Math 6010, Fall 2004: Midterm 1 Solutions

(1) *Let* $S = \{\boldsymbol{x} \in \mathbf{R}^n : \ x_1 + x_2 = c\}$ *where* $c \in \mathbf{R}$ *is fixed.*

(a) *Find all constants* c *that make* S *a subspace of* $\mathbf{R}^n$.

Solution. For S to be a subspace we need to know that: (i) whenever $\boldsymbol{x}, \boldsymbol{y} \in S$ then so is $\boldsymbol{x} + \boldsymbol{y}$; (ii) for all $\alpha \in \mathbf{R}$ and $\boldsymbol{x} \in S$, $\alpha\boldsymbol{x} \in S$. The unique solution is $c = 0$.

(b) *Compute the projection matrix* $\mathbf{P}_S$. *Use this to project the vector* $\boldsymbol{x} = (1, 0, \ldots, 0)'$ *onto the subspace* S *(with an appropriate choice of* c*).*

Solution. First, we need a basis for S: If $\boldsymbol{x} \in S$ then

$$\boldsymbol{x} = x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Therefore, the following $n \times (n-1)$ matrix is a basis-matrix for S:

$$\boldsymbol{V} = \left(\begin{array}{c|c} 1 & \\ -1 & \\ \hline & \mathbf{I}_{n-2} \end{array}\right),$$

where the blank spaces are all zeros. Thus, $\boldsymbol{V}'\boldsymbol{V} = \left(\begin{array}{c|c} 2 & \\ \hline & \mathbf{I}_{n-2} \end{array}\right)$. [This is $(n-1) \times (n-1)$.] This is easy to invert:

$$(\boldsymbol{V}'\boldsymbol{V})^{-1} = \left(\begin{array}{c|c} \frac{1}{2} & \\ \hline & \mathbf{I}_{n-2} \end{array}\right).$$

Therefore,

$$\mathbf{P}_S = \boldsymbol{V}(\boldsymbol{V}'\boldsymbol{V})^{-1}\boldsymbol{V}' = \left(\begin{array}{cc|c} \frac{1}{2} & -\frac{1}{2} & \\ -\frac{1}{2} & \frac{1}{2} & \\ \hline & & \mathbf{I}_{n-2} \end{array}\right).$$

In particular, $\mathbf{P}_S(1, 0, \ldots, 0)' = (\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0)'$.

(2) *Consider the model,*

$$y = \beta_1 + \beta_2 \sin(x) + \varepsilon.$$

(a) *Compute the design matrix to obtain the linear model* $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

Solution. Let $s_i = \sin(x_i)$ for $i = 1, \ldots, n$. Then, the answer is the following $n \times 2$ matrix:

$$\boldsymbol{X} = \begin{pmatrix} 1 & s_1 \\ \vdots & \vdots \\ 1 & s_n \end{pmatrix}.$$

(b) *Compute the LSE* $\widehat{\boldsymbol{\beta}}$ *of* $\widehat{\boldsymbol{\beta}}$.

Solution. Evidently,

$$\boldsymbol{X}'\boldsymbol{X} = n\begin{pmatrix} 1 & \overline{s} \\ \overline{s} & \overline{s^2} \end{pmatrix}, \text{ so that } (\boldsymbol{X}'\boldsymbol{X})^{-1} = \frac{1}{n\mathrm{Var}(\boldsymbol{s})}\begin{pmatrix} \overline{s^2} & -\overline{s} \\ -\overline{s} & 1 \end{pmatrix},$$

where $\mathrm{Var}(\boldsymbol{s}) = \overline{s^2} - (\overline{s})^2$ is the (sample) variance of $s_1, \ldots, s_n$. Now $\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$. But

$$(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' = \frac{1}{n\mathrm{Var}(\boldsymbol{s})}\begin{pmatrix} \overline{s^2} - s_1\overline{s} & \cdots & \overline{s^2} - s_n\overline{s} \\ s_1 - \overline{s} & \cdots & s_n - \overline{s} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \widehat{\boldsymbol{\beta}} &= \frac{1}{n\mathrm{Var}(\boldsymbol{s})}\begin{pmatrix} \left(\overline{s^2} - s_1\overline{s}\right) y_1 + \cdots + \left(\overline{s^2} - s_n\overline{s}\right) y_n \\ (s_1 - \overline{s})\, y_1 + \cdots + (s_n - \overline{s})\, y_n \end{pmatrix} \\ &= \frac{1}{\mathrm{Var}(\boldsymbol{s})}\begin{pmatrix} \overline{s^2} \cdot \overline{y} - \overline{s} \cdot \overline{sy} \\ \overline{sy} - \overline{s} \cdot \overline{y} \end{pmatrix}. \end{aligned}$$

This is more than enough. But it can be simplified into more familiar objects. Because $\overline{s^2} = \mathrm{Var}(\boldsymbol{s}) + (\overline{s})^2$,

$$\frac{\overline{s^2} \cdot \overline{y} - \overline{s} \cdot \overline{sy}}{\mathrm{Var}(\boldsymbol{y})} = \overline{y} - \overline{s}\frac{\overline{s} \cdot \overline{y} - \overline{sy}}{\mathrm{Var}(\boldsymbol{s})},$$

which is equal to $\overline{y} - \overline{s}\mathrm{Corr}(\boldsymbol{s}, \boldsymbol{y})\mathrm{SD}(\boldsymbol{s})/\mathrm{SD}(\boldsymbol{y})$. Therefore,

$$\widehat{\boldsymbol{\beta}} = \begin{pmatrix} \mathrm{Corr}(\boldsymbol{s}, \boldsymbol{y})\frac{\mathrm{SD}(\boldsymbol{s})}{\mathrm{SD}(\boldsymbol{y})} \\ \overline{y} - \overline{s}\mathrm{Corr}(\boldsymbol{s}, \boldsymbol{y})\frac{\mathrm{SD}(\boldsymbol{s})}{\mathrm{SD}(\boldsymbol{y})} \end{pmatrix}.$$

This is the more familar form of simple linear regression, but with the x_i's replaced everywhere by $s_i = \sin(x_i)$.