# LECTURE 10: ITÔ's FORMULA AND THE WRIGHT–FISCHER MODEL

## §1. ITÔ's FORMULA

**(1.1) An Itô Formula.** Suppose $Y$ solves the stochastic differential equation,

$$dY(t) = a(Y(t))dW(t) + b(Y(t))dt, \tag{1.2}$$

and recall from (2.11) of Lecture 10 that for any nice function $f$,

$$f(Y(t)) = f(Y(0)) + \int_0^t f'(Y(s))a(Y(s))\, dW(s) + \frac{1}{2}\int_0^t f''(Y(s))\, [a(Y(s))]^2\, ds. \tag{1.3}$$

From this, and a few lines, one can show the following.

**(1.4) Probabilistic Interpretation of $a$ and $b$.** As $h \downarrow 0$,

$$E\left\{ \frac{Y(t+h) - Y(t)}{h} \,\middle|\, Y(t) = x \right\} \to b(Y(t))$$

$$E\left\{ \frac{\left[Y(t+h) - Y(t)\right]^2}{h} \,\middle|\, Y(t) = x \right\} \to a(Y(t)).$$

This gives further credance to our intuition that $a(x)$ determines the strength of the fluctuation if $Y$ enters the value $x$, and $b(x)$ determines the drift (or push) if $Y$ enters $b(x)$.

## §2. THE WRIGHT–FISCHER GENE FREQUENCY MODEL

**(2.1) A Haploid Model.** The haploid model is the simplest model for asexual gene reproduction; here, there are no genetic effects due to genetic mutation or selection for a specific gene.

Let $2N$ denote a fixed population size comprised of two types of individuals (more aptly, genes): Type A and Type B. If the parent consists of $i$ type-A individuals (and hence $2N - i$ type-B), then in the next generation, each gene becomes type-A with probability $\frac{i}{2N}$ and type-B with the remaining probability $1 - \frac{i}{2N}$. All genes follow this prescription independently, and this works to construct a random process that evolves from generation to generation.

Let $X_n :=$ the number of type-A individuals in generation $n$. Then, given that we have simulated the process until time $(n-1)$ and observed $X_{n-1} = j$, we have:

$$P\{X_n = j \,|\, X_{n-1} = i\} = \binom{2N}{j}\left(\frac{i}{2N}\right)^j \left(1 - \frac{i}{2N}\right)^{2N-j}, \qquad \forall j = 0, \ldots, 2N. \tag{2.2}$$

A question arises that is the genetics' analogue of the maze-problem from Robert Thorn's talk:

**(2.3) Question.** What is the probability that starting with $i$ type-A individuals for some $i = 0, \ldots, 2N$, $X_n$ is eventually equal to 0? Can you answer this by simulation when $N$ is large? ♣

**(2.4) A Diffusion-Approximation.** Consider the entire random process $\frac{X_k}{2N}$ where $k = 1, \ldots, 2N$, and $N$ is fixed but large. Then, one can show that when $N$ is large, this process looks like the solution to the following stochastic differential equation (called Feller's equation) run until time one:

$$d(Y(t)) = Y(t)\{1 - Y(t)\}dW(t). \tag{2.5}$$

Thinking of this SDE as we did in (2.3, Lecture 10), you should convince yourself that when the solution $Y$ hits 0 or 1, it sticks there forever.

**(2.6) An Argument to Convince you of (2.5).** This is not a rigorous argument, but its intuitively convincing: Based on the conditional-binomial formula (2.2) above, and a few calculations involving the means and variances of binomials, we have the following: As $h \to 0$, and for each $0 \leq t \leq 1$,

$$\begin{aligned} E\left\{ \frac{X_{2N(t+\frac{1}{N})} - X_{2Nt}}{2N} \,\middle|\, X_{2Nt} = i \right\} &= 0 \to 0 \\ E\left\{ \frac{\left[X_{2N(t+\frac{1}{N})} - X_{2Nt}\right]^2}{2N} \,\middle|\, X_{2Nt} = i \right\} &= \frac{1}{2N}\left(\frac{i}{2N}\right)\left(1 - \frac{i}{N}\right). \end{aligned} \tag{2.7}$$

So let $h = \frac{1}{2N}$ and consider the process $Y_N(t) := \frac{1}{2N} X_{\lfloor 2Nt \rfloor}$ to "see" that $Y_N$ should look like $Y$ in light of (1.4). ♣

**(2.9) Simulation Project.** Simulate the Wright–Fischer haploid model, as well as Feller's diffusion, and "compare." You should think hard about what this means, since we are talking about different random processes. ♣