Math 6010, Fall 2004: Homework

Homework 4

#1, page 136: As in the hint let $I_i = 1$ if E_i is incorrect; else, let $I_i = 0$. Then $\sum_i I_i$ denotes the number of incorrect statements, and $E[\sum_i I_i] = \sum_i E[I_i] = \sum_i P(E_i) = \sum_i \alpha_i$ is the corresponding expectation.

#2, page 136: Fix $k > 1$, and define

$$f(\alpha) = \left(1 - \frac{\alpha}{k}\right)^k - (1-\alpha) \qquad 0 \le \alpha \le 1.$$

Evidently,

$$f'(\alpha) = 1 - \left(1 - \frac{\alpha}{k}\right)^{k-1} > 0,$$

for all $\alpha > 0$. This proves that the minimum of f occurs uniquely at $\alpha = 0$; i.e., $f(\alpha) > f(0)$, which is the desired result.

#4, page 136: A solution will be posted soon.

#5, page 136: Every time we add a new variable we increase the variance (§5.4). Here, however, is a direct argument: Suppose we have the new model,

$$G: \quad \boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} + \boldsymbol{\varepsilon} = \boldsymbol{W\delta} + \boldsymbol{\varepsilon},$$

where $\boldsymbol{W} = (\boldsymbol{X}, \boldsymbol{Z})$ columnwise, and $\boldsymbol{\delta} = (\boldsymbol{\beta}', \boldsymbol{\gamma}')'$. The least-squares predictor, under G, is

$$\hat{\boldsymbol{\delta}}_G = (\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{Y}.$$

Thus, the new predictor at $(\boldsymbol{x}_0', \gamma_0')'$ is:

$$\hat{Y}_{0G} = (\boldsymbol{x}_0', \gamma_0')' \, \hat{\boldsymbol{\delta}}_G.$$

By Theorem 3.6(iv) (p. 55),

$$\text{Var}\left(\hat{\boldsymbol{\delta}}_G\right) = \begin{pmatrix} (\boldsymbol{X}'\boldsymbol{X})^{-1} + \boldsymbol{LML}' & -\boldsymbol{LM}' \\ -\boldsymbol{ML}' & \boldsymbol{M} \end{pmatrix},$$

where $\boldsymbol{L} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Z}$, $\boldsymbol{M} = (\boldsymbol{Z}'\boldsymbol{RZ})^{-1}$, and $\boldsymbol{R} = \mathbf{I}_n - \boldsymbol{W}(\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'$. Therefore,

$$\begin{aligned}
\text{Var}\left[\hat{Y}_{0G}\right] &= (\boldsymbol{x}_0', \gamma_0')' \,\text{Var}\left(\hat{\boldsymbol{\delta}}_G\right) (\boldsymbol{x}_0', \gamma_0') \\
&= \sigma^2 \left(\boldsymbol{x}_0' \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{x}_0 + \boldsymbol{x}_0'\boldsymbol{LML}'\boldsymbol{x}_0 - \gamma_0'\boldsymbol{ML}'\boldsymbol{x}_0 - \boldsymbol{x}_0'\boldsymbol{LM}'\gamma_0 + \boldsymbol{\gamma_0}'\boldsymbol{M}\boldsymbol{\gamma_0} \right) \\
&= \sigma^2 \left(\boldsymbol{x}_0' \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{x}_0 + \boldsymbol{x}_0'\boldsymbol{LML}'\boldsymbol{x}_0 - 2\gamma_0'\boldsymbol{ML}'\boldsymbol{x}_0 + \boldsymbol{\gamma_0}'\boldsymbol{M}\boldsymbol{\gamma_0} \right) \\
&= \sigma^2 \left(\boldsymbol{x}_0' \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{x}_0 + \left[\boldsymbol{L}'\boldsymbol{x}_0 - \gamma_0\right]' \boldsymbol{M} \left[\boldsymbol{L}'\boldsymbol{x}_0 - \gamma_0\right] \right) \\
&\ge \sigma^2 \boldsymbol{x}_0' \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{x}_0,
\end{aligned}$$

as long as $\boldsymbol{M}$ is positive definite. It remains to prove that $\boldsymbol{M}$ is p.d. We can note that $\boldsymbol{M}^{-1}$ is positive definite, because $\boldsymbol{w}'\boldsymbol{Mw} = \|\boldsymbol{RZw}\|^2$ for all $\boldsymbol{w}$. (This uses $\boldsymbol{R}'\boldsymbol{R} = \boldsymbol{R}$.) Therefore, $\boldsymbol{M}^{-1}$ has all strictly positive eigenvalues, which means that the same is true for $\boldsymbol{M}$. This proves that $\boldsymbol{M}$ is p.d. and the result follows.

#6, page 136: Let $\boldsymbol{a} = (a_0, a_1)'$; we are to find simultaneous $100(1-\alpha)$%-CI's for all linear combinations $\boldsymbol{a}'\boldsymbol{\beta}$. An answer is,

$$\boldsymbol{a}'\hat{\boldsymbol{\beta}} \pm \sqrt{2S^2\boldsymbol{a}'\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{a}F_{2,n-2}(\alpha)}.$$

But here,

$$\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} = \frac{1}{ns_x^2}\begin{pmatrix} \overline{x^2} & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}.$$

Therefore,

$$\boldsymbol{a}'\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{a} = \frac{1}{ns_x^2}\left(a_0^2\overline{x^2} - 2a_0a_1\bar{x} + a_1^2\right).$$

#5, page 196: Recall that $H_0: \mu_1 = \cdots = \mu_p$, which imposes $q = p-1$ linear restrictions. In class we proved that

$$\text{RSS}_{H_0} - \text{RSS} = \sum_{i=1}^{p} J_i\left(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}\right)^2 = \sum_{i=1}^{p}\sum_{j=1}^{J_i}\left(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}\right)^2.$$

Therefore, by Theorem 4.1(ii) (p. 100),

$$\begin{aligned} E\left[\sum_{i=1}^{p}\sum_{j=1}^{J_i}\left(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}\right)^2\right] &= \sigma^2(p-1) + \left(\text{RSS}_{H_0} - \text{RSS}\right)_{\boldsymbol{Y}=E[\boldsymbol{Y}]} \\ &= \sigma^2(p-1) + \sum_{i=1}^{p}\sum_{j=1}^{J_i}\left(\mu_i - \frac{1}{p}\sum_{i=1}^{p} J_i\mu_i\right)^2. \end{aligned}$$

This answers (a). For (b), note that

$$\begin{aligned} E\left[\sum_{i=1}^{n}\sum_{j=1}^{J_i}\left(Y_{ij} - \bar{Y}_{i\cdot}\right)^2\right] &= \sum_{i=1}^{n}\sum_{j=1}^{J_i} E\left[\left(Y_{ij} - \bar{Y}_{i\cdot}\right)^2\right] \\ &= \sum_{i=1}^{n} J_i E\left[\left(Y_{i1} - \bar{Y}_{i\cdot}\right)^2\right] \qquad \text{(why?)} \\ &= \sum_{i=1}^{n} J_i\frac{\sigma^2}{J_i - 1} = \sigma^2\sum_{i=1}^{n}\frac{J_i}{J_i - 1}. \end{aligned}$$