mathscr{J}_{1,a}\|^2_{L^k(\Omega)} \\
&\leqslant 4k \int_0^{t-\varepsilon^a} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \ [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]^2\, \|f(c_\alpha u_s(y))\|^2_{L^k(\Omega)} \\
&\leqslant A_{k,T} \int_{\varepsilon^a}^T \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \ [p_{s+\varepsilon}(y) - p_s(y)]^2\,.
\end{aligned} \tag{4.26}$$

We first bound $\int_{\varepsilon^a}^T \mathrm{d}s$ from above by $\mathrm{e}^T \cdot \int_{\varepsilon^a}^\infty \mathrm{e}^{-s}\,\mathrm{d}s$, and then apply (2.8) and Plancherel's formula in order to deduce the following bounds:

$$\begin{aligned}
\|\mathscr{I}_{1,a}\|_{L^k(\Omega)}^2 &\leqslant A_{k,T}\int_{\varepsilon^a}^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_{-\infty}^\infty \mathrm{d}\xi\ \mathrm{e}^{-2s|\xi|^\alpha}\left|1-\mathrm{e}^{-\varepsilon|\xi|^\alpha}\right|^2 \\
&\leqslant A_{k,T}\int_{\varepsilon^a}^\infty \mathrm{e}^{-s}\,\mathrm{d}s\int_0^\infty \mathrm{d}\xi\ \mathrm{e}^{-2s\xi^\alpha}\left(1\wedge\varepsilon^2\xi^{2\alpha}\right) \qquad (4.27)\\
&= A_{k,T}\int_0^\infty \left(1\wedge\varepsilon^2\xi^{2\alpha}\right)\mathrm{e}^{-2\varepsilon^a\xi^\alpha}\frac{\mathrm{d}\xi}{1+\xi^\alpha},
\end{aligned}$$

since $0\leqslant 1-\mathrm{e}^{-z}\leqslant 1\wedge z$ for all $z\geqslant 0$. Clearly,

$$\begin{aligned}
\int_0^{\varepsilon^{-1/\alpha}}\left(1\wedge\varepsilon^2\xi^{2\alpha}\right)\mathrm{e}^{-2\varepsilon^a\xi^\alpha}\frac{\mathrm{d}\xi}{1+\xi^\alpha} &\leqslant \varepsilon^2\int_0^{\varepsilon^{-1/\alpha}}\xi^\alpha\mathrm{e}^{-2\varepsilon^a\xi^\alpha}\mathrm{d}\xi \\
&= \varepsilon^{(\alpha-1)/\alpha}\int_0^1 x^\alpha\exp\left(-\frac{2x^\alpha}{\varepsilon^{1-a}}\right)\mathrm{d}x \\
&\leqslant \varepsilon^{(\alpha-1)/\alpha}\int_0^\infty x^\alpha\exp\left(-\frac{2x^\alpha}{\varepsilon^{1-a}}\right)\mathrm{d}x \\
&= \mathrm{const}\cdot\varepsilon^{(2\alpha-a-\alpha a)/\alpha}. \qquad (4.28)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_{\varepsilon^{-1/\alpha}}^\infty\left(1\wedge\varepsilon^2\xi^{2\alpha}\right)\mathrm{e}^{-2\varepsilon^a\xi^\alpha}\frac{\mathrm{d}\xi}{1+\xi^\alpha} &\leqslant \int_{\varepsilon^{-1/\alpha}}^\infty \mathrm{e}^{-2\varepsilon^a\xi^\alpha}\mathrm{d}\xi \qquad (4.29)\\
&\leqslant \mathrm{const}\cdot\exp\left(-2\varepsilon^{-(1-a)}\right),
\end{aligned}$$

uniformly for all $\varepsilon\in(0\,,1)$. The preceding two paragraphs together imply that

$$\mathrm{E}\left(|\mathscr{I}_{1,a}|^k\right)\leqslant A_{a,k,T}\,\varepsilon^{(2\alpha-a-a\alpha)k/(2\alpha)}, \qquad (4.30)$$

which proves the lemma, due to the relation (2.16) between H and α. □

Lemma 4.8. *For all $k\in[2\,,\infty)$ and $a\in(0\,,1)$ there exists a finite constant $A_{a,k,T}$ such that uniformly for all $\varepsilon\in(0\,,1)$,*

$$\sup_{x\in\mathbf{R}}\sup_{t\in[0,T]}\mathrm{E}\left(\left|\mathscr{I}_{1,a}'-\mathscr{I}_{1,a}''\right|^k\right)\leqslant A_{a,k,T}\,\varepsilon^{2aHk}. \qquad (4.31)$$

Proof. We proceed as we did for (4.11), using the BDG inequality (4.10), in order to find that

$$\begin{aligned}
\left\|\mathscr{I}_{1,a}'-\mathscr{I}_{1,a}''\right\|_{L^k(\Omega)}^2 &\leqslant A_{k,T}\cdot\int_0^{\varepsilon^a} s^{(\alpha-1)/\alpha}\|p_{s+\varepsilon}-p_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s \qquad (4.32)\\
&= A_{k,T}\varepsilon^{(2\alpha-1)/\alpha}\cdot\int_0^{\varepsilon^{a-1}} r^{(\alpha-1)/\alpha}\|p_{\varepsilon(1+r)}-p_{\varepsilon r}\|_{L^2(\mathbf{R})}^2\,\mathrm{d}r,
\end{aligned}$$

after a change of variables $[r := s/\varepsilon]$. The scaling property (4.12) can be written in the following form:

$$p_{\varepsilon\tau}(y) = \varepsilon^{-1/\alpha} p_\tau(y/\varepsilon^{1/\alpha}), \tag{4.33}$$

valid for all $\tau, \varepsilon > 0$ and $y \in \mathbf{R}$. Consequently,

$$\|p_{\varepsilon(1+r)} - p_{\varepsilon r}\|_{L^2(\mathbf{R})}^2 = \varepsilon^{-1/\alpha} \cdot \|p_{1+r} - p_r\|_{L^2(\mathbf{R})}^2. \tag{4.34}$$

Eq. (2.8) and the Plancherel theorem together imply that

$$\begin{aligned} \|p_{1+r} - p_r\|_{L^2(\mathbf{R})}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-r|z|^\alpha} \left(1 - \mathrm{e}^{-|z|^\alpha/2}\right)^2 \mathrm{d}z \\ &\leqslant \int_0^\infty \mathrm{e}^{-rz^\alpha}\, \mathrm{d}z \\ &= \frac{\Gamma(1/\alpha)}{\alpha r^{1/\alpha}}, \end{aligned} \tag{4.35}$$

for all $r > 0$. Therefore, (4.32) implies that

$$\left\| \mathscr{I}_{1,a}' - \mathscr{I}_{1,a}'' \right\|_{L^k(\Omega)}^2 \leqslant A_{k,T} \varepsilon^{2(\alpha-1)/\alpha} \cdot \int_0^{\varepsilon^{a-1}} r^{(\alpha-2)/\alpha}\, \mathrm{d}r, \tag{4.36}$$

which readily implies the lemma. □

Lemma 4.9. *For all $k \in [2\,, \infty)$ and $a \in (0\,, 1)$ there exists a finite constant $A_{a,k,T}$ such that uniformly for all $\varepsilon \in (0\,, 1)$,*

$$\sup_{x \in \mathbf{R}} \sup_{t \in [0,T]} \mathrm{E}\left(\left| \mathscr{I}_{1,a}'' - \widetilde{\mathscr{I}}_{1,a} \right|^k \right) \leqslant A_{k,T}\, \varepsilon^{2aHk}. \tag{4.37}$$

Proof. We proceed as we did for (4.11), apply the BDG inequality (4.10), and obtain the following bounds:

$$\begin{aligned} &\left\| \mathscr{I}_{1,a}'' - \widetilde{\mathscr{I}}_{1,a} \right\|_{L^k(\Omega)}^2 \\ &\leqslant A_k \int_{t-\varepsilon^a}^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \left[p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)\right]^2 \|u_t(y) - u_t(x)\|_{L^k(\Omega)}^2 \\ &\leqslant A_{k,T} \int_0^{\varepsilon^a} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \left[p_{s+\varepsilon}(y) - p_s(y)\right]^2 \left(|y|^{\alpha-1} \wedge 1\right) \\ &\leqslant A_{k,T} \varepsilon \int_0^{\varepsilon^{a-1}} \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}y\ \left[p_{\varepsilon(r+1)}(y) - p_{\varepsilon r}(y)\right]^2 |y|^{\alpha-1}. \end{aligned} \tag{4.38}$$

Thanks to the scaling property (4.33), we may obtain the following after a change of variables $[w := y/\varepsilon^{1/\alpha}]$:

$$\begin{aligned} &\left\| \mathscr{I}_{1,a}'' - \widetilde{\mathscr{I}}_{1,a} \right\|_{L^k(\Omega)}^2 \\ &\qquad \leqslant A_{k,T} \varepsilon^{2(\alpha-1)/\alpha} \int_0^{\varepsilon^{a-1}} \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}w \left[p_{r+1}(w) - p_r(w)\right]^2 |w|^{\alpha-1}. \end{aligned} \tag{4.39}$$

Next we notice that

$$\begin{aligned}\int_0^{\varepsilon^{a-1}} \mathrm{d}r \int_0^\infty \mathrm{d}w\, [p_r(w)]^2\, w^{\alpha-1} &= \int_0^{\varepsilon^{a-1}} r^{-2/\alpha}\, \mathrm{d}r \int_0^\infty \mathrm{d}w \left[p_1(w/r^{1/\alpha})\right]^2 w^{\alpha-1}\\ &= \int_0^{\varepsilon^{a-1}} r^{(\alpha-2)/\alpha}\, \mathrm{d}r \int_0^\infty \mathrm{d}x\, [p_1(x)]^2\, x^{\alpha-1},\\ &\leqslant \text{const}\cdot \varepsilon^{2(a-1)(\alpha-1)/\alpha}, \end{aligned}\tag{4.40}$$

where the last inequality uses the facts that: (i) $\alpha > 1$; and (ii) $p_1(x) \leqslant \text{const}\cdot (1+|x|)^{-1-\alpha}$ (see [22, Proposition 3.3.1, p. 380]). Therefore,

$$\left\| \mathscr{I}''_{1,a} - \widetilde{\mathscr{I}}_{1,a} \right\|^2_{L^k(\Omega)} \leqslant A_{k,T}\varepsilon^{2a(\alpha-1)/\alpha}, \tag{4.41}$$

which proves the lemma, due to the relation (2.16) between H and α. □

Proof of Proposition 4.6. So far, the parameter a has been an arbitrary real number in $(0\,,1)$. Now we choose and fix it as follows:

$$a := \frac{1}{1+H}. \tag{4.42}$$

Thus, for this particular choice of a,

$$1 - a(1-H) = 2aH = \mathcal{G}_H, \tag{4.43}$$

where $\mathcal{G} := 2H/(1+H)$ was defined in (4.2). Because $\mathcal{G}_H < 2H$ and because of (4.20), Lemmas 4.7, 4.8, and 4.9 together imply that, for this choice of a,

$$\mathrm{E}\left(\left| \mathscr{I}_1 - \widetilde{\mathscr{I}}_{1,a}\right|^k\right) \leqslant A_{k,T}\, \varepsilon^{\mathcal{G}_H k}, \tag{4.44}$$

uniformly for all $\varepsilon \in (0\,,1)$, $x \in \mathbf{R}$, and $t \in [0\,,T]$. Thanks to the definition (4.23) of $\widetilde{\mathscr{I}}_{1,a}$, it suffices to demonstrate the following with the same parameter dependencies as above:

$$\mathrm{E}\left(\left| \widetilde{\mathscr{I}}_{1,a} - f(c_\alpha u_t(x))\cdot \Lambda([0\,,t])\right|^k\right) \leqslant A_{k,T}\, \varepsilon^{\mathcal{G}_H k}; \tag{4.45}$$

where $\Lambda(Q) := \int_{Q\times\mathbf{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] W(\mathrm{d}s\,\mathrm{d}y)$ for every interval $Q \subset [0\,,T]$.

Because $\widetilde{\mathscr{I}}_{1,a} = f(c_\alpha u_t(x)) \times \Lambda([t-\varepsilon^a\,,t])$, Lemma 3.1 shows that the left-hand side of (4.45)

$$\mathrm{E}\left(\left| \widetilde{\mathscr{I}}_{1,a} - f(c_\alpha u_t(x))\cdot \Lambda([0\,,t])\right|^k\right) \leqslant A_{k,T}\sqrt{\mathrm{E}\left(|\Lambda([0\,,t-\varepsilon^a])|^{2k}\right)}. \tag{4.46}$$

Since $\Lambda([0\,,t-\varepsilon^a])$ is the same as the quantity $\mathscr{I}_{1,a}$ in the case that $f \equiv 1$, we may apply Lemma 4.7 to the linear equation (2.5) with $f \equiv 1$ in order to see that

$$\sqrt{\mathrm{E}\left(|\Lambda([0\,,t-\varepsilon^a])|^{2k}\right)} \leqslant A_{k,T}\, \varepsilon^{\mathcal{G}_H k}, \tag{4.47}$$

which implies (4.45). □

5 Proof of Theorem 1.1

We conclude this article by proving Theorem 1.1.

Let us define a Lipschitz-continuous function f by

$$f(x) := \frac{2^H}{\kappa_H^2\sqrt{2}}\, g(x) \qquad (x \in \mathbf{R}), \tag{5.1}$$

where κ_H was defined in (1.4). Let us also define a stochastic process

$$Y_t := c_\alpha u_t(0) \qquad (t \geqslant 0), \tag{5.2}$$

where the constant $c_\alpha [= \kappa_H]$ was defined in (2.14) and u denotes the solution to the stochastic PDE (3.1). Because of Remark 2.3 and the definition of f, we can see that:

(i) $Y_t = \kappa_H u_t(0)$; and

(ii) u solves the stochastic PDE (1.5).

We also remark that $g(x) = c_\alpha^2 2^{1/(2\alpha)} f(x)$.

We are assured by (3.4) that $Y \in \cap_{\gamma \in (0,H)} C^\gamma([0\,,t])$, up to a modification [in the usual sense of stochastic processes]. Recall from (2.11) the solution v to the linear SPDE (2.5).

Let X be the fBm(H) from Proposition 2.2 and choose and fix $t \in (0\,,T]$. Then

$$\begin{aligned}
\Theta &:= Y_{t+\varepsilon} - Y_t - g(Y_t)(X_{t+\varepsilon} - X_t) \\
&= Y_{t+\varepsilon} - Y_t - c_\alpha^2 2^{1/(2\alpha)} f(Y_t)(X_{t+\varepsilon} - X_t) \\
&= c_\alpha \left(u_{t+\varepsilon}(0) - u_t(0) - f(c_\alpha u_t(0)) \left[c_\alpha 2^{1/(2\alpha)} X_{t+\varepsilon} - c_\alpha 2^{1/(2\alpha)} X_t \right] \right) \\
&= c_\alpha \left(u_{t+\varepsilon}(0) - u_t(0) - f(c_\alpha u_t(0))(v_{t+\varepsilon}(0) - v_t(0)) \right) \\
&\qquad\qquad + c_\alpha f(c_\alpha u_t(0))(R_{t+\varepsilon} - R_t).
\end{aligned} \tag{5.3}$$

We proved, earlier in Remark 2.4, that $\|R_{t+\varepsilon} - R_t\|_k \leqslant A_{k,t}\,\varepsilon$. Because f is Lipschitz continuous, Hölder's inequality and (3.3) together imply that $\|c_\alpha f(c_\alpha u_t(0))(R_{t+\varepsilon} - R_t)\|_k \leqslant A_{k,t}\,\varepsilon$, whence we obtain the bound,

$$\sup_{t \in (0,T]} \mathrm{E}(\Theta^2) \leqslant A_T \varepsilon^{2\mathcal{G}_H}, \tag{5.4}$$

from Theorem 4.1. Since $\mathcal{G}_H > H$—see Remark 4.2—the preceding displayed bound and Chebyshev's inequality together imply that for every $\varepsilon \in (0\,,1)$, $\delta > 0$, and $b \in (H, \mathcal{G}_H)$,

$$\begin{aligned}
\mathrm{P}\left\{ \left| \frac{Y_{t+\varepsilon} - Y_t}{X_{t+\varepsilon} - X_t} - g(Y_t) \right| > \delta \right\} &= \mathrm{P}\left\{ \frac{|\Theta|}{|X_{t+\varepsilon} - X_t|} > \delta \right\} \\
&\leqslant A_T\, \varepsilon^{2(\mathcal{G}_H - b)} + \mathrm{P}\left\{ |X_{t+\varepsilon} - X_t| < \frac{\varepsilon^b}{\delta} \right\}.
\end{aligned} \tag{5.5}$$

The first term converges to zero as $\varepsilon \to 0^+$ since $b < \mathcal{G}_H$. It remains to prove that the second term also vanishes as $\varepsilon \to 0^+$. But since X is fBm(H), the increment $X_{t+\varepsilon} - X_t$ has the same distribution as $\varepsilon^H Z$ where Z is a standard normal random variable. Therefore,

$$\sup_{t\in(0,T]} \mathrm{P}\left\{|X_{t+\varepsilon} - X_t| < \frac{\varepsilon^b}{\delta}\right\} = \mathrm{P}\left\{|Z| \leqslant \frac{\varepsilon^{b-H}}{\delta}\right\}, \tag{5.6}$$

which goes to zero as $\varepsilon \to 0^+$ since $b > H$. □

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