ON RATIONALITY OF COMPLEX ALGEBRAIC VARIETIES

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Lecture Notes

Contents

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Introduction

Rationality has been a central topic in the field since the Lüroth problem was formulated at the end of the 18th century. The only rational curve is the projective line, and Castelnuovo's criterion settles the two-dimensional case. However, determining which varieties are rational in higher dimensions can be a challenging problem.

The purpose of these lectures is to explore two topics related to rationality of algebraic varieties, touching on the history behind these topics and introducing the necessary tools. These lectures are not meant to provide a comprehensive overview of the problem of rationality in algebraic geometry, and many related topics are here omitted; this includes topics that have recently witnessed significant advances.

These lectures will focus on the following two topics: birational rigidity and deformations of rational varieties. We will prove Iskovskikh–Manin's theorem on quartic threefolds and its generalization to higher dimensions, and Kontsevich–Tschinkel's specialization result on rationality. Along the way, we will also review some classical theorems on surfaces such as Noether–Castelnuovo's factorization theorem of Cremona transformations and Segre and Manin's theorems on rationality of cubic surfaces over non-closed fields, and discuss some interesting examples on families of rational (and nonrational) varieties.

Unless otherwise specified, we will restrict to working over the complex numbers. Some of the results hold over more general fields, but others rely on resolution of singularities and vanishing theorems hence require working over an algebraically closed field of characteristic 0, so there is no much loss in just assuming that the ground field is $\mathbb{C}$.

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1. Lecture 1: The Lüroth problem and Noether's factorization

This first lecture is devoted to a quick discussion of the Lüroth problem, which has prompted a lot of research on the subject of rationality, and a review of the proof of the factorization theorem of planar Cremona transformations.

1.1. **Rationality and unirationality.**

Definition 1.1. A variety X is *rational* if it is birational to a projective space $\mathbb{P}^n$, and is *unirational* is there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$.

These notions can be interpreted in terms of the field of rational functions $\mathbb{C}(X)$ of X. Rationality is equivalent to the fact that $\mathbb{C}(X) \simeq \mathbb{C}(x_1, \dots, x_n)$ is a purely transcendental extension of the ground field, and unirationality to the existence of a field extension $\mathbb{C}(X) \subset \mathbb{C}(x_1, \dots, x_n)$ over $\mathbb{C}$.

A well-known theorem of Lüroth (stated here over the complex numbers) says that any field K with sits in between $\mathbb{C}$ and $\mathbb{C}(x)$ is equal to $\mathbb{C}(y)$ for some element $y \in K$. Geometrically, this corresponds to the basic fact that $\mathbb{P}^1$ is the only smooth projective unirational curve, an elementary consequence of the Hurvitz formula. Note that, over the complex numbers, this a purely topological statement: there are no covering maps from a sphere to a compact orientable surface of positive genus that ramify over a finite set, which can be easily checked by comparing Euler characteristics.

Lüroth's theorem motivated the following problem.

Problem 1.2. *Is every unirational variety actually rational?*

The answer to this question is affirmative in dimension 2, thanks to Castelnuovo's criterion of rationality. Recall that the latter states that a smooth projective surface X is rational if and only if $q(X) = P_2(X) = 0$, where $q(X) = h^1(X, \mathcal{O}_X) = h^0(X, \Omega_X)$ is the irregularity of X and $P_2(X) = h^0(X, \omega_X^{\otimes 2})$ is the second plurigenus.

Now, suppose X is a smooth projective surface that is unirational, and fix a dominant rational map $\mathbb{P}^2 \dashrightarrow X$. After resolving indeterminacies, we obtain a dominant morphism $f\colon Y \to X$ where Y is a rational smooth projective surface. Since X is smooth, f induces injections $H^0(X, \Omega_X) \hookrightarrow H^0(Y, \Omega_Y)$ and $H^0(X, \omega_X^{\otimes 2}) \hookrightarrow H^0(Y, \omega_Y^{\otimes 2})$, hence $q(X) \le q(Y)$ and $P_2(X) \le P_2(Y)$. As these invariants vanish on Y, they must vanish on X, and we conclude that X is rational by Castelnuovo's criterion.

In higher dimensions the question remained open for several decades. The problem was finally solved, in the negative, by three independent teams in the early seventies. The following results imply that the Lüroth problem fails in dimension three.

Theorem 1.3 (Iskovskikh–Manin [IM71])**.** *Smooth quartic threefolds $X_4 \subset \mathbb{P}^4$ are nonrational.*

Theorem 1.4 (Clemens–Griffiths [CG72])**.** *Smooth cubic threefolds $X_3 \subset \mathbb{P}^4$ are nonrational.*

Theorem 1.5 (Artin–Mumford [AM72])**.** *There are examples of unirational but not rational three-dimensional conic bundles.*

All smooth cubic threefolds $X = X_3 \in \mathbb{P}^4$ are unirational. Pick two general hyperplane sections $S_i = H_i \cap X$, and define $S_1 \times S_2 \dashrightarrow X$ to be the map that associates to a general pair of points $(q_1, q_2) \in S_1 \times S_2$ the third point of intersection of the line $\overline{q_1 q_2}$ with X. Since S_i are rational surfaces, this shows that X is unirational.[1] It is unknown whether all smooth quartic threefolds $X_4 \in \mathbb{P}^4$ are unirational, but some of them are known to be, as examples were constructed by Segre [Seg60]. Therefore, each of the above theorems provides a counterexample to the Lüroth problem.

Rationality is disproved in these theorems in three very different ways. The common strategy is to look for a birational invariant and prove that such invariant of the variety at hand differs from the corresponding invariant computed from the projective space. The invariants used in the above theorems are, in order:

- The group of birational automorphisms of X.
- The intermediate Jacobian of X, modulo Jacobians of curves.
- The Brauer group of X.

Each method has its own benefits and limitations.

There are several related problems that remain open, the most famous asking whether every rationally connected variety is unirational. The answer is generally expected to be

[1] This argument extends to higher dimensions by taking as hyperplanes H_i the embedded tangent spaces of two general points of X. It was further extended to cubic hypersurfaces with a rational point over any field in [Kol02].

negative. While some strategies to tackle this question have been proposed, it seems that we are still far from answering this question.

In this lectures, we will look at the theorem of Iskovshikh–Manin and its higher dimensional generalization. The nonrationality of smooth quartic threefolds $X_4 \subset \mathbb{P}^4$ was first claimed by Fano [Fan07, Fan15], and while his arguments were incomplete, his strategy was eventually picked up and successfully implemented by Iskovskikh and Manin. Fano's approach was inspired by the proof of Noether's theorem on the structure of the Cremona group of $\mathbb{P}^2$, which we are going to review next.

1.2. **Factorization of planar Cremona maps.**

Theorem 1.6 (Noether Factorization [Noe72, Cas01])**.** *The Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by linear transformations and the standard quadratic transformation*

$$\chi\colon (x:y:z) \dashrightarrow (yz:xz:xy).$$

For ease of exposition, we shall work without fixing coordinates, but allowing instead to take standard quadratic transformations centered at any triple of distinct non-collinear points of $\mathbb{P}^2$. The freedom in choosing the base points absorbs the role of the linear transformations among the generators of the Cremona group.

Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. This map is defined by a two-dimensional linear system $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^2}(r)|$ of curves of degree r with no fixed components. Note that ϕ is an automorphism if and only if $r = 1$.

Suppose that ϕ is not an isomorphism. A minimal sequence of point-blowups

$$f\colon Y = X_{k+1} \xrightarrow{f_k} X_k \to \cdots \to X_1 \xrightarrow{f_1} X_0 = \mathbb{P}^2$$

resolving the indeterminacies of ϕ determines a series of *base points* $p_0, p_1, \dots, p_k$, possibly some infinitely near to others: the centers p_i of the blowups $f_i\colon X_{i+1} \to X_i$. We denote by m_i the multiplicity at the point p_i of the proper transform of $\mathcal{H}$ to X_i. We can assume that the sequence of blowups is ordered so that

$$m_0 \ge m_1 \ge \cdots \ge m_k > 0.$$

Noether's idea to prove the theorem is that taking a standard quadratic tranformation χ centered at points of large multiplicity should lower the degree of the map [Noe72]. The basic computation is the following. Suppose for a moment that the three points p_0, p_1, p_2 are distinct on $\mathbb{P}^2$. The key observation in Noether's argument is that

$$m_0 + m_1 + m_2 > r.$$

This inequality is obtained by showing that if one writes $m_0 = r - 2a$ and let b be the largest index such that $m_b > a$, then $b \ge 2$ (a detailed proof of this will be given later in Lemma 1.7 without assuming that the points are distinct). Note that the three points p_0, p_1, p_2 cannot be collinear, so we can take the standard quadratic transformation χ centered at these points. By precomposing ϕ with χ^{-1} (which is the same as χ), one obtains a new birational map

$$\phi' = \phi \circ \chi^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

of degree

$$r' = 2r - m_0 - m_1 - m_2 < r.$$

In particular, this operation lowers the degree of the map. One says that χ *untwists* the map ϕ. A recursive application of this process would eventually reduce ϕ to a linear transformation, thus providing the required factorization.

The issue with this approach is that, in general, p_0, p_1, p_2 may fail to be distinct in $\mathbb{P}^2$, and one may not be able to find three distinct points whose multiplicities exceed, together, the degree of the map. One must work with infinitely near points. After several attempted proofs, including those of Noether and Clifford which turned out to be fallacious as pointed out by Segre [Seg01], a complete proof of Noether's theorem was finally given by Castelnuovo [Cas01].

Here we present a later proof, due to Alexander [Ale16], which is in some sense closer to the original idea of Noether. We present it here in a slight reformulation, more in the logical structure than in the computations. We first prove that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by linear transformations, the standard quadratic transformation χ, and the quadratic transformation

$$\omega\colon (x:y:z) \dashrightarrow (x^2:xy:yz).$$

Theorem 1.6 will then follow by observing that ω itself factors as a composition of linear and standard quadratic transformations.

Note that ω has three base points q_0, q_1, q_2, with q_1 infinitely near q_0 and q_2 not lying on the line passing through q_0 with tangent direction q_1. Specifically, $q_0 = (0:0:1)$ and q_1 is the tangent direction at q_0 along the line $y = 0$ (which can be check by setting $z = 1$ and observing that the linear system has base ideal (x^2, y) in the (x, y)-chart), and $q_2 = (0:1:0)$ (which can be check by setting $y = 1$ and observing that the linear system has base ideal (x, z) in the (x, z)-chart). If n_0, n_1, n_2 are the multiplicities of $\mathcal{H}$ at these points, then the map $\phi' = \phi\circ\omega^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ has degree $r' = 2r - n_0 - n_1 - n_2$. As we are already doing for χ, we will work without fixing coordinates and allow ω to be centered to any triple of points q_0, q_1, q_2 with the above properties.

Proof of Theorem 1.6. Keeping the above notation, let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational transformation of degree $r > 1$, defined by a linear system $\mathcal{H}$. Let $p_0, \dots, p_k$ the base points of $\mathcal{H}$, and $m_0, \dots, m_k$ be their multiplicities, ordered as before:

$$m_0 \ge m_1 \ge \cdots \ge m_k > 0.$$

Let E_i be the exceptional divisor of the blowup $f_i\colon X_{i+1} \to X_i$ centered at p_i, and let F_i be the pullback of E_i to $Y = X_{k+1}$. Finally, let $D \in \mathcal{H}$ be a general member, and let D_Y denote its proper transform on Y. Note that the rational map ϕ lifts, via $f\colon Y \to X_0 = \mathbb{P}^2$, to a morphism $g = \phi\circ f\colon Y \to \mathbb{P}^2$, and D_Y is the pullback, via g, of a general line in $\mathbb{P}^2$.

We set

$$a = a(\phi) := \frac{r - m_0}{2},$$

and define

$$b = b(\phi) := \max\{i \mid m_i > a\}.$$

Lemma 1.7. $b \ge 2$.

Proof. The fact that a general member of $\mathcal{H}$ comes from a general line of $\mathbb{P}^2$ translates into the following two equations:

$$\sum_{i=0}^{k} m_i^2 = r^2 - 1 \quad\text{and}\quad \sum_{i=0}^{k} m_i = 3r - 3$$

Indeed, writing $f^*D = D_Y + \sum_{i=0}^k m_i F_i$ and observing that $F_i \cdot D_Y = m_i$ for every i and $D_Y^2 = 1$, we have

$$r^2 = D^2 = D \cdot f_* D_Y = f^* D \cdot D_Y = D_Y^2 + \sum_{i=0}^k m_i (F_i \cdot D_Y) = 1 + \sum_{i=0}^k m_i^2,$$

since since $K_Y = f^* K_X + \sum_{i=0}^k F_i$ and $K_Y \cdot D_Y = -3$, we have

$$3r = -K_X \cdot D = -K_Y \cdot D_Y + \sum_{i=0}^k (F_i \cdot D_Y) = 3 + \sum_{i=0}^k m_i.$$

Subtracting a times the second equation from the first, we get

$$\sum_{i=0}^k m_i(m_i - a) = (r-1)(r-3a+1).$$

Dropping all the nonpositive terms in the left-hand-side (namely, those indexed from $b+1$ to k) and subtracting $3a-1$ from the right-hand-side, we get

$$\sum_{i=0}^b m_i(m_i - a) > r(r-3a) = r(m_0 - a).$$

Here we used the fact that $3a - 1 > 0$, which holds because $m_0 \le r$. Moving the first term of the sum to the other side, this gives

$$\sum_{i=1}^b m_i(m_i - a) > (r - m_0)(m_0 - a).$$

Note that we have $m_i \le 2a$ for all $i \neq 0$, as otherwise we would have $m_0 + m_i > r$, forcing the line through p_0 and p_i to be in the base locus of $\mathcal{H}$, which is impossible since $\mathcal{H}$ is free in codimension one. Using this and the identity $r - m_0 = 2a$, the last formula implies that

$$\sum_{i=1}^b (m_i - a) > (m_0 - a),$$

and since $m_i \le m_0$ for all i, this forces $b \ge 2$. Therefore

$$m_0 + m_1 + m_2 > (r - 2a) + a + a = r,$$

which proves the lemma. $\square$

This lemma says that the first three points p_0, p_1, p_2 have multiplicities

$$m_0 \ge m_1 \ge m_2 > a.$$

The proof now goes by induction on the vector $(a, b) \in \frac{1}{2}\mathbb{N} \times \mathbb{N}$ with respect to the lexicographic order. We think of this vector as a measure of the complexity of ϕ. We study two cases, according to the relative position of p_0, p_1, p_2.

Case 1. Suppose that p_0, p_1, p_2 are distinct points in $\mathbb{P}^2$. Note that they cannot be collinear, since $m_0 + m_1 + m_2 > m_0 + 2a = r$. Let $\phi' := \phi \circ \chi^{-1}$ where χ is the standard quadratic transformation centered at these three points, and let $(a', b') := (a(\phi'), b(\phi'))$.

We denote by p_0', p_1', p_2' the base points of χ^{-1}, and let m_0', m_1', m_2' be the multiplicities at these points of the linear system $\mathcal{H}'$ defining ϕ'. Note that $\mathcal{H}'$ is the homaloidal transform of $\mathcal{H}$, it has degree

$$r' = 2r - m_0 - m_1 - m_2,$$

and

$$m_h' = r - m_i - m_j \quad \text{for} \quad \{h, i, j\} = \{0, 1, 2\}.$$

Each point p_i, for $3 \le i \le k$ is either mapped to one of p_0', p_1', p_2', or it remains a distinct point of multiplicity m_i of $\mathcal{H}'$. No other base points of $\mathcal{H}'$ are created. The question now is whether $\mathcal{H}'$ achieves its largest multiplicity at p_0'.

If the largest multiplicity of $\mathcal{H}'$ is not achieved at p_0', then it is larger than m_0' and we have

$$2a' < r' - m_0' = r - m_0 = 2a.$$

On the contrary, if m_0' is the largest multiplicity of $\mathcal{H}'$, then $a' = a$. In this case, however, we get

$$m_i' = r - m_0 - m_j = 2a - m_j < a \quad \text{for} \quad \{i, j\} = \{1, 2\},$$

and therefore $b' < b$. Either way, we have $(a', b') < (a, b)$, and we can apply induction.

Case 2. Suppose now that p_0, p_1, p_2 are not distinct points in $\mathbb{P}^2$. We fix a general point $q \in \mathbb{P}^2$.

If p_1 not is infinitely near p_0, then we let $\phi' := \phi \circ \chi^{-1}$ where χ is the standard quadratic transformation centered at p_0, p_1, q, and denote by p_0', p_1', q' the base points of χ^{-1}. If p_1 is infinitely near p_0, then we let $\phi' := \phi \circ \omega^{-1}$ where ω is the quadratic transformation centered at p_0, p_1, q, and denote by p_0', p_1', q' the base points of ω^{-1}.

Let $\mathcal{H}'$ denote the linear system defining ϕ', let r' be its degree, and let m_0', m_1', n' be the multiplicities of $\mathcal{H}'$ at the points p_0', p_1', q'. Note that $r' = 2r - m_0 - m_1$, $m_i' = r - m_i$ for $\{i, j\} = \{1, 2\}$, and

$$n' = r - m_0 - m_1 = 2a - m_1 < a.$$

Furthermore, as in Case 1, ϕ' does not create new base points, and those p_i, for $3 \le i \le k$, that are not mapped to any of p_0', p_1', q' maintain the same multiplicity m_i in $\mathcal{H}'$.

Let $(a', b') := (a(\phi'), b(\phi'))$. If the largest multiplicity of $\mathcal{H}'$ is larger than m_0, then we get $a' < a$. Otherwise, we have $a' = a$, but then $b' < b$ since $n' < a = a'$. Therefore, $(a', b') < (a, b)$, and induction applies.

To conclude the proof, we are left to verify that ω, given in some fixed coordinates by $(x : y : z) \dashrightarrow (x^2 : xy : yz)$, is the composition of linear transformations and the standard quadratic transformation χ given, in the same coordinates, by $(x : y : z) \dashrightarrow (yz : xz : xy)$. This is well explained in [KSC04, Page 200], which we follow in our computations. Changing coordinates in ω using the automorphism α defined by $(x : y : z) \mapsto (x : x + y : z)$, we obtain the transformation

$$\alpha^{-1} \circ \omega \circ \alpha : (x : y : z) \dashrightarrow (x^2 : xy : (x + y)z).$$

Untwisting this with χ, we get

$$\alpha^{-1} \circ \omega \circ \alpha \circ \chi : (x : y : z) \dashrightarrow (yz : xz : x(x + y)).$$

This is a standard quadratic transformation in suitable coordinates, since its base locus is three distinct points. In fact, it is equal to $\beta \circ \chi \circ \alpha$ where β is the linear transformation given by $(x : y : z) \mapsto (x - y : x : z)$. Therefore we have

$$\omega = \alpha^{-1} \circ \beta \circ \chi \circ \alpha \circ \chi \circ \alpha^{-1},$$

which gives the required factorization. □

2. Lecture 2: The method of maximal singularities

In this lecture, we discuss how Fano's work on quartic threefolds and the theorems of Segre and Manin on cubic surfaces over non-closed fields eventually led to the formulation of the method of maximal singularities. The material is mainly based on [dF14].

2.1. **Fano's intuition.** Fano believed that an approach similar to the proof of the factorization of planar Cremona maps, applied to a smooth quartic threefold $X = X_4 \subset \mathbb{P}^4$, should prove that any given birational automorphism $\phi\colon X \dashrightarrow X$ must be a regular automorphism of X. This means that the group of birational automorphisms $\mathrm{Bir}(X)$ of X is just the automorphism group of X:

$$\mathrm{Bir}(X) = \mathrm{Aut}(X).$$

Note that this is a strong condition, as in general there is only an inclusion and the group of birational automorphisms of a variety can be much larger than the automorphism group. Since the automorphism group of X is finite [MM63] whereas the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ of $\mathbb{P}^3$ is not, proving the above equality would immediately imply that X is not rational. We will see that the same approach can be adapted to show directly that there cannot be any birational map $\phi\colon X \dashrightarrow \mathbb{P}^3$, which will prove in a more straightforward way that X is not rational. In fact, the proof will show that X is *birationally superrigid* a stronger notion than rationality which will be introduced later.

Fano's intuition turned out to be correct in the end, but his tools were inadequate for the task. The main difference between the setting of Noether's theorem and the one considered by Fano has to do with dimension: while to study birational maps on surfaces it suffices to use multiplicities as a measure of their singularities, measuring singularities of birational maps on threefolds and higher dimensional varieties requires more sophisticated tools that were simply not available during Fano's times.

Before the correct method to study birational maps on quartic threefolds was finally identified by Iskovkikh and Manin, Fano's more naïve approach via multiplicities was successfully applied by Segre and Manin to study the birational geometry of cubic surfaces over nonclosed fields. This not only provided a testing ground for the method, but also led the way to the correct formulation in arbitrary dimensions, which became known as the *method of maximal singularities*. At the center of this method is the *Noether–Fano inequality*.

2.2. **Cubic surfaces of Picard number one.** Let κ be a perfect field, and let $X_\kappa \subset \mathbb{P}^3_\kappa$ be a smooth cubic surface. Since the canonical class of X_κ is defined over κ, the Picard group $\mathrm{Pic}(X_\kappa)$ contains the hyperplane class $\mathcal{O}_{X_\kappa}(1)$. The surface has Picard number one if and only if $\mathrm{Pic}(X_\kappa)$ is generated by $\mathcal{O}_{X_\kappa}(1)$.

Segre proved that if the Picard number is one then X_κ is not rational. This result was later revisited by Manin who proved that if two such cubics are birational to each other, then they are projectively equivalent. A nice survey of these results can be found in the recent treatment [KSC04]. A further adaptation of the proof gives the following more precise result.

Theorem 2.1 ([Seg51, Man66, dF14])**.** *Let $X_\kappa \subset \mathbb{P}^3_\kappa$ be a smooth cubic surface of Picard number one over a perfect field κ. Suppose that there is a birational map*

$$\phi_\kappa\colon X_\kappa \dashrightarrow X'_\kappa$$

where X'_κ is a smooth projective surface that is either a Del Pezzo surface of Picard number one, or a conic bundle over a curve S'_κ. Then X'_κ is a cubic surface of Picard number one, and there is a birational automorphism $\beta_\kappa \in \mathrm{Bir}(X_\kappa)$ such that $\phi_\kappa \circ \beta_\kappa \colon X_\kappa \to X'_\kappa$ is a projective equivalence. In particular, X_κ is nonrational.

Proof. If X'_κ is a conic bundle over a curve S'_κ then we fix a divisor A'_κ on X'_κ given by the pullback of a very ample divisor on S'_κ. If X'_κ is a Del Pezzo surface of Picard number one, then we set $S'_\kappa = \operatorname{Spec}\kappa$ and $A'_\kappa = 0$. We fix an integer $r' \geq 1$ such that $-r'K_{X'_\kappa} + A'_\kappa$ is very ample. Since the Picard group of X_κ is generated by the class of $-K_{X_\kappa}$, there is a positive integer r such that

$$(\phi_\kappa)_*^{-1}(-r'K_{X'_\kappa} + A'_\kappa) \sim -rK_{X_\kappa}.$$

Let $\overline{\kappa}$ be the algebraic closure of κ, and denote $X = X_{\overline{\kappa}}$, $X' = X'_{\overline{\kappa}}$, $S' = S'_{\overline{\kappa}}$, $A' = A'_{\overline{\kappa}}$ and $\phi = \phi_{\overline{\kappa}}$. Let $D' \in |-r'K_{X'} + A'|$ be a general element, and let

$$D = \phi_*^{-1}D' \in |-rK_X|.$$

We split the proof in two cases.

Case 1. Assume that $\mathrm{mult}_x(D) > r$ for some $x \in X$.

We use the existence of such points of high multiplicity to construct a suitable birational involution of X (defined over κ) that, pre-composed to ϕ, untwists the map. This part of the proof is similar, in spirit, to the proof of Noether's theorem on $\mathrm{Bir}(\mathbb{P}^2)$.

The Galois group of $\overline{\kappa}$ over κ acts on the base points of ϕ and preserves the multiplicities of D at these points. Since D belongs to a linear system with zero-dimensional base locus and $\deg D = 3r$ (as a cycle in $\mathbb{P}^3$), there are at most two points at which D has multiplicity larger than r, and the union of these points is preserved by the Galois action. If there is only one point $x \in X$ (not counting infinitely near ones), then x is defined over κ. Otherwise, we have two distinct points x, y on X whose union $\{x, y\} \subset X$ is defined over κ.

We now untwist ϕ by pre-composing with a suitable birational involution $\alpha_1 \in \mathrm{Bir}(X)$, constructed as follows. Let $g \colon \widetilde{X} \to X$ be the blowup of X at the points of multiplicty larger than r. If there is only one such point x, the blowup resolves the indeterminacies of the rational map $X \dashrightarrow \mathbb{P}^2$ given by the linear system $|\mathcal{O}_X(1) \otimes \mathfrak{m}_x|$, which lifts to a double cover $h \colon \widetilde{X} \to \mathbb{P}^2$. The Galois group of this cover is generated by an involution $\widetilde{\alpha}_1$ of $\widetilde{X}$, which descends to a birational involution α_1 of X. If there are two points x, y of multiplicity greater than r, then g resolves the indeterminacies of the rational map $X \dashrightarrow \mathbb{P}^3$ given by the linear system $|\mathcal{O}_X(2) \otimes \mathfrak{m}_x^2 \otimes \mathfrak{m}_y^2|$, which lifts to a double cover $h \colon \widetilde{X} \to Q \subset \mathbb{P}^3$ where Q is a smooth quadric surface. In this case, we denote by $\widetilde{\alpha}_1$ the Galois involution of the cover and by α_1 the birational involution induced on X. In both cases, α_1 is defined over κ. Therefore the composition

$$\phi_1 = \phi \circ \alpha_1^{-1} \colon X \dashrightarrow X'$$

is defined over κ and hence is given by a linear system in $|-r_1K_X|$ for some r_1 (note that $\alpha_1^{-1} = \alpha_1$).

In either case, we have $r_1 < r$. To see this, let E be the exceptional divisor of $g \colon \widetilde{X} \to X$, and let L be the pullback to $\widetilde{X}$ of the hyperplane class of $\mathbb{P}^2$ (resp., of $Q \subset \mathbb{P}^3$) by h. Note that $L \sim g^*(-K_X) - E$ by construction, and $g_*\widetilde{\alpha}_{1*}E \sim -sK_X$ for some s, since this cycle is defined over κ. We observe that there are no lines in X passing through a point of multiplicty larger than r, since D belongs to a movable linear system cut out by forms of degree r. It follows that the involution $\widetilde{\alpha}_1$ does not stabilize the divisor E. This means that $g_*\widetilde{\alpha}_{1*}E$ is supported on a nonempty curve, and therefore $s \geq 1$. If m is the multiplicity of D at x (and hence at y in the second case) and $\widetilde{D}$ is the proper transform of D on $\widetilde{X}$, then $\widetilde{D} + (m-r)E \sim rL$. Applying $(\widetilde{\alpha}_1)_*$ to this divisor and pushing down to X, we obtain $\alpha_{1*}D \sim -r_1K_X$ where $r_1 = r-(m-r)s < r$ since $m > r$. Therefore, this operation lowers the degree of the equations defining the map.

Let $D_1 = \phi_{1*}^{-1}D' \in |-r_1K_X|$. If $\mathrm{mult}_x(D_1) > r_1$ for some $x \in X$, then we proceed as before to construct a new involution α_2, and proceed from there. Since the degree decreases each time, this process stops after finitely many steps. It stops precisely when, letting

$$\phi_i = \phi \circ \alpha_1^{-1} \circ \ldots \circ \alpha_i^{-1} \colon X \dashrightarrow X'$$

and $D_i = \phi_{i*}^{-1}D' \in |-r_iK_X|$, we have $\mathrm{mult}_x(D_i) \leq r_i$ for every $x \in X$. Note that ϕ_i is defined over κ. Then, replacing ϕ by ϕ_i, we reduce to the next case.

Case 2. Assume that $\mathrm{mult}_x(D) \leq r$ for every $x \in X$.

Taking a sequence of blow-ups, we obtain a resolution of indeterminacy

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$

with Y smooth. Write

$$\begin{aligned} K_Y + \tfrac{1}{r'}D_Y &= p^*(K_X + \tfrac{1}{r'}D) + E' \\ &= q^*(K_{X'} + \tfrac{1}{r'}D') + F' \end{aligned}$$

where E' is p-exceptional, F' is q-exceptional, and $D_Y = p_*^{-1}D = q_*^{-1}D'$. Since X' is smooth and D' is a general hyperplane section, we have $F' \geq 0$ and $\mathrm{Supp}(F') = \mathrm{Ex}(q)$. Note that $K_{X'} + \frac{1}{r'}D'$ is nef. Intersecting with the image in Y of a general complete intersection curve $C \subset X$ we see that $(K_X + \frac{1}{r'}D) \cdot C \geq 0$, and this implies that $r \geq r'$.

Next, we write

$$\begin{aligned} K_Y + \tfrac{1}{r}D_Y &= p^*(K_X + \tfrac{1}{r}D) + E \\ &= q^*(K_{X'} + \tfrac{1}{r}D') + F \end{aligned}$$

where, again, E is p-exceptional and F is q-exceptional. The fact that $\mathrm{mult}_x(D) \leq r$ for all $x \in X$ implies that $E \geq 0$. Intersecting this time with the image in Y of a general complete intersection curve C' in a general fiber of $X' \to S'$, we get $(K_{X'} + \frac{1}{r}D') \cdot C' \geq 0$, and therefore $r = r'$. Note also that $E = E'$ and $F = F'$.

The difference $E - F$ is numerically equivalent to the pullback of A'. In particular, $E-F$ is nef over X and is numerically trivial over X'. Since $p_*(E-F) \leq 0$, the Negativity Lemma, applied to p, implies that $E \leq F$. Similarly, since $q_*(E-F) \geq 0$, the Negativity Lemma, applied to q, implies that $E \geq F$. Therefore $E = F$. This means that A' is numerically trivial, and hence $S' = \mathrm{Spec}\,\overline{\kappa}$. Furthermore, we have $\mathrm{Ex}(q) \subset \mathrm{Ex}(p)$, and therefore the inverse map $\sigma = \phi^{-1} \colon X' \dashrightarrow X$ is a morphism.

To conclude, just observe that if $S'_\kappa = \mathrm{Spec}\,\kappa$ then X'_κ must have Picard number one. But σ, being the inverse of ϕ, is defined over κ. It follows that σ is an isomorphism, as otherwise it would increase the Picard number. Therefore X'_κ is a smooth cubic surface of Picard number one.

Since we can assume without loss of generality to have picked $r' = 1$ to start with, we conclude that, after the reduction step performed in Case 1, ϕ is a projective equivalence defined over κ. The second assertion of the theorem follows by taking $\beta_\kappa = \alpha_1^{-1} \circ \dots \circ \alpha_i^{-1}$, which is defined over κ. □

2.3. **The Noether–Fano inequality.** The proof of Theorem 2.1 already captures, in the simplest possible setting, the main ideas behind the *method of maximal singularities*, a sophisticated method to study birational links among Fano manifolds of Picard number 1 and, more generally, among Mori fiber spaces. A key step is to correctly interpret the condition on multiplicities of divisors on surfaces used in the second part of the proof so that the same argument works in higher dimensions. What is needed is to replace multiplicity with a more sophisticated measure of singularity of a divisor on a variety, which we define next.

Definition 2.2. Let X be a normal variety and Δ a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that the pair (X, Δ) is *canonical* if for any proper birational morphism $p\colon Y \to X$ with Y smooth, the $\mathbb{Q}$-divisor $K_Y - p^*(K + \Delta) + p_*^{-1}\Delta$ is linearly equivalent to an effective p-exceptional divisor. Taking $K_X = p_*K_Y$, this is the same as requiring that $K_Y - p^*(K + \Delta) + p_*^{-1}\Delta$ is effective.

Remark 2.3. One can prove that if X is a smooth surface, then (X, Δ) is canonical if and only if $\mathrm{mult}_x(\Delta) \le 1$ for every point $x \in X$.

Mori fiber spaces are the terminal objects produced by the minimal model program within the class of uniruled varieties, and are defined as follows.

Definition 2.4. A *Mori fiber space* is a normal projective variety X with $\mathbb{Q}$-factorial terminal singularities, equipped with an extremal Mori contraction $f\colon X \to S$ of fiber type. This means that f is a proper morphism with connected fibers and relative Picard number $\rho(X/S) = 1$, the anticanonical class $-K_X$ is f-ample, and $\dim S < \dim X$.

In dimension two, they consists of $\mathbb{P}^2$ and ruled surfaces, and any birational equivalence among them can be factored as a sequence of elementary transformations. In higher dimensions, the factorization process is more complicated, and is studied via the *Sarkisov program*. This consists of a series of elementary links which are used, very much in spirit as in Case 1 of the proof of Theorem 2.1, to untwist the map. We shall not discuss the Sarkisov program here. For an introduction to the program, we recommend [Cor00].

A new phenomenon occurring in higher dimensions is that some Mori fiber structures are unique in their birational class. This leads to the notions of birational rigidity and birational superrigidity. Here we focus on the latter.

Definition 2.5. A Mori fiber space $f\colon X \to S$ is *birationally superrigid* if every birational map $\phi\colon X \dashrightarrow X'$ from X to another Mori fiber space $f'\colon X' \to S'$ is a fiberwise isomorphism (i.e., ϕ is an isomorphism, and there is an isomorphism $\psi\colon S \to S'$ such that $f' \circ \phi = \psi \circ f$).

The arguments in Case 2 of the proof of Theorem 2.1 extend to give sufficient conditions to establish birational superrigidity. The following theorem, which lies at the heat of the menthod of maximal singularities, is proven in [IM71] in the special case where X is a smooth quartic threefold in $\mathbb{P}^4$ and $X' = X$. The general statement is due to [Cor00], whose proof relies of some results from the minimal model program.

Theorem 2.6 (Noether–Fano Inequality [IM71, Cor00]). *Let $\phi\colon X \dashrightarrow X'$ be a birational map between two Mori fiber spaces $f\colon X \to S$ and $f'\colon X' \to S'$. Fix a projective embedding of X' given by a linear system $\mathcal{H}' \subset |-r'K_{X'} + A'|$ where r' is a positive rational number and A' is the pullback of an ample divisor on S' (we set $A' = 0$ if S' is a point), so that ϕ is defined by a movable linear system $\mathcal{H}$. Let r the positive rational number such that*

$$\mathcal{H} \subset |-rK_X + A|$$

where A is the pull-back of a $\mathbb{Q}$-divisor on S. Assume that:

(1) *A is nef, and*
(2) *for general $D \in \mathcal{H}$, the pair $(X, \frac{1}{r}D)$ is canonical.*

Then ϕ is an isomorphism, and there is an isomorphism $\psi\colon S \to S'$ such that $f' \circ \phi = \psi \circ f$.

Proof. First note that we have $\mathcal{H} \subset |-rK_X + A|$ for some r and A because $\rho(X/S) = 1$, and $r > 0$ because $-K_X$ is ample over S. The projective embedding is given by a very ample linear system $\mathcal{H}'$. Note that $D = \phi_*^{-1}(D')$ for a general $D' \in \mathcal{H}'$.

Let

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$

be a resolution of singularities. Note that the exceptional loci $\mathrm{Ex}(p)$ and $\mathrm{Ex}(q)$ have pure codimension 1. Let $D_Y = q_*^{-1}D$ (which is the same as q^*D). Note that $D = p_*D_Y$ and $D_Y = p_*^{-1}D$. Write

$$\begin{aligned} K_Y + \tfrac{1}{r'}D_Y &= p^*(K_X + \tfrac{1}{r'}D) + E' \\ &= q^*(K_{X'} + \tfrac{1}{r'}D') + F' \end{aligned}$$

where E' is p-exceptional and F' is q-exceptional. Since X' has terminal singularities and D' is a general hyperplane section, we have $F' \geq 0$ and $\mathrm{Supp}(F') = \mathrm{Ex}(q)$. Since $K_{X'} + \frac{1}{r'}D'$ is numerically equivalent to the pullback of A', which is nef, we have $(K_X + \frac{1}{r'}D) \cdot C \geq 0$ for a general complete intersection curve C in a general fiber of f. This implies that $r \geq r'$.

Next, we write

$$\begin{aligned} K_Y + \tfrac{1}{r}D_Y &= p^*(K_X + \tfrac{1}{r}D) + E \\ &= q^*(K_{X'} + \tfrac{1}{r}D') + F \end{aligned}$$

where E is p-exceptional and F is q-exceptional. Assume that the pair $(X, \frac{1}{r}B)$ is canonical. Since D is defined by a general element of the linear system of divisors cutting out B, and $r \geq 1$, it follows that $(X, \frac{1}{r}D)$ is canonical. This means that $E \geq 0$. Since $K_X + \frac{1}{r}D$ is numerically equivalent to the pullback of A, which is nef by hypothesis, we have $(K_{X'} + \frac{1}{r}D') \cdot C' \geq 0$ for a general complete intersection curve C' in a general fiber of f', and therefore $r = r'$. Note, in particular, that $E = E'$ and $F = F'$, and hence

$$E - F \sim_{\mathbb{Q}} q^*A' - p^*A.$$

Since $E - F$ is p-nef and $p_*(E - F) \leq 0$, we have $E \leq F$ by the Negativity Lemma. Similarly, since $F - E$ is q-nef and $q_*(F - E) \leq 0$, we have $F \leq E$. Therefore $E = F$. This means that $p^*A \sim_{\mathbb{Q}} q^*A'$, and therefore, since A' is the pullback of a very ample divisor on S', there is a (proper) morphism $\psi\colon S \to S'$ fitting in a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[d, "f"'] & & X' \arrow[d, "f'"] \\ S \arrow[rr, "\psi"] & & S' \end{tikzcd}$$

Computing the Picard number of Y in two ways, we get

$$\begin{aligned} \rho(Y) &= \rho(Y/X) + 1 + \rho(S/S') + \rho(S) \\ &= \rho(Y/X') + 1 + \rho(S). \end{aligned}$$

Note that $\mathrm{Ex}(q) \subset \mathrm{Ex}(p)$ since F contains every q-exceptional divisor in its support, and therefore $\rho(Y/X') \le \rho(Y/X)$. It follows that $\rho(Y/X') = \rho(Y/X)$ and $\rho(S/S') = 0$. The second identity implies that ψ is an isomorphism, since S' is normal. The first identity implies that $\mathrm{Ex}(p) = \mathrm{Ex}(q)$, and thus the difference $p^*D - q^*D'$ is q-exceptional. Since D is ample, this implies that ϕ is a (proper) morphism. Keeping in mind that X and X' have the same Picard number and X' is normal, it follows that ϕ is an isomorphism too. $\square$

As we already mentioned in the previous lecture, the method of maximal singularities was introduced by Iskovskikh and Manin in [IM71] to prove that any smooth quartic threefold $X = X_4 \subset \mathbb{P}^4$ has $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ and hence is not rational (see Theorem 1.3). Essentially the same proof yields the stronger property that X is birationally superrigid.

Iskovshikh–Manin's theorem was later extended to higher dimensions, to the statement that every smooth hypersurface $X \subset \mathbb{P}^N$ of degree N, for $N \ge 4$, is birationally superrigid. The proof requires techniques that will be introduced in the next few lectures, but we shall give a quick proof of the theorem of Iskovskikh and Manin as soon as we have enough tools in place to settle the three-dimensional case.

3. Lecture 3: Multiplicities

In order to apply the Noether–Fano inequality to concrete situations (for example, to Fano hypersurfaces in projective spaces, the case of interest in these lectures), one needs to relate conditions on singularities of pairs to other measures of singularities such as multiplicities, which can be controlled in terms of the degrees of the equations. In this lecture we review some basic definitions and properties related to multiplicities.

3.1. **Multiplicities.** There are historically different approaches to multiplicities. Here we follow [Ful98].

Definition 3.1. If $(R, \mathfrak{m})$ is a Noetherian local ring of dimension n and $\mathfrak{a} \subset R$ is an $\mathfrak{m}$-primary ideal, then the Hilbert polynomial of $\mathfrak{a}$ has degree n. Writing the leading term in the form $\frac{e}{n!} \cdot m^n$, the coefficient $e =: e(\mathfrak{a})$ is called the *Hilbert–Samuel multiplicity* (or just *Samuel multiplicity*) of $\mathfrak{a}$.

By definition,

$$e(\mathfrak{a}) = \lim_{m\to\infty} \frac{l(R/\mathfrak{a}^m)}{m^n/n!}.$$

where the length is computed as an R-module.

If $X = \operatorname{Spec} R$ and $\widetilde{X} = \operatorname{Bl}_{\mathfrak{a}} X \to X$ is the blow-up of $\mathfrak{a}$, then

$$e(\mathfrak{a}) = (-1)^{n-1} A^n$$

where A is the Cartier divisor on $\widetilde{X}$ such that $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-A)$. This can be seen by comparing Hilbert polynomials. Let $P_{\mathfrak{a}}$ be the Hilbert polynomial of $\mathfrak{a}$, given by $P_{\mathfrak{a}}(m) = l(R/\mathfrak{a}^m)$ for m large enough, and let P_A be the Hilbert polynomial of A in the embedding given by $\mathcal{O}_A(-A)$, so that $P_A(m) = \dim_k(\mathfrak{a}^m/\mathfrak{a}^{m+1})$ for m large enough, where k is the residue field of R (note that $A = \operatorname{Proj} \oplus_{m\ge 0} \mathfrak{a}^m/\mathfrak{a}^{m+1}$ and $\mathcal{O}_A(-A) = \mathcal{O}_A(1)$ in this projective embedding). By definition, $e(\mathfrak{a})$ is the normalized leading coefficient of $P_{\mathfrak{a}}$ (normalized by $n!$, where n is the degree of the polynomial), and the normalized leading coefficient of P_A (normalized by $(n-1)!$) is the degree of A with respect to $\mathcal{O}_A(-A)$, which is equal to $(-A)^{n-1} \cdot A$. Using the relation

$$l(R/\mathfrak{a}^m) = \sum_{t=0}^{m-1} l(\mathfrak{a}^t/\mathfrak{a}^{t+1})$$

we see that $P_{\mathfrak{a}}(m) \sim \sum_{k=0}^{m-1} P_A(k)$, hence $\Delta P_{\mathfrak{a}}(m) \sim P_A(m)$ for $m \to \infty$, which implies that $P_{\mathfrak{a}}$ and P_A have the same normalized leading coefficients (cf. [Har77, Proposition I.7.3]).

Example 3.2. If $R = k[x_1, \dots, x_n]$ and $\mathfrak{a} = (x_1^{d_1}, \dots, x_n^{d_n}) \subset R$, then

$$e(\mathfrak{a}) = d_1 \dots d_n.$$

This can be seen directly using the property that since $\mathfrak{a}$ is generated by a regular sequence, we have $e(\mathfrak{a}) = l(R/\mathfrak{a})$ [Ful98, Example 4.3.5(c)]. Alternatively, it can be deduced as a special case of the next example.

Example 3.3. Let $R = k[x_1, \dots, x_n]$ and $\mathfrak{a} \subset R$ is monomial ideal. Any monomial $x^u = \prod x_i^{u_i} \in \mathfrak{a}$ determines a point $u = (u_1, \dots, u_n) \in \mathbb{Z}^n_{\ge 0}$. The convex hull of these points in $\mathbb{R}^n_{\ge 0}$ is called the *Newton polyhedron* of $\mathfrak{a}$, and is denoted by $P(\mathfrak{a})$. Assume that $\mathfrak{a}$ is $\mathfrak{m}$-primary where $\mathfrak{m} = (x_1, \dots, x_n)$. This is equivalent to the condition that the region $\mathbb{R}^n_{\ge 0} \setminus P(\mathfrak{a})$ is bounded. Then

$$e(\mathfrak{a}) = n! \operatorname{vol}\big(\mathbb{R}^n_{\ge 0} \setminus P(\mathfrak{a})\big)$$

where the volume is computed with respect to the Euclidean measure. The fact that $e(\mathfrak{a})$ is greater than or equal to the right-hand-side can be seen easily by observing that for every m

$$l(R/\mathfrak{a}^m) = |\{u \in \mathbb{Z}^n_{\ge 0} \mid x^u \notin \mathfrak{a}\}| \ge |\mathbb{Z}^n_{\ge 0} \setminus P(\mathfrak{a}^m)| \ge \operatorname{vol}\big(\mathbb{R}^n_{\ge 0} \setminus P(\mathfrak{a}^m)\big) = m^n \operatorname{vol}\big(\mathbb{R}^n_{\ge 0} \setminus P(\mathfrak{a})\big).$$

The reverse inequality can be proven by showing that in both inequalities the difference is in $o(m^n)$. We refer to [Tei88] for more details.

Definition 3.4. The *multiplicity* $e_p(X)$ of a variety X at a point p is defined to be the Samuel multiplicity $e(\mathfrak{m}_p)$ of the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathcal{O}_{X,p}$. More generally, for any closed subscheme Z of a pure-dimensional scheme X and any irreducible component T of Z, the *multiplicity of X along Z at T* is defined to be the Samuel multiplicity of the ideal $\mathcal{I}_Z \cdot \mathcal{O}_{X,T}$ determined by Z in the local ring $\mathcal{O}_{X,T}$. It is denoted by $e_Z(X)_T$. If $Z = T$, then we just write $e_T(X)$.

Using the formula for Samuel multiplicity as an intersection number, one can see that the multiplicity $e_Z(X)_T$ of a pure dimensional scheme X along a closed subscheme Z at an irreducible component T of Z is the coefficient of $[T]$ in the Segre class of the normal cone $C_Z X := \operatorname{Spec} \bigoplus_{m \geq 0} \mathfrak{a}^m/\mathfrak{a}^{m+1}$ to Z in X (cf. [Ful98, Example 4.3.4]).

Example 3.5. If D is an effective Cartier divisor on a variety X of dimension at least 2, and $p \in X$ is a regular point, then $e_p(D)$ is simply the multiplicity of a generator of the ideal of D in the local ring at p [Ful98, Example 4.3.9].

Example 3.6. If $X \subset \mathbb{A}^n$ is a pure dimensional Cohen–Macaulay scheme and $p \in X$ is a closed point, then

$$e_p(X) = i(p, X \cdot L; \mathbb{A}^n),$$

the intersection multiplicity of X with a general linear subspace $L \subset \mathbb{A}^n$ of codimension equal to the dimension of X. This follows from [Ful98, Example 4.3.5(c) and Proposition 7.1]. It relates the definition of multiplicity adopted here with more classical definitions. Note, however, that equality does not hold in general if X is not Cohen–Macaulay.

Multiplicities are semicontinuous, in the following sense. If X be a pure-dimensional scheme, then for every irreducible closed set $T \subset X$ there is a nonempty open set $T^\circ \subset T$ such that $e_p(X) \geq e_T(X)$ for every point $p \in X$, and equality holds if $p \in T^\circ$ [Ben70, Theorem (4)]. We will tacitly use this fact without further mention.

Proposition 3.7. *Let Z be a pure-dimensional closed Cohen-Macaulay subscheme of $\mathbb{P}^m$ of positive dimension.*

(1) *If H meets properly the embedded tangent cone of Z at a point p, then $e_p(Z \cap H) = e_p(Z)$.*
(2) *Given a hyperplane in the dual space $\mathcal{H} \subset (\mathbb{P}^m)^\vee$, if $H \in \mathcal{H}$ is general enough, then $e_p(Z \cap H) = e_p(Z)$ for every $p \in Z \cap H$.*

Proof. We can assume that $Z \neq \mathbb{P}^m$. Consider any linear subspace $L \subset \mathbb{P}^m$ of dimension $\dim L = m - \dim Z$ that meets properly the embedded tangent cone of Z at p. Then the component of $Z \cap L$ at p is zero-dimensional, and we have $e_p(Z) = l(\mathcal{O}_{Z \cap L, p})$ by [Ful98, Proposition 7.1 and Corollary 12.4]. This implies (1).

At any point $p \in Z$, the fiber over p of the conormal variety of Z, viewed as a linear subspace of $(\mathbb{P}^n)^\vee$, contains the dual variety of every component of the embedded projective tangent cone $C_p Z$ of Z at p (e.g., see [Kle86, page 219]). It follows then by (1) that $e_p(Z \cap H) = e_p(Z)$ as long as H is chosen outside the dual variety $Z_i^\vee$ of each irreducible component Z_i of Z. To conclude, it suffices to observe that $Z_i^\vee$ cannot contain any hyperplane of $(\mathbb{P}^m)^\vee$, since it is irreducible of dimension $\leq m-1$, and $Z_i^{\vee\vee} = Z_i$ is not a point. □

3.2. **Bezout-type multiplicity bounds.** We will use the following two bounds on multiplicities coming from Bezout's theorem.

Proposition 3.8. *Let $Z = V \cap H_1 \cap \cdots \cap H_r$ be the complete intersection of a variety $V \subset \mathbb{P}^n$ with r hypersurfaces $H_i \subset \mathbb{P}^n$. Then for every irreducible component T of Z we have*

$$e_Z(V)_T \leq \deg V \cdot \prod \deg H_i.$$

Proof. Letting $D_i = H_i \cap V$, we have

$$e_Z(V)_T = i(T, D_1 \cdot \ldots \cdot D_r; V)$$

by [Ful98, Example 7.1.10(a)], where the right-hand-side is the intersection multiplicity of the divisors D_i at the generic point of T. By the definition of intersection product, this is bounded above by the intersection product $(D_1 \cdot \ldots \cdot D_r)_V$, which in turns is equal to $(H_1 \cdot \ldots \cdot H_r \cdot [V])_{\mathbb{P}^n}$, see [Ful98, Example 2.4.3]. Therefore the proposition follows by Bezout's theorem [Ful98, Propostion 8.4]. □

Proposition 3.9. *Let $V, W \subset \mathbb{P}^n$ be two subvarieties of complementary dimension intersecting properly. Then*

$$\sum_{p \in V \cap W} e_p(V) \leq (V \cdot W) = \deg V \cdot \deg W.$$

Proof. We have $\sum e_p(V) \cdot e_p(W) \leq (V \cdot W)$ by [Ful98, Corollary 12.4] and $(V \cdot W) = \deg V \cdot \deg W$ by [Ful98, Propostion 8.4]. We conclude by observing that $e_p(W) \geq 1$ for every $p \in V \cap W$. □

3.3. **Pukhlikov's multiplicity bound.** The next result gives a multiplicity bound on divisors on smooth hypersurfaces that we will use in the proof of Iskivskikh–Manin's theorem. It provides a simplification to the original proof which required a careful analysis of multiplicities along low degree curves on quartic threefolds.

Proposition 3.10 ([Puk98])**.** *Let $X \subset \mathbb{P}^N$ be a smooth hypersurface and $D \in |\mathcal{O}_X(m)|$ an effective divisor on X cut out by a form of degree m. Then $e_C(D) \leq m$ for every irreducible curve C on X.*

Proof. First, note that the bound is trivially satisfied if either $C \not\subseteq \operatorname{Supp}(D)$ or $\deg X = 1$. Thus, we may assume that $C \subseteq \operatorname{Supp}(D)$ and $\deg X = d \geq 2$.

Let $\pi_p \colon \mathbb{P}^N \smallsetminus \{p\} \to H_p \cong \mathbb{P}^{N-1}$ be the linear projection from a point $p = (a_0, \ldots, a_N) \in \mathbb{P}^N \smallsetminus X$, and set $f_p = \pi_p|_X \colon X \to H_p$. If $F(x_0, \ldots, x_N) = 0$ is the homogeneous equation defining X, then the relative canonical divisor K_{X/H_p} is cut on X by the equation $\sum_{i=0}^N a_i \frac{\partial F}{\partial x_i} = 0$, and moves freely in a base point free linear system since X is smooth.

Given a curve $C \subset X$, pick p general so that $C \to f_p(C)$ is a birational map. We can also assume that $f_p^{-1}(f_p(C))$ is generically reduced, and write

$$\operatorname{Supp}(f_p^{-1}(f_p(C))) = C \cup R$$

where R is a curve of degree $(d-1)\deg C$. We call R the residual curve of C with respect to π_p. This curves meets C at every point where C intersects the locus where f_p ramifies. By taking a general projection, we may assume that the ramification divisor K_{X/H_p} intersects C in $(d-1)\deg C$ points, $d-1$ being the degree of the form cutting this divisor on X. Then, using Proposition 3.9, we get

$$m(d-1)\deg C = (D \cdot R) \geq \sum_{q \in C \cap R} e_q(D) \geq e_C(D)(d-1)\deg C.$$

This implies that $e_C(\alpha) \leq m$, proving the proposition. □

In order to tackle the higher dimensional analogue of Iskovskikh–Manin's theorem, we will need the following generalization of Proposition 3.10 to higher codimensional subschemes.

Proposition 3.11 ([Puk02])**.** *Let $X \subset \mathbb{P}^N$ be a smooth hypersurface and $Z \subset X$ a pure-dimensional closed subscheme of codimension $k \le \frac{1}{2}\dim X$. Assume that the cycle $[Z]$ associated to Z is numerically equivalent to $m \cdot c_1(\mathcal{O}_X(1))^k$ for some $m \in \mathbb{N}$. Then*

$$\dim\{x \in X \mid e_x(Z) > m\} < k.$$

Proof. For simplicity, we will prove the case $k = 2$ of this proposition which is the only case, along with Proposition 3.10, that we will use. The proof of the general case is just an iteration of the same process.

So, assume that $Z \subset X$ is a scheme of pure codimension 2 with $[Z] \equiv m \cdot c_1(\mathcal{O}_X(1))^2$. We need to prove that $e_S(Z) \le m$ for any irreducible surface $S \subset X$.

Fix any irreducible surface S, and assume that $S \subset Z$ as otherwise there is nothing to prove. Keeping the notation introduced in Proposition 3.10, we fix a general point $p \in \mathbb{P}^N$ so that $f_p^{-1}(f_p(S))$ is generically reduced, and write

$$\mathrm{Supp}(f_p^{-1}(f_p(S))) = S \cup R,$$

where this time R is the residual surface of S under the projection f_p. Note that R has degree $(d-1)\deg S$. Then we fix a second general point $p' \in \mathbb{P}^N$ and write

$$\mathrm{Supp}(f_{p'}^{-1}(f_{p'}(R)) = R \cup R',$$

where R' is the residual surface of R under the projection $f_{p'}$. By constriction, $\deg R = (d-1)\deg S$ and hence $\deg R' = (d-1)^2 \deg S$.

Let for short $K = K_{X/H_p}$ be the relative canonical divisor of f_p and $K' = K_{X/H_{p'}}$ be the relative canonical divisor of $f_{p'}$. Since K and K' are two general members of a base-point-free linear system on X, the intersection $S \cap K \cap K'$ is a reduced finite set of cardinality $(d-1)^2 \deg S$. As we already observed in the proof of Proposition 3.10, every point q in the support of $S \cap K$ will belong to $S \cap R$, and similarly, if q is in the support of $R \cap K'$ then it belongs to $R \cap R'$. This implies that

$$S \cap K \cap K' \subset S \cap R',$$

hence $S \cap R'$ contains at least $(d-1)^2 \deg S$ distinct points.

By taking the first projection general, one can ensure that $R \not\subset Z$, so that $R \cap Z$ is 1-dimensional. Taking the second projection general will then ensure that $R' \cap Z$ is a finite set, meaning that R' intersects properly every irreducible component of Z. These assertions require a computation with secant varieties and joints that we omit here.

Using that $Z \equiv m \cdot c_1(\mathcal{O}_X(1))^k$ and $S \cap R'$ contains a set of cardinality $(d-1)^2 \deg S$, we conclude by Proposition 3.9 that

$$m(d-1)^2 \deg S = (Z \cdot R') \ge \sum_{q \in S \cap R'} e_q(Z) \ge e_S(Z)(d-1)^2 \deg S,$$

hence $e_S(Z) \le m$. □

4. Lecture 4: Singularities of pairs and log canonical thresholds

In this lecture we prove Iskovskikh–Manin's theorem. We will follow the proof given in [Cor00], which relies on inversion of adjunction to translate the information coming out from the Noether–Fano inequality into a condition on log canonical thresholds and deduce from there a bound on multiplicity that can be compared to the degrees of equations involved using Bezout's theorem.

4.1. **Log discrepancies.** For simplicity, we restrict to the case of a smooth ambient variety X, since this is the only setting we will need to consider. Analogous definitions hold when we replace X with the spectrum of the local ring of a variety at a regular point.

Definition 4.1. A divisor E *over* X is a prime divisor on some smooth birational model $f\colon Y \to X$. Its image $f(E) \subset X$ is called the *center* and denoted by $c_X(E)$. The divisor E is said to be *exceptional* over X if $\operatorname{codim}(c_X(E), X) \geq 2$.

If $Y' \to Y$ is another smooth birational model and E has a proper transform E' in Y' (e.g., Y' is proper over Y), then we sometime identify E with E', as they define the same valuation. We recall that a divisor E over X defines a divisorial valuation ord_E on X which can be applied to non-zero ideal sheaves $\mathfrak{a} \subset \mathcal{O}_X$ by setting $\operatorname{ord}_E(\mathfrak{a}) := \operatorname{ord}_E(\mathfrak{a} \cdot \mathcal{O}_{Y,E})$. For any rational number r, we consider the formal expression $\mathfrak{a}^r$, and set $\operatorname{ord}_E(\mathfrak{a}^r) = r \operatorname{ord}_E(\mathfrak{a})$.

Definition 4.2. For us, a *pair* (X, Z) consists of a smooth variety X and a $\mathbb{Q}$-*scheme* Z on X, namely, a formal $\mathbb{Q}$-linear combination $Z = \sum c_i Z_i$ of proper closed subschemes $Z_i \subset X$. A pair (X, Z) is *effective* if $c_i \geq 0$ for all i. A *log resolution* of a pair (X, Z) is a proper birational morphism $f\colon Y \to X$, with Y smooth, such that $\operatorname{Ex}(f)$ and $f^{-1}(Z_i)$ are divisors and $\operatorname{Ex}(f) + \sum f^{-1}(Z_i)$ has simple normal crossings. We denote $f^{-1}(Z) := \sum c_i f^{-1}(Z_i)$.

Note that for any $\mathbb{Q}$-divisor Δ, (X, Δ) is a pair in this sense. Sometime we write $(X, \mathfrak{a}^r)$ for the pair $(X, rZ(\mathfrak{a}))$. If $X = \operatorname{Spec} R$, then we also consider pairs of the form $(R, \mathfrak{a}^r)$ where $\mathfrak{a} \subset R$ is an ideal. The definitions given below extend to this setting in the obvious way.

Definition 4.3. Let E be a divisor over X as above. Given a pair (X, Z), we define the *log discrepancy* of E over (X, Z) to be the number

$$a_E(X, Z) := \operatorname{ord}_E(K_{Y/X}) + 1 - \operatorname{ord}_E(Z)$$

where $K_{Y/X}$ is the relative canonical divisor of f (the unique exceptional divisor linearly equivalent to $K_Y - f^*K_X$) and $\operatorname{ord}_E(Z) := \sum c_i \operatorname{ord}(\mathcal{I}_{Z_i})$.

Definition 4.4. We say that a pair (X, Z) is *terminal* (resp., *canonical*) if $a_E(X, Z) > 1$ (resp., $a_E(X, Z) \geq 1$) for all exceptional divisors E over X. The pair (X, Z) is said to be *klt* (resp., *log canonical*) if $a_E(X, Z) > 0$ (resp., $a_E(X, Z) \geq 0$) for all divisors E over X.

Theorem 4.5 (Inversion of Adjunction)**.** *Let (X, Z) be an effective pair, $H \subset X$ a smooth prime divisor that is not contained in the support of Z, and $Z|_H = \sum c_i Z_i \cap H$. Suppose there is an exceptional divisor E over X such that $c_X(E) \cap H \neq \emptyset$ and $a_E(X, Z + H) \leq 0$. Then for every irreducible component W of $c_X(E) \cap H$ there exists a divisor F over Y with $c_Y(F) \supset W$ such that $a_F(Y, Z|_H) \leq 0$.*

Sketch of Proof. We may assume that E is a divisor on a log resolution $f\colon Y \to X$ of $(X, Z + H)$. If E intersects the proper transform H' of H, then we can take F to be an irreducible component of $E \cap H'$ dominating W. Then $\operatorname{ord}_F(Z|_H) = \operatorname{ord}_E(Z)$, and using the adjunction formulas $\omega_H = \omega_X \otimes \mathcal{O}_H(H)$ and $\omega_{H'} = \omega_Y \otimes \mathcal{O}_{H'}(H')$ it is easy to check that $a_F(H, Z|_H) = a_E(X, Z + H)$.

In general, E may be disjoint from H'. Nonetheless, if we write

$$\lceil K_{Y/X} - f^{-1}(Z + H) \rceil = P - N$$

where P and N are effective $\mathbb{Q}$-divisors with no common components, then one can prove using vanishing theorems (a relative version of Kawamata–Viehweg's vanishing theorem, to be precise) that the map $\mathrm{Supp}(N) \to X$ has connected fiber. This property is known as the *connectedness theorem* and is due to Shokurov in dimension 2 [Sho92] and Kollár in all dimensions [Kol92]. Note that both E and the proper transform H' of H are contained in the support of N. Using the connectedness of the fiber over the generic point η_W of W, which intersects both E and H', we can find a prime divisor $E' \neq H'$ in the support of N such that $E' \cap H' \cap f^{-1}(\eta_W) \neq \emptyset$. The condition that E' is an irreducible component of N implies that $a_{E'}(X, Z + H) \leq 0$. Taking F to be an irreducible component of $E' \cap H'$ intersecting $f^{-1}(\eta_W)$, we conclude as before that $a_F(H, Z|_H) \leq 0$. Note also that $c_H(F) \subset W$ since $F \cap f^{-1}(W) \neq \emptyset$. □

Remark 4.6. Theorem 4.5 applies for instance if there is a divisor E over X with $c_X(E) \subset H$ such that $a_E(X, Z) \leq 1$. Indeed in this case we have $\mathrm{ord}_E(H) \geq 1$, hence $a_E(X, Z+H) \leq 0$.

Proposition 4.7. *Let Δ be an effective $\mathbb{Q}$-divisor on a smooth variety X, and suppose that $a_E(X, \Delta) \leq 1$ for some prime exceptional divisor E over X. If T is the center of E in X, then $e_T(\Delta) \geq 1$.*

Proof. We can assume that E is a divisor of a log-resolution $f \colon X' \to X$ of (X, Δ). Pick a general point $p \in T$, and let $Y \subset X$ be a general complete intersection subvariety of codimension $\mathrm{codim}(Y, X) = \dim T$, passing through p. Then the proper transform Y' of Y meets E transversally, and we have $a_{E'}(Y, \Delta|_Y) \leq 1$ for any irreducible component E' of $E|_{Y'}$. Notice that $\dim Y \geq 2$. If $H \subset Y$ is a general hyperplane section through p, then $(H, \Delta|_H)$ is not klt near p by Theorem 4.5. Taking a general complete intersection curve $C \subset H$ through p, we see that $(C, \Delta|_C)$ is not klt at p. This is equivalent to $e_p(\Delta|_C) \geq 1$. On the other hand, by taking the hyperplanes cutting out C generally enough, we have $e_p(\Delta|_C) = e_p(\Delta)$. We conclude that $e_T(\Delta) \geq 1$. □

4.2. **Log canonical thresholds.** As before, let X be a smooth variety (or the spectrum of the local ring of a variety at a regular point).

Definition 4.8. The *log canonical threshold* of a nonzero ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ is defined by

$$\mathrm{lct}(\mathfrak{a}) = \sup\{t > 0 \mid (X, \mathfrak{a}^t) \text{ is log canonical}\},$$

If $Z \subset X$ is a proper closed subscheme and $x \in X$ is a point, then we denote $\mathrm{lct}(X, Z) := \mathrm{lct}(\mathcal{I}_Z)$ and $\mathrm{lct}_x(X, Z) := \mathrm{lct}(\mathcal{I}_Z \cdot \mathcal{O}_{X,x})$.

It follows by the definition that

$$\mathrm{lct}(\mathfrak{a}) = \inf_E \frac{\mathrm{ord}_E(K_{Y/X}) + 1}{\mathrm{ord}_E(\mathfrak{a})}$$

where the infimum is taken over all divisors E over X (note that the model Y depends on E). One can show, in fact, that it suffices to restrict the infimum on prime divisors on a log resolution of the pair $(X, \mathfrak{a})$, which shows that if $\mathfrak{a}$ is a proper ideal then the infimum is a minimum.

Example 4.9. Passing to the algebraic notation, let $R = k[x_1, \dots, x_n]$ and $\mathfrak{a} \subset R$ be a nonzero monomial ideal. For $t > 0$, we consider the polyhedron $tP(\mathfrak{a}) \subset \mathbb{R}^n_{\geq 0}$ obtained by rescaling the Newton polyhedron $P(\mathfrak{a})$ by t. Then, denoting $e = (1, \dots, 1)$, we have

$$\mathrm{lct}(\mathfrak{a}) = \sup\{t > 0 \mid e \in tP(\mathfrak{a})\}.$$

The proof uses toric geometry. Let $N = \mathbb{Z}^n$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, so that $P(\mathfrak{a}) \subset N_\mathbb{R}$. Denote by $M = \mathrm{Hom}(N, \mathbb{Z}) = \mathbb{Z}^n$ the dual lattice, and let $\langle \, , \rangle \colon N_\mathbb{R} \times M_\mathbb{R} \to \mathbb{R}$ be the canonical pairing. We consider $\mathbb{A}^n = \operatorname{Spec} R$ with the standard toric structure. The ideal $\mathfrak{a}$ defines an ideal sheaf on $\mathbb{A}^n$ which we will denote by the same symbol. Since $\mathfrak{a}$ is toric invariant, it suffices to check log discrepancies on toric invariant divisors over $\mathbb{A}^n$, or, equivalently, on monomial divisorial valuations on $\mathbb{A}^n$. The latter are parameterized precisely by the primitive points $v \in \mathbb{Z}^n_{\geq 0} \subset N$. That is, for any such v there exists a toric invariant divisor E over X such that $\operatorname{ord}_E(x^u) = \langle u, v\rangle$ for all $u \in N$, and conversely. For short, we write $v = \operatorname{ord}_E$. By definition we have

$$\operatorname{ord}_E(\mathfrak{a}) = \min\{\langle u, v\rangle \mid u \in P(\mathfrak{a})\},$$

and

$$\operatorname{ord}_E(K_{Y/X}) + 1 = \langle e, v\rangle$$

where $Y \to \mathbb{A}^n$ is any model given by a sequence of toric blow-ups where E appears as a divisor, which follows from the fact that, on a toric variety X, we have $K_X \sim -\sum D_i$ where the sum is over the toric invariant prime divisors on X. Therefore, $\operatorname{lct}(\mathfrak{a})$ is the supremum of the numbers t such that $\langle e, v\rangle \geq \langle tu, v\rangle$ for all prime elements $v \in \mathbb{Z}^n_{\geq 0}$ and all $u \in P(\mathfrak{a})$, and this is equivalent to the condition that $e \in tP(\mathfrak{a})$.

Example 4.10. For a concrete example, if $\mathfrak{a} = (x_1^{d_1}, \dots, x_n^{d_n}) \subset k[x_1, \dots, n_x]$ then

$$\operatorname{lct}(\mathfrak{a}) = \frac{1}{d_1} + \dots + \frac{1}{d_n}.$$

The following theorem relates Samuel multiplicity to log canonical threshold. We focus on the local setting, and let $R = \mathcal{O}_{X,x}$ be the local ring of a variety X at a (non-necessarily closed) regular point $x \in X$. Let $n = \dim R$, and denote by $\mathfrak{m} \subset R$ the maximal ideal.

If $n = 1$, then an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subset R$ is principal, generated by an element $h \in R$, and $e(\mathfrak{a}) = \operatorname{mult}(h) = 1/\operatorname{lct}(h) = 1/\operatorname{lct}(\mathfrak{a})$. In higher dimension there are two natural ways to generalize this relation, by either considering principal ideals or looking at $\mathfrak{m}$-primary ideals. In the first case one can show that

$$n \cdot \operatorname{mult}(h) \geq \frac{n}{\operatorname{lct}(h)} \geq \operatorname{mult}(h)$$

for any $h \in \mathfrak{m}$. As for the $\mathfrak{m}$-primary case, we have the following result which gives a lower-bound on Samuel multiplicity in terms of the log canonical threshold.[2]

Theorem 4.11 ([dFEM04])**.** *Let $(R, \mathfrak{m})$ be as above. Then, for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$,*

$$l(R/\mathfrak{a}) \geq \frac{1}{n!}\left(\frac{n}{\operatorname{lct}(\mathfrak{a})}\right)^n$$

and

$$e(\mathfrak{a}) \geq \left(\frac{n}{\operatorname{lct}(\mathfrak{a})}\right)^n.$$

Proof. The second formula in the theorem follows directly from the first using

$$e(\mathfrak{a}) = \lim_{m\to\infty} \frac{l(R/\mathfrak{a})}{m^n/n!}$$

and $\operatorname{lct}(\mathfrak{a}^m) = \frac{1}{m}\operatorname{lct}(\mathfrak{a})$. So, it suffices to prove the bound on the length of $R/\mathfrak{a}$.

[2]Examples show, on the contrary, that there cannot be upper bounds on Hilbert–Samuel multiplicity only in terms of the log canonical threshold if $n \geq 2$. Taking for instance $\mathfrak{a} = (x^2, xy, y^m) \subset \mathbb{C}[x, y]$, we see that $e(\mathfrak{a})$ can be made arbitrarily large while keeping $\operatorname{lct}(\mathfrak{a}) = 1$.

Passing to the completion, we can fix an isomorphism $\widehat{R} \cong k[[x_1, \dots, x_n]]$ such that each D_i is locally defined by $x_i = 0$, where k is the residue field of $(R, \mathfrak{m})$, hence restrict to the polynomial ring $k[x_1, \dots, x_n]$. After extending scalars, we can also replace k by its algebraically closed field. We can therefore assume without loss of generality that $R = k[x_1, \dots, x_n]$ where k is an algebraically closed field of characteristic zero. Note that $\mathfrak{a}$ carries through this reduction because it is $\mathfrak{m}$-primary. Furthermore, each step of the reduction preserves lengths and log canonical thresholds, so it suffices to prove the theorem in this setting.

We shall start by verifying that the inequality holds in the special case of monomial ideals. Suppose therefore for a moment that $\mathfrak{a}$ is a monomial ideal. Let $P(\mathfrak{a}) \subset \mathbb{R}^n_{\geq 0}$ be the Newton polytope of $\mathfrak{a}$, and let $(u_1, \dots, u_n)$ be the coordinates in $\mathbb{R}^n_{\geq 0}$. By Example 4.9, there is a bounded facet of $P(\mathfrak{a})$ such that, if $\sum_{i=1}^n a_i u_i = 1$ is the equation of the hyperplane supporting it, then

$$\operatorname{lct}(\mathfrak{a}) = \sum_{i=1}^n a_i.$$

On the other hand, the length of $R/\mathfrak{a}$ is equal to the number of integral points of $\mathbb{R}^n_{\geq 0} \setminus P(\mathfrak{a})$, hence it is bounded below by the number of integral points contained in the region cut out by $\sum_{i=1}^n a_i u_i < 1$ in $\mathbb{R}^n_{\geq 0}$. This number is greater or equal to the volume of this region, something that is easy to verify by covering the region by translates of the unit cube by the integral points in the region. This gives

$$l(R/\mathfrak{a}) \geq \frac{1}{n!} \cdot \prod_{i=1}^n \frac{1}{a_i}.$$

Applying the inequality between the arithmetic mean and the geometric mean of the set of numbers $\{a_i\}$, we get

$$\Big(\prod a_i\Big)^{1/n} \leq \frac{1}{n} \cdot \sum a_i.$$

Combining these formulas, we get

$$l(R/\mathfrak{a}) \geq \frac{1}{n!} \cdot \frac{1}{\prod a_i} \geq \frac{1}{n!}\Big(\frac{n}{\sum a_i}\Big)^n = \frac{1}{n!}\Big(\frac{n}{\operatorname{lct}(\mathfrak{a})}\Big)^n,$$

as claimed.

The proof of the general case consists in reducing to the monomial case, via a flat degeneration to monomial ideals. To this end, we fix a monomial order in R, and let $\operatorname{in}(\mathfrak{a})$ denote the monomial initial ideal of $\mathfrak{a}$. By semi-continuity of the log canonical threshold, we have $\operatorname{lct}(\operatorname{in}(\mathfrak{a})) \leq \operatorname{lct}(\mathfrak{a})$, and flatness ensures that $l(R/\operatorname{in}(\mathfrak{a})) = l(R/\mathfrak{a})$. Therefore we have

$$l(R/\mathfrak{a}) = l(R/\operatorname{in}(\mathfrak{a})) \geq \frac{1}{n!}\Big(\frac{n}{\operatorname{lct}(\operatorname{in}(\mathfrak{a}))}\Big)^n \geq \frac{1}{n!}\Big(\frac{n}{\operatorname{lct}(\mathfrak{a})}\Big)^n,$$

from the monomial case. □

Remark 4.12. From a geometric point of view, the theorem has the following formulation. Let X be a smooth variety, let $Z \subset X$ be a closed subscheme, and let T be an irreducible component of Z, of codimension n in X. Let $c = \operatorname{lct}_{\eta_T}(X, Z)$ be the log canonical threshold of (X, Z) at the generic point of T. Then

$$l(\mathcal{O}_{Z,T}) \geq \frac{1}{n!}\Big(\frac{n}{c}\Big)^n$$

and

$$e_Z(X)_T \geq \left(\frac{n}{c}\right)^n.$$

4.3. **Proof of Iskovskikh–Manin's theorem.** The properties proved thus far are enough to prove Iskovskikh–Manin's theorem (see Theorem 1.3), which we restated here in a stronger form. The proof follows the one given in [Cor00].

Theorem 4.13 ([IM71, Cor00]). *Every smooth quartic threefold $X_4 \subset \mathbb{P}^4$ is birationally superrigid.*

Proof. First note that $\mathrm{Pic}(X)$ is generated by the hyperplane class, which by adjunction is linearly equivalent to $-K_X$. This follows by the Lefschetz hyperplane theorem.

Suppose that $\phi\colon X \dashrightarrow X'$ is a birational map from a smooth quartic threefold $X \subset \mathbb{P}^4$ to a Mori fiber space $X' \to S'$, and assume by way of contradiction that ϕ is not an isomorphism. With the same notation as in Theorem 2.6, if D is a general member of the linear system $\mathcal{H} \subset |-rK_X|$ defining ϕ, then $(X, \frac{1}{r}D)$ is not canonical (note that in our setting $A = 0$). By Proposition 3.10, we have $e_C(D) \leq r$ for every curve $C \in X$, and this implies that $(X, \frac{1}{r}D)$ is canonical in dimension one by Proposition 4.7. Therefore there is a divisor E over X, with center equal to a point $p \in X$, such that $a_E(X, \frac{1}{r}D) < 1$.

Let now $Z = D_1 \cap D_2 \subset X$ be the complete intersection of two general divisors $D_1, D_2 \in \mathcal{H}$. Note that $a_E(X, \frac{1}{r}Z) = a_E(X, \frac{1}{r}D) < 1$. Let $S \subset X$ be the surface cut out by a general hyperplane through p, and denote by $Z|_S$ the intersection $Z \cap S$. Since $a_E(X, \frac{1}{r}Z + S) \leq a_E(X, \frac{1}{r}Z) - 1 < 0$, it follows by Theorem 4.5 that there is a divisor F over S, with center p, such that $a_F(S, \frac{1}{r}Z|_S) < 0$. This means that $\mathrm{lct}_p(S, Z|_S) < 1/r$, hence

$$e_{Z|_S}(S)_p > 4r^2.$$

by Theorem 4.11. On the other hand, since S has degree 4 and $Z|_S$ is a zero dimensional complete intersection scheme cut out on S by two equations of degree r, we have

$$e_{Z|_S}(S)_p \leq 4r^2$$

by Proposition 3.8, hence we get a contradiction. □

5. Lecture 5: Multiplier ideals

The main tool needed to extend Iskovskikh–Manin's theorem to higher dimensions is multiplier ideals. We start this section by defining these ideals and reviewing some properties. We will then turn to following higher dimensional version of the theorem of Iskovshikh and Manin.

Theorem 5.1. *For any $N \geq 4$, every smooth hypersurface $X = X_N \subset \mathbb{P}^N$ of degree N is birationally superrigid.*

5.1. **Multiplier ideals.** As before, we consider pairs of the form (X, Z) where X is a smooth variety and $Z = \sum c_i Z_i$ is a $\mathbb{Q}$-scheme.

Definition 5.2. The *multiplier ideal sheaf* of a pair (X, Z) is

$$\mathcal{J}(X, Z) := f_*\mathcal{O}_Y(\lceil K_{Y/X} - f^{-1}(Z)\rceil)$$

where f is a log resolution of the pair and the round-up of a $\mathbb{Q}$-divisor is taken componentwise. If $X = \mathrm{Spec}\, R$, Z is the subscheme defined by a nonzero ideal $\mathfrak{a} \subset R$, and $c \geq 0$, then we denote by $J(\mathfrak{a}^c) \subset R$ be the ideal of global sections of $\mathcal{J}(X, cZ)$. We call it the *multiplier ideal* of $(R, \mathfrak{a}^c)$.

Though this is not an obvious fact, the definition of $\mathcal{J}(X,Z)$ is independent of the choice of log resolution. In general, $\mathcal{J}(X,Z)$ is a coherent sheaf of fractional ideals. However, if (X,Z) is an effective pair then $\mathcal{J}(X,Z) \subset \mathcal{O}_X$. Note that, as $K_{Y/X}$ is an integral divisor, we can equivalently write

$$\mathcal{J}(X,Z) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^{-1}(Z) \rfloor)$$

Proposition 5.3. *If (X,Z) is an effective pair, then $\mathcal{J}(X,Z) = \mathcal{O}_X$ if and only if (X,Z) is klt. In particular, if $Z \subset X$ is a proper closed subscheme, then*

$$\operatorname{lct}(X,Z) = \sup\{t > 0 \mid \mathcal{J}(X,tZ) = \mathcal{O}_X\}.$$

Proof. The proposition is clear from the definition once one observes that if D is a $\mathbb{Q}$-divisor on a log resolution $f\colon Y \to X$ such that $f_*D \le 0$, then $f_*\mathcal{O}_Y(D) = \mathcal{O}_X$ if and only if $D \ge 0$. □

Example 5.4. For any $g \in \mathbb{C}[x_1,\dots,x_n]$ and $c > 0$, we have

$$J((g)^c) = \{h \in \mathbb{C}[x_1,\dots,x_n] \mid |h|^2|g|^{-2c} \text{ is locally integrable on } \mathbb{C}^2\}.$$

This explains the terminology, as it describes $J((g)^c)$ as an ideal of multipliers: the multipliers that make $|g|^{-c}$ locally $\mathcal{L}^2$. One can prove this using a log resolution to reduce to check local integrability for monomials. The factor of 2 in the exponents is consistent with fact that the unit of volume on $\mathbb{C}^2$ is $\omega \wedge \overline{\omega}$ where $\omega = dx_1 \wedge \cdots \wedge dx_n$, hence the change-of-variable formula along a log resolution $f\colon Y \to \mathbb{C}^2$ involves $|\mathrm{Jac}_f|^2$ as the Jacobian factor.

Example 5.5. If $\mathfrak{a} \subset k[x_1,\dots,x_n]$ is a monomial ideal, then a similar computation as in the formula for the log canonical threshold of $\mathfrak{a}$ shows that

$$J(\mathfrak{a}^c) = (x^u \mid u + e \in \operatorname{Int}(cP(\mathfrak{a})))$$

where $\operatorname{Int}(cP(\mathfrak{a}))$ is the interior of the Newton polygon of $\mathfrak{a}$ rescaled by c. This formula is proved in [How01], to which we refer for more details.

5.2. **Nadel vanishing theorem.** Multiplier ideals satisfy a very useful vanishing theorem. In fact, their definition can be viewed as a the result of 'forcing' the Kawamata–Viehweg vanishing theorem to hold in more general settings. The following theorem was orignially proved by Nadel in the analytic setting, and it first appeared in the algebro-geometric setting in the work of Esnault and Viehweg. For a general reference, we recommend [Laz04]. We state it here in the smooth setting, but we mention that the theorem holds under less restrictive conditions.

Theorem 5.6 (Nadel vanishing theorem)**.** *Let (X,Z) be a pair where X is a smooth projective variety and $Z = \sum c_j Z_j$ is an effective $\mathbb{Q}$-scheme. Assume that A_j are Cartier divisors on X such that $\mathcal{O}_X(A_j) \otimes \mathcal{I}_{Z_j}$ is globally generated for every j, and let A be a Cartier divisor such that the $\mathbb{Q}$-divisor $A - K_X - \sum c_j A_j$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(A) \otimes \mathcal{J}(X,Z)) = 0 \quad \textit{for all } i \ge 1.$$

Sketch of Proof. Let $f\colon Y \to X$ be a log resolution of (X,Z). Write $\mathcal{I}_{Z_j} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E_j)$, and let $E = \sum c_i f^{-1}(Z)$. By our assumption, $\mathcal{O}_Y(f^*A_j - E_j)$ is globally generated. Let $B = f^{-1}(Z) - K_{Y/X}$. Recall that $\mathcal{J}(X,Z) = f_*\mathcal{O}_Y(-\lfloor B \rfloor)$. Then, by projection formula, we have

$$f_*\mathcal{O}_Y(f^*A - \lfloor B \rfloor) = \mathcal{O}_X(A) \otimes \mathcal{J}(X,Z)$$

and, applying relative vanishing [Laz04, Theorem 9.4.1],

$$R^p f_*\mathcal{O}_Y(f^*A - \lfloor B \rfloor) = 0 \quad \text{for all } q \geq 1.$$

We deduce by the Leray spectral sequence that

$$H^i(X, \mathcal{O}_X(A) \otimes \mathcal{J}(X,Z)) = H^i(Y, \mathcal{O}_Y(f^*A - \lfloor B \rfloor)).$$

Writing

$$f^*A - \lfloor B \rfloor = K_Y + \lceil f^*(A - K_X - \sum c_j A_j) + \sum c_j (f^*A_j - E_j) \rceil$$

we see that $f^*A - \lfloor B \rfloor$ satisfies the hypotheses of Kawamata–Viehweg vanishing theorem, since the $\mathbb{Q}$-divisor under the round-up is nef and big and its fractional part is supported on a simple normal crossing divisor. Therefore we have vanishing for $i \geq 1$. □

5.3. **Colength of multiplier ideals.** The last property we need is the following bound on colength of multiplier ideals. It is based on a simple but clever observation due to [Zhu20] which allows to replaces a quadratic bound obtained at the end of the proof of [dF16] with an exponential one. This is especially relevant when one wants to extend the proof to other Fano varieties of index one such as singular Fano hypersurfaces and complete intersections, where the stronger bound yields significantly stronger results.

Recall that, according to our definition, a pair (X, Z) is always assumed to be on a smooth variety X.

Proposition 5.7 ([Zhu20, Kol19])**.** *Let (X, Z) be an effective pair with* $\operatorname{lct}(X, Z) \leq 1$*, and let $\Sigma \subset X$ be the subscheme defined by the multiplier ideal $\mathcal{J}(X, 2Z) \subset \mathcal{O}_X$. If $n = \dim X$, then*

$$l(\mathcal{O}_\Sigma) \geq \tfrac{1}{2} 3^n.$$

We start with two lemmas. The first is the main observation behind the proposition.

Lemma 5.8. *Let (X, Z) be an effective pair. Then for every divisor E over X we have*

$$a_E(X, \mathcal{J}(X, 2Z)) < 2a_E(X, Z).$$

In particular, if (X, Z) is not klt, then $(X, \mathcal{J}(X, 2Z))$ is not log canonical.

Proof. Denote for short $\mathcal{J} = \mathcal{J}(X, 2Z)$. We may assume that E is a prime divisor on a log resolution $f \colon Y \to X$ of (X, Z). Set $k_E = \operatorname{ord}_E(K_{Y/X})$, $z_E = \operatorname{ord}_E(Z)$, and $j_E = \operatorname{ord}_E(\mathcal{J})$. By definition, $\mathcal{J} = f_*\mathcal{O}_Y(-\lfloor 2f^{-1}(Z) - K_{Y/X} \rfloor)$, hence

$$j_E \geq \operatorname{ord}_E(\lfloor 2f^{-1}(Z) - K_{Y/X} \rfloor) = \lfloor 2z_E - k_E \rfloor > 2z_E - k_E - 1.$$

This implies that

$$a_E(X, \mathcal{J}) = k_E + 1 - j_E < 2k_E + 2 - 2z_E = 2a_E(X, Z),$$

as claimed. □

The second lemma gives a lower-bound on colength that is sharper, for small values of n, compared to the one in Theorem 4.11. Only using the latter in the proof of Proposition 5.7 would give $l(\mathcal{O}_\Sigma) \geq n^n/n!$, a bound that is not sharp enough to settle the theorem in all dimensions, leaving out a few lower dimensional cases for which we would have to rely on a different argument. The fact that the bound on colength could be sharpened for small n was first noted in [Zhu20] and later revisited in [Kol19]. The bound given here follows [Kol19].

Lemma 5.9. *Let $R = \mathcal{O}_{X,x}$ be the local ring of a variety X at a regular point x. Assume that $\dim R = n \geq 2$, and let $\mathfrak{a} \subset R$ be an $\mathfrak{m}$-primary ideal with $\operatorname{lct}(\mathfrak{a}) < 1$. Then*

$$l(R/\mathfrak{a}) \geq \tfrac{1}{2}3^n.$$

Proof. As in the proof of Theorem 4.11, we can reduce to the case where $\mathfrak{a}$ is a monomial ideal in a polynomial ring $k[x_1, \dots, x_n]$. The condition that $\operatorname{lct}(\mathfrak{a}) < 1$ implies that there are positive numbers a_i such that the tetrahedron

$$T = \left\{ u \in \mathbb{R}^n_{\geq 0} \mid \sum a_i u_i \leq \sum a_i \right\}$$

is disjoint from the Newton polyhedron $P(\mathfrak{a})$. Hence $l(R/\mathfrak{a}) \geq |T \cap \mathbb{Z}^n|$. The upperbound $\frac{1}{2}3^n$ accounts for the fact that for any $(u_1, \dots, u_n) \in \mathbb{Z}^n$ with $u_i \in \{0, 1, 2\}$ for all i, either $(u_1, \dots, u_n)$ or $(2 - u_1, \dots, 2 - u_n)$ belongs to T. $\square$

Proof of Proposition 5.7. We may assume that Σ is zero dimensional as otherwise $l(\mathcal{O}_\Sigma) = \infty$. Since $(X, \mathfrak{b}^c)$ is not klt, it follows by Lemma 5.8 that (X, Σ) is not log canonical, that is, $\operatorname{lct}(X, \Sigma) < 1$. Then there is a connected component Σ' of Σ such that $\operatorname{lct}(X, \Sigma') < 1$, hence $l(\mathcal{O}_{\Sigma'})$ satisfies the claimed bound by Lemma 5.9. Since $l(\mathcal{O}_\Sigma) \geq l(\mathcal{O}_{\Sigma'})$, this proves the proposition. $\square$

5.4. **Birational rigidity of Fano hypersurfaces.** We are now ready to prove Theorem 5.1. The proof builds upon the work of several people since [IM71], each contributing to a different piece of the puzzle. We recommend [Kol19] for an excellent survey on the history of the proof. Just to mention some of the main contributions throughout the years, low dimensional cases were settled in [Puk87, Che00, dFEM03], and a proof that works for hypersurfaces that are general in moduli was given in [Puk98]. Incomplete proofs were given in [Puk02] and [dF13] before a complete proof was finally found in [dF16]. The final step of the proof was later simplified in [Zhu20]. While no new cases of this theorem were proved in [Cor00], the application of inversion of adjunction to reprove the three dimensional case using log canonical thresholds has played an important role in finding the proof of the general result.

Proof of Theorem 5.1. Let $X = X_N \subset \mathbb{P}^N$ be a smooth hypersurface of degree N, with $N \geq 4$. Note that $-K_X \sim \mathcal{O}_X(1)$ and, by Lefschetz hyperplane theorem, it generates the Picard group of X.

If X is not birationally superrigid, then we can find a birational map $\phi\colon X \dashrightarrow X'$ from X to a Mori fiber space X' that is not an isomorphism. We fix a suitable projective embedding of X' (as in Theorem 2.6) and let $\mathcal{H} \subset |-rK_X|$ be the corresponding linear system defining ϕ. By the Noether–Fano inequality, for a general member $D \in \mathcal{H}$ the pair $(X, \frac{1}{r}D)$ is not canonical. On the other hand, Proposition 3.10 implies that $e_C(D) \leq r$ for every curve $C \subset X$, and by Proposition 4.7 this implies that $(X, \frac{1}{r}D)$ is canonical in dimension one (i.e., away from a finite set). Therefore there exists a divisor E over X, with center $C_E(X)$ equal to a closed point $p \in X$, such that

$$a_E(X, \tfrac{1}{r}D) < 1.$$

We can fix $c < \frac{1}{r}$ such that $a_E(X, cD) \leq 1$.

Fix now two general elements $D, D' \in \mathcal{H}$, and let $Z = D \cap D' \subset X$ their schematic intersection. For any subvariety $V \subset X$, we will denote by $Z|_V$ the intersection $Z \cap V$. Note that we still have

$$a_E(X, cZ) \leq 1$$

since $\operatorname{ord}_E(Z) = \operatorname{ord}_E(D)$, and Proposition 3.11 implies that

$$\dim\{x \in X \mid e_x(Z) > r^2\} \le 1.$$

Let $Y \subset X$ be a general hyperplane section through p (the center of E). Since $\operatorname{ord}_E(Y) \ge 1$, we have $a_E(X, cZ + Y) \le 0$, hence inversion of adjunction (Theorem 4.5) implies that the restricted pair $(Y, cZ|_Y)$ is not log terminal. Note, on the other hand, that the pair is log terminal in dimension one.

In fact, the pair $(Y, 2cZ|_Y)$ (where we doubled the contribution of the 'boundary') is also log terminal in dimension one. To see this, first notice that, by Proposition 3.7, we can ensure that the set $\{y \in Y \mid e_y(Z|_Y) > r^2\}$ is zero dimensional. Let $U \subset Y$ be the complement of a finite set such that $e_q(Z|_Y) \le r^2$ for all $q \in U$. Let $S \subset Y$ be a smooth surface cut out by general hyperplanes through q. Applying again Proposition 3.7, we get $e_q(Z|_S) \le r^2$. Since $Z|_S$ is a complete intersection and p is an irreducible component of it, we have $e_q(Z|_S) = e(\mathcal{I}_{Z|_S} \cdot \mathcal{O}_{S,q})$, hence Theorem 4.11 implies that

$$\operatorname{lct}_q(S, Z|_S) \ge \frac{2}{\sqrt{e_q(Z|_S)}} \ge \frac{2}{r} > 2c.$$

Therefore $(S, 2cZ|_S)$ is log terminal near q. By inversion of adjunction, this implies that $(Y, 2cZ|_Y)$ is log terminal near q. This proves that $(Y, 2cZ|_Y)$ is log terminal in dimension one, as claimed.

It follows that the multiplier ideal $\mathcal{J}(Y, 2cZ|_Y)$ defines a zero-dimensional subscheme $\Sigma \subset Y$. We have

$$H^1(Y, \mathcal{J}(Y, 2cZ|_Y) \otimes \mathcal{O}_Y(2)) = 0$$

by Nadel's vanishing theorem, since ω_Y is trivial, $Z|_Y$ is cut out by forms of degree r, and $2cr < 2$. Hence there is a surjection

$$H^0(Y, \mathcal{O}_Y(2)) \twoheadrightarrow H^0(\Sigma, \mathcal{O}_\Sigma).$$

Keeping in mind that $H^0(Y, \mathcal{O}_Y(2)) \cong H^0(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(2))$, it follows that

$$h^0(\Sigma, \mathcal{O}_\Sigma) \le h^0(Y, \mathcal{O}_Y(2)) = \binom{N+1}{2}.$$

On the other hand Proposition 5.7 gives the lower-bound

$$l(\mathcal{O}_\Sigma) \ge \tfrac{1}{2} 3^{N-2}.$$

By contrasting the two inequality, we get a contradiction as soon as $N \ge 6$.

With the case $N = 4$ already settled, this leaves open only the case $N = 5$. We treat this case using generic projections, as in [dFEM03]. The necessary formula on generic projection will be proved in the next lecture; here will will borrow from it. The same argument given below to treat the case $N = 5$ can also be used to deal with the case $N = 4$, instead of the argument based on Bezout's theorem we outlined earlier, hence one can think of the proof of the theorem as splitting into two parts rather than three.

If $N = 5$, then Y is a threefold in $\mathbb{P}^4$ and $Z|_Y$ is one dimensional. Let $f \colon Y \dashrightarrow \mathbb{P}^2$ be the map induced by a general linear projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2$. We may assume that the indeterminacies of f are disjoint from the support of $Z|_Y$, hence we can define the $\mathbb{Q}$-divisor

$$\Delta = \frac{c^2}{4} \cdot f_*[Z|_Y].$$

Since the set $\{y \in Y \mid e_y(Z|_Y) > r^2\}$ is zero dimensional and, for a general projection, f restricts to a birational morphism on the support of $Z|_Y$, it follows that the pair $(\mathbb{P}^2, \Delta)$ is log terminal in dimension one. By contrast, Theorem 6.5 (from the next lecture) implies that the pair is not log terminal at $f(p)$. Therefore, $\mathcal{J}(\mathbb{P}^2, \Delta)$ defines a zero dimensional scheme $W \subset \mathbb{P}^2$. Note that $\deg(\Delta) < 2$. We have $H^1(\mathbb{P}^2, \mathcal{O}(-1) \otimes \mathcal{J}(\mathbb{P}^2, \Delta)) = 0$ by Nadel's vanishing theorem, and this yields a surjection $H^0(\mathbb{P}^2, \mathcal{O}(-1)) \twoheadrightarrow H^0(\mathcal{O}_W)$, which is impossible since $H^0(\mathcal{O}_W) \neq \{0\}$.

□

6. Lecture 6: Hypersurfaces with ordinary double points

In general, singularities on the variety may affect properties like birational superrigidity. For example, if $X \subset \mathbb{P}^4$ is a quartic threefold with a point of multiplicity 2, then the projection from the singular point defines a rational map $X \dashrightarrow \mathbb{P}^3$ of degree 2, and the birational involution swapping the two sheets is a birational map of X that is not a regular automorphism. This case was studies in details in [Puk88, CM04, Mel04] where it is shown among other things that under suitable conditions on the class group, X is birationally rigid, a weaker notion compared to birational superrigidity.

In this section we look at higher dimensional Fano hypersurfaces with ordinary double points. We recall that a point $p \in X$ is an *ordinary double point* if it is an isolated singularity and the tangent cone $C_\pi(X)$ is a quadric cone of maximal rank (i.e., the affine cone over a smooth projective quadric).

Theorem 6.1 ([dF22]). *For every $N \geq 6$, every hypersurface $X \subset \mathbb{P}^N$ of degree N with only ordinary double points as singularities is birationally superrigid.*

As we already discussed, if X is singular then the conclusions of the theorem cannot hold for $N = 4$, but we do not know whether they may hold for $N = 5$. The case $N = 6$ was proved in [Che07] under the technical assumption that X does not contain any plane. In higher dimensions, the result was first proved in [Puk88] assuming that X is general in moduli, and the main theorem of [dF17] implies the result for arbitrary X for $N \geq 20$.

We will give a proof of the theorem that does not rely on these previous results. We will adapt the proof we gave in the smooth case to deal with ordinary double points; the resulting argument will cover all cases $N \geq 9$. The remaining few cases will be settled using a similar argument as in [dFEM03]. We should stress that the methods to deal with the low dimensional cases are not strong enough to cover all cases left out by [dF17], hence we need a new argument for the higher dimensional cases.

6.1. Log discrepancies over singular varieties.

The first thing to do is to extend the definition to singular varieties. We assume that X is normal and the canonical class K_X is Cartier (or $\mathbb{Q}$-Cartier). This condition allows us to pull-back K_X along a log resoution $f\colon Y \to X$ and define the relative canonical divisor $K_{Y/X}$ as the unique exceptional divisor (or $\mathbb{Q}$-divisor) that is linear equivalent to $K_Y - f^*K_X$. (Taking $K_X = f_*K_X$, one can just define $K_{Y/X} = K_Y - f^*K_X$.) Log discrepancies are defined as before, by setting

$$a_E(X, Z) = \operatorname{ord}_E(K_{Y/X}) + 1 - \operatorname{ord}_E(Z)$$

for any divisor E over X and any $\mathbb{Q}$-scheme Z on X. The definitions of singularities and multiplier ideals given before on a smooth variety extend to this setting in the obvious way, and so do some of their properties. In particular, Nadel vanishing theorem and inversion of adjunction for klt singularities, as stated in Theorem 4.5, still hold in this setting where X is allowed to be singular.

It will be convenient to introduce the following notation. We will only use it on smooth varieties, but the definition can be given in general.

Definition 6.2. The *minimal log discrepancy* of a pair (X, Z) at a point $p \in X$ is the infimum of the discrepancies $a_E(X, Z)$ of all divisors E over X with center p. It is denoted by $\mathrm{mld}_p(X, Z)$.

A different way of measuring singularities is to use the Jacobian ideal sheaf $\mathrm{Jac}_f = \mathrm{Fitt}^0(\Omega_{Y/X})$ of a log resolution in place of the relative canonical divisor.

Definition 6.3. With the above notation, we define the *Mather log discrepancy* of E over a pair (X, Z) to be

$$\widehat{a}_E(X, Z) = \mathrm{ord}_E(\mathrm{Jac}_f) + 1 - \mathrm{ord}_E(Z).$$

If X is smooth then $\widehat{a}_E(X, Z) = a_E(X, Z)$. However, as soon as X is singular at the generic point of the center of E we have a strict inequality $\widehat{a}_E(X, Z) > a_E(X, Z)$. For example, if X has locally complete intersection singularities, then

$$\widehat{a}_E(X, Z) = a_E(X, Z) + \mathrm{ord}_E(\mathrm{Jac}_X)$$

where $\mathrm{Jac}_X = \mathrm{Fitt}^{\dim X}(\Omega_X)$ is the Jacobian ideal sheaf of X. This follows from the following property.

Lemma 6.4. *With the above notation, if X is a variety with locally complete intersection singularities then*

$$\mathrm{ord}_E(\mathrm{Jac}_f) = \mathrm{ord}_E(K_{Y/X}) + \mathrm{ord}_E(\mathrm{Jac}_X).$$

Proof. Let $n = \dim X$. Following the proof of [Pie79, Proposition 1], we start by showing that the image of the natural map $c_X \colon \Omega^n_{X/k} \to \omega_X$ is equal to $\mathrm{Jac}_X \otimes \omega_X$. Working locally on X, we may assume that X is embedded in $A := \mathbb{A}^N$ and that its ideal $\mathcal{I}_X$ is generated by a regular sequence $h_1, \dots, h_r \in k[x_1, \dots, x_N]$, where $r = N - n$. The exactness of the Koszul complex defined by $(h_1, \dots, h_r)$ yields a natural isomorphism

$$\omega_X = \mathcal{E}xt^r_A(\mathcal{O}_X, \omega_A) \cong \mathcal{H}om_{\mathcal{O}_X}(\wedge^r \mathcal{I}_X/\mathcal{I}_X^2, \omega_A|_X).$$

which is independent of the choice of the regular sequence generating $\mathcal{I}_X$. For every index set $I = \{i_1, \dots, i_n\} \subset \{1, \dots, N\}$, the form $dx_{i_1} \wedge \dots \wedge dx_{i_n}|_X$ is sent, via c_X and the above isomorphism, to the element $\phi_I \in \mathcal{H}om_{\mathcal{O}_X}(\wedge^r \mathcal{I}_X/\mathcal{I}_X^2, \omega_A|_X)$ defined by

$$\phi_I(\overline{h}_1 \wedge \dots \wedge \overline{h}_r) = dh_1 \wedge \dots \wedge dh_r \wedge dx_{i_1} \wedge \dots \wedge dx_{i_n}|_X.$$

Up to sign, the form on the right-hand-side is equal to $\Delta_I dx_1 \wedge \dots \wedge dx_N|_X$ where Δ_I be the $r \times r$ minor obtained from the Jacobian matrix $[\partial h_j/\partial x_i]$ by removing the rows indexed by I. Tracing back through the isomorphisms, we conclude that the image of c_X is equal to $\mathrm{Jac}_X \otimes \omega_X$.

Let $f \colon Y \to X$ be a resolution of singularity, and assume that E is a prime divisor on Y. We have a commutative diagram

$$\begin{array}{ccc} f^*\Omega^n_{X/k} & \longrightarrow & f^*\omega_X \\ \downarrow & & \downarrow \\ \Omega^n_{Y/k} & \xrightarrow{\cong} & \omega_Y \end{array} .$$

By comparing images, we see that $\mathrm{Jac}_f = \mathrm{Jac}_X \cdot \mathcal{O}_Y(-K_{Y/X})$, and the lemma follows by applying ord_E to both sides of this equation. ☐

6.2. **Singularities under generic projections.** When dealing with the low dimensional cases, we will follow a strategy similar to the proof in [dFEM03]. The main idea is to take general linear projections whose purpose is to realize the complete intersection subscheme cut out by two general members of the linear system as a divisor in a smaller dimensional projective space. The following is the key technical result used for this purpose.

Let $X \subset \mathbb{A}^N$ be a smooth variety of dimension n, $Z \subset X$ be a complete intersection subscheme of pure codimension k, and E a divisor over X. Consider a general linear projections

$$\phi\colon X \subset \mathbb{A}^N \to Y = \mathbb{A}^m,$$

where $m = n - k + 1$. We assume that $\phi|_Z$ is a proper finite morphism. Note that $\phi_*[Z]$ is a cycle of codimension one in Y, so we can regard it as a Cartier divisor on Y. We can write $\mathrm{val}_E|_{\mathbb{C}(Y)} = d \cdot \mathrm{val}_F$ where F is a divisor over Y and d is a positive integer.

Theorem 6.5 ([dFEM03])**.** *With the above notation, for every $c \geq 0$ such that $a_E(X, cZ) \geq 0$ we have*

$$d \cdot a_F(V, \tfrac{c^k}{k^k} \cdot \phi_*[Z]) \leq a_E(X, cZ).$$

When Z is zero dimensional (i.e. $k = n$), this result, with $c = \mathrm{lct}(X, Z)$, is equivalent to Theorem 4.11. Indeed in this case, assuming for simplicity that Z_{red} is just one point $p \in X$, we have $[Z] = \mu\{p\}$ where $\mu = l(\mathcal{O}_{Z,p})$ is the multiplicity of Z at p, hence $\phi_*[Z] = \mu\{f(p)\}$ in $Y = \mathbb{A}^1$. Since Z is complete intersection, we have $\mu = e(\mathcal{I}_Z \cdot \mathcal{O}_{X,p}) = e_Z(X)_p$, and the statement that $a_F(V, \frac{c^n}{n^n} \cdot \phi_*[Z]) \leq 0$ is equivalent to the inequality

$$e_Z(X)_p \geq \frac{n^n}{c^n}.$$

We will also use the following version of the theorem that can be applied to singular varieties by replacing log discrepancies with Mather log discrepancies.

Theorem 6.6 ([dFM15])**.** *Consider the same assumption as in Theorem 6.5 with the only difference that we now allow X to be singular. Then for every $c \geq 0$ such that $\widehat{a}_E(X, cZ) \geq 0$ we have*

$$d \cdot a_F(V, \tfrac{c^k}{k^k} \cdot \phi_*[Z]) \leq \widehat{a}_E(X, cZ).$$

6.3. **Birational rigidity of singular hypersurfaces.** Here we prove Theorem 6.1. Let therefore $X \subset \mathbb{P}^N$ be a hypersurface of degree N with only ordinary double points as singularities, and assume that $N \geq 6$.

The first remark is that X is indeed a Mori fiber space, so that it makes sense asking about its birational superrigidity. The definition of Mori fiber space for a singular variety is the same as the one given under the smoothness condition, with the only requirement that the singularities are terminal and the variety is $\mathbb{Q}$-factorial. In case of ordinary double points in dimension ≥ 3, is easy to see that the singularities are terminal and locally factorial in the analytic topology. However, $\mathbb{Q}$-factoriality is a local property in the Zariski topology, so it needs to be verified. In our situation, it follows from the next lemma.

Lemma 6.7. *Any normal hypersurface $V \subset \mathbb{P}^N$ whose singular locus has codimension at least 4 is factorial.*

Proof. If $\dim V \leq 3$ then V is smooth and hence factorial. Assume then that $\dim V \geq 4$. The hypersurface $W \subset \mathbb{P}^4$ cut out by V on a general linear 4-space $\mathbb{P}^4 \subset \mathbb{P}^N$ is smooth. By the Lefschetz hyperplane theorem, both $\mathrm{Pic}(V)$ and $\mathrm{Pic}(W)$ are generated by the respective hyperplane classes, and so the restriction map $\mathrm{Pic}(V) \to \mathrm{Pic}(W)$ is an isomorphism. Since

W is smooth, the class map $\mathrm{Pic}(W) \to \mathrm{Cl}(W)$ is an isomorphism. On the other hand, the restriction of Weil divisors (which is well-defined in our setting) induces an isomorphism $\mathrm{Cl}(V) \to \mathrm{Cl}(W)$ by an inductive application of [RS06, Theorem 1]. It follows that $\mathrm{Pic}(V) \to \mathrm{Cl}(V)$ is an isomorphism. □

A second remark is that, because of the singularities of X, in order to apply Pukhlikov's multiplicity bounds we will first need to cut X with a general hyperplane section in order to reduce to a smooth hypersurface where the result can be applied. This weakens the condition on dimension on the statement of Pukhlikov's result by one unit. The following example shows that this is unavoidable.

Example 6.8. Let $Q \subset \mathbb{P}^3$ be a quadric cone over a smooth conic. If $D \in \mathcal{O}_Q(1)$ is cut out by any plane tangent to Q along a line $L \subset Q$, then $e_L(D) = 2$.

Proof of Theorem 6.1. The starting point is the same as in the smooth case: assuming X is not birationally superrigid, we construct a movable linear system $\mathcal{H} \subset |-rK_X|$ such that if $D \in \mathcal{H}$ is a general member, then there exists an exceptional divisor G over X such that $a_G(X, cD) \le 1$ for some

$$c < \frac{1}{r}.$$

Let $T \subset X$ be the center of G and $p \in T$ be a general point.

Since X has isolated singularities, we can apply Propositions 3.10 and 3.11 to a general hyperplane section of X to conclude, using Proposition 3.7, that

$$\dim\{x \in X \mid e_x(D) > r\} \le 1$$

for a general $D \in \mathcal{H}$, and

$$\dim\{x \in X \mid e_x(Z) > r^2\} \le 2$$

for $Z = D \cap D'$ the intersection of two general $D, D' \in \mathcal{H}$. Note that $T \subset \{x \in X \mid e_x(D) > r\}$, hence $\dim T \le 1$. Note that if $\dim T = 1$ then we take $p \in T$ to be a smooth point of X, and if $\dim T = 0$ then p may be a singular point.

Let $Y = V \cap V' \subset X$ be intersection of two general hyperplane sections $V, V' \subset X$ through p. After cutting down with one hyperplane, we get a pair $(V, cZ|_V)$ that is canonical in dimension one. After cutting down with the second hyperplane, we get a pair $(Y, cZ|_Y)$ that is not klt at p. Furthermore, the set $\{y \in Y \mid e_y(Z|_Y) > r^2\}$ is zero dimensional, hence $\mathcal{J}(Y, 2cZ|_Y)$ defines a zero dimensional scheme $\Sigma \subset Y$.

Since $Z|_Y$ is cut out on Y by forms of degree r and $c > \frac{1}{r}$, we have $H^1(Y, \omega_Y(2) \otimes \mathcal{J}(Y, 2cZ|_Y)) = 0$ by Nadel vanishing. Note that $\omega_Y \cong \mathcal{O}_Y(1)$. Then

$$h^0(\mathcal{O}_\Sigma) \le h^0(Y, \mathcal{O}_Y(3)) = h^0(\mathbb{P}^{N-2}, \mathcal{O}(3)) = \binom{N+1}{3}. \tag{6.1}$$

Lemma 5.8 still holds in our setting, so we have $\mathrm{lct}(Y, \Sigma) \le 1$. If p is a smooth point, then we can apply Lemma 5.9 to get a lower bound. However, if p is a singular point then we cannot apply that lemma on the nose, since the lemma requires in its computations that the variety is smooth. Nonetheless, we can take a degeneration that will allow us to apply Lemma 5.9 in lower dimension.

Let us discuss the case where p is a singular point, the other case being easier and leading to a stronger bound. We restrict to affine chart $\mathbb{A}^{N-2}$ containing p, and fix coordinates $(x_1, \dots, x_{N-2})$ centered at p such that the tangent cone $C_p(X)$ (a cone over a smooth quadric) is defined by $\sum x_i^2 = 0$. For simplicity, we still denote by Y its restriction to

$\mathbb{A}^{N-2}$ and let Σ denote now the connected component of the zero dimensional scheme defined by $\mathcal{J}(Y, 2cZ|_Y)$ that is supported at p.

We take a flat degeneration of $Y \subset \mathbb{A}^{N-2}$ to the union of two hyperplanes $H_1 \cap H_2$, and let $\Sigma' \subset H_1 \cap H_2$ be the zero dimensional scheme supported at p obtained by flat degeneration from Σ. Concretely, if $f(x_1, \ldots, x_{N-2}) = 0$ is the equation of Y in $\mathbb{A}^{N-2}$, then degeneration is constructed using the $\mathbb{C}^*$-action on $\mathbb{A}^{N-2}$ given by $x_i \mapsto \lambda x_i$ for $i = 1, 2$ and $x_j \mapsto \lambda^2 x_j$ for $j > 2$ sending $\lambda \to 0$.

Recall that $\operatorname{lct}(Y, \Sigma) \leq 1$, or, equivalently, $\operatorname{mld}_p(Y, \Sigma) \leq 0$. By adjunction, this implies that

$$\operatorname{mld}_p(\mathbb{A}^{N-2}, \Sigma + Y) \leq 0.$$

By semicontinuity of log canonical thresholds, we have

$$\operatorname{mld}_p(\mathbb{A}^{N-2}, \Sigma' + H_1 + H_2) \leq 0.$$

By inversion of adjunction, this implies that

$$\operatorname{mld}_p(H_1, \Sigma'_1 + H_{12}) \leq 0$$

where $\Sigma'_1 = \Sigma' \cap H_1$ and $H_{12} = H_1 \cap H_2$. Applying inversion of adjunction again, we get

$$\operatorname{mld}_p(H_{12}, \Sigma'_{12}) \leq 0$$

where $\Sigma'_{12} = \Sigma'_1 \cap H_{12} = \Sigma' \cap H_{12}$. Note that $H_{12} = \mathbb{A}^{N-4}$. Then Lemma 5.9 implies that

$$l(\mathcal{O}_{\Sigma'_{12}}) \geq \tfrac{1}{2} 3^{N-4}.$$

Since $l(\mathcal{O}_\Sigma) = l(\mathcal{O}_{\Sigma'})$ by flatness and $l(\mathcal{O}_{\Sigma'}) \geq l(\mathcal{O}_{\Sigma'_{12}})$ from the inclusion $\Sigma'_{12} \subset \Sigma'$, we conclude that

$$l(\mathcal{O}_\Sigma) \geq \tfrac{1}{2} 3^{N-4}. \tag{6.2}$$

Comparing the two inequalities (6.1) and (6.2), we get a contradiction as soon as $N \geq 9$. So the theorem is proved in this case.

We now address the remaining cases $6 \leq N \leq 8$.

Suppose first that $\dim T = 1$. We take a general linear projection $\mathbb{P}^{N-2} \dashrightarrow \mathbb{P}^{N-4}$, let $f\colon Y \to \mathbb{P}^{N-4}$ be the induced map, and define

$$\Delta = \frac{c^2}{4} \cdot \pi_*[Z|_Y].$$

This is an effective $\mathbb{R}$-divisor of degree $\deg(\Delta) < N/4$. Since $\deg(\Delta) < 2$, Nadel vanishing theorem implies that

$$H^1(\mathbb{P}^{N-4}, \mathcal{O}(5-N) \otimes \mathcal{J}(\mathbb{P}^{N-4}, \Delta)) = 0.$$

By Theorem 6.5, the pair $(\mathbb{P}^{M-4}, \Delta)$ is not klt at $f(p)$. On the other hand, using that the set $\{y \in Y \mid e_y(Z|_Y) > r^2\}$ is zero dimensional and arguing as in [dFEM03], one can check if $N = 6, 7$ then $\mathcal{J}(\mathbb{P}^{N-4}, \Delta)$ defines a zero dimensional scheme $W \subset \mathbb{P}^{N-4}$, and the surjection

$$H^0(\mathbb{P}^{N-4}, \mathcal{O}(5-N) \twoheadrightarrow H^0(\mathcal{O}_W)$$

forces $N \leq 5$. If $N = 8$, then one can only conclude that $\mathcal{J}(\mathbb{P}^{N-4}, \Delta)$ defines a scheme of dimension at most 1, but after cutting down one more time, and using the vanishing

$$H^1(\mathbb{P}_3, \mathcal{O}(-2) \otimes \mathcal{J}(\mathbb{P}^3, \Delta|_{\mathbb{P}^3})) = 0,$$

we get to a similar contradiction.

Therefore T must be zero dimensional. In this case p could be a singular point. If p is a smooth point, then the same argument above (and in fact the same proof of the theorem given when X is smooth) can be applied here. Let us assume that p is a singular point. After cutting down with just one hyperplane section, we already get a pair $(V, cZ|_V)$, with $V \subset \mathbb{P}^{N-1}$, that is klt in dimension one but is not klt at p. So, we can find a divisor E over V with center p such that $a_E(V, cZ|_V) \le 0$. Observing that $\mathrm{Jac}_V \cdot \mathcal{O}_{V,p} = \mathfrak{m}_{V,p}$ (since p is an ordinary double point), we have

$$\widehat{a}_E(V, cZ|_V) \le \mathrm{val}_E(\mathfrak{m}_{V,p})$$

by Theorem 6.6. Take a general linear projection $g\colon V \to \mathbb{P}^{N-3}$ and let

$$\Theta = \frac{c^2}{4} \cdot \pi_*[Z|_V].$$

By Theorem 6.6, there exists a divisor F over $\mathbb{P}^{N-3}$ with center $q = g(p)$ such that,

$$a_F(\mathbb{P}^{N-3}, \Theta) \le \mathrm{val}_F(\mathfrak{m}_q).$$

Restricting to a general hyperplane $\mathbb{P}^{N-4}$ through q and letting Δ be the restriction of Θ, we get a pair $(\mathbb{P}^{N-4}, \Delta)$ which satisfies the same properties as in the discussion of the case $\dim T = 1$. We can therefore repeat the same argument to get a contradiction and finish the proof of the theorem. □

7. Lecture 7: Rationality in families of varieties

As before, we work over $\mathbb{C}$. Let $f\colon X \to T$ be a *family of projective varieties*, namely, a projective equidimensional morphism onto a connected reduced scheme T of finite type such that the fibers $X_t := f^{-1}(t)$ are integral schemes for all $t \in T$. Let n denote the relative dimension of f. We are interested in understanding the algebraic structure of the *rational locus*

$$\mathrm{Rat}(f) := \{\, t \in T \mid X_t \text{ is geometrically rational } \}$$

of the family.

Once singularities are allowed, it is easy to pick up examples of families of rational varieties that specialize to nonrational ones, and conversely. Restricting to smooth families, it is harder to find examples of either phenomenon. In dimension $n \le 2$, for a smooth family f we have that $\mathrm{Rat}(t)$ is both open and closed (hence either $\mathrm{Rat}(f) = \emptyset$ or $\mathrm{Rat}(f) = T$ if T is connected). In general, we have the following result.

Theorem 7.1. *For any smooth family $f\colon X \to T$ of projective varieties, the rational locus $\mathrm{Rat}(f)$ is a countable union of closed subsets of T.*

We deduce this theorem from the following two properties. The first is a general (and rather elementary) property that gives an algebraic structure to the rational locus of arbitrary families.

Proposition 7.2 ([dFF13])**.** *For any family of projective varieties $f\colon X \to T$, the rational locus $\mathrm{Rat}(f)$ is a countable union of locally closed subsets.*

The second implies that smooth specializations of rational varieties are always rational.

Theorem 7.3 ([KT19])**.** *If $f\colon X \to C$ is a smooth family of projective varieties over a smooth curve and the generic fiber is rational, then every closed fiber is rational.*

It is generally expected that families of smooth cubic fourfolds in $\mathbb{P}^5$ where the rational locus $\mathrm{Rat}(t)$ is not closed (meaning, it is really an infinite countable union of closed subsets), but this is still open. Countable subfamilies of rational cubic fourfolds were constructed in [Has99, Has00], and a conjecture of Kuznetsov [Kuz10] predicts exactly which smooth cubic fourfolds are rational in terms of their derived categories would imply this. However, at the moment it remains unknown whether there are smooth cubic fourfolds that are not rational.

Nonetheless, a different example of a smooth family of projective varieties where the rational locus $\mathrm{Rat}(t)$ is not closed (meaning, it is an actual infinite countable union of closed subsets) was recently found in [HPT18]. Their result shows that Theorem 7.1 is indeed optimal.

7.1. **Countable constructibility of the rational locus.** Here we prove Proposition 7.2. The following proof was suggested by Claire Voisin. We start with a lemma.

Lemma 7.4. *Let $U \to W \to V$ be morphisms of schemes of finite type, with $U \to V$ flat and $W \to V$ projective. Then the set $v \in V$ such that U_v is irreducible and $U_v \to W_v$ is birational is constructible.*

Proof. Assuming without loss of generality that H is irreducible, these properties hold at the generic point of H if and only if they hold over a nonempty open set of H. The statement then follows by Noetherian induction. ☐

Proof of Proposition 7.2. Let $P := \mathbb{P}^n_T$ where n is the relative dimension of f. Given any morphism $W \to T$, for any $w \in W$, we denote by X_w and P_w the base change of X and P along $\{w\} \to W \to T$. Observe that, given any morphism $W \to T$, every closed subscheme $Z \subset X \times_T P \times_T W$ determines a birational map $X_w \dashrightarrow P_w \cong \mathbb{P}^n$ for every $w \in W$ such that Z_w is irreducible and both projections $Z_w \to X_w$ and $Z_w \to P_w$ are birational. Conversely, all birational maps from base changes of fibers of f to a projective space arise in this way.

Consider the relative Hilbert scheme $\mathrm{Hilb}(X \times_T P/T)$ of $X \times_T P$ over T. Let

$$V \subset \mathrm{Hilb}(X \times_T P/T)$$

be an irreducible component and $U \to V$ its universal family. By construction, U is a closed subscheme of $X \times_T P \times_T V$ and is flat over V. We have the following diagram:

$$\begin{tikzcd}
U \arrow[rr, hook] \arrow[dd] & & X \times_T P \times_T V \arrow[rrr] \arrow[dl] \arrow[dr] & & & X \times_T P \arrow[dl] \arrow[dr] & \\
& X \times_T V \arrow[dr] & & P \times_T V \arrow[dl] & X \arrow[dr] & & P \arrow[dl] \\
V \arrow[rr, equal] & & V \arrow[rrr] & & & T &
\end{tikzcd}$$

Consider then the set

$$S = \{v \in V \mid U_v \text{ is irreducible and } U_v \to X_v \text{ and } U_v \to P_v \text{ are birational}\}.$$

By applying Lemma 7.4 to $U \to X \times_T V \to V$ and $U \to P \times_T V \to V$, we see that S is a constructible subset of V. By Chevalley's theorem, the image of S in T is also constructible, and as such can be written as a finite union of locally closed subsets. The union of all these sets, as V varies among the irreducible components of the Hilbert scheme, is $\mathrm{Rat}(f)$. The statement then follows by the fact that the Hilbert scheme has countably many irreducible components. □

Remark 7.5. An analogous property is satisfied by the locus of unirational varieties: the argument easily adjusts to this case by relaxing the condition on $U_v \to X_v$ from being birational to being dominant. A related result concerning the behavior of uniruledness in families is proven in [Kol96, Therem IV.1.8], where it is shown that the locus of uniruled varieties in an equidimentional proper family is a countable union of closed subsets of the base. Note, however, that is not known whether the analogue of Theorem 7.3 holds for unirationality.

7.2. **Smooth specializations of rational varieties.** We now address Theorem 7.3. We start with a definition.

Definition 7.6. For any field k, we define the *Burnside group* $\mathrm{Burn}(k)$ of k to be the Grothendieck group generated the set of k-isomorphism classes $[L/k]$ of finitely generated field extensions L of k.

Any element of $\mathrm{Burn}(k)$ can be written as a finite sum $\sum_i n_i[L_i/k]$ where each L_i is a finitely generated field extension of k and $n_i \in \mathbb{Z}$, and every such expression defines an element of $\mathrm{Burn}(k)$. Furthermore, given two finitely generated field extensions L and L', we have $[L/k] = [L'/k]$ in $\mathrm{Burn}(k)$ if and only if $L \simeq L'$ over k. Note, in particular, that the Burnside ring carries a natural grading

$$\mathrm{Burn}(k) = \bigoplus_{n \ge 0} \mathrm{Burn}_n(k)$$

given by transcendence degree over k.

The idea of the proof is to construct, given a smooth curve C and a close point $0 \in C$, a specialization map from the Burnside semi-ring of the function field of C to the Burnside semi-ring of the residue field at 0. Such specialization map should have the property that if $f\colon X \to C$ is a smooth projective family of varieties, then the class of the function field of the generic fiber X_η should specialize to the class of the function field of the fiber over 0. As we will see later, this property (and the existence of such specialization map) will automatically imply the theorem.

Let therefore C be a smooth curve, and let $0 \in C$ be a closed point. We denote by $R = \mathcal{O}_{C,0}$ the local ring and by K the quotient field of R (i.e., the function field of C).[3]

Theorem 7.7 ([KT19])**.** *For every n, there is a group homomorphism*

$$\rho_n\colon \mathrm{Burn}_n(K) \to \mathrm{Burn}_n(\mathbb{C}),$$

associated to the germ $(C, 0)$, with the following property:

> *If $f\colon \mathcal{X} \to \operatorname{Spec} R$ is any smooth proper morphism of relative dimension n, and $K(\mathcal{X}_\eta)$ and $\mathbb{C}(\mathcal{X}_0)$ are, respectively, the function fields of the generic fiber $\mathcal{X}_\eta$ and the closed fiber $\mathcal{X}_0$ of f, then $\rho_n([K(\mathcal{X}_\eta)/K]) = [\mathbb{C}(\mathcal{X}_0)/\mathbb{C}]$.*

Definition 7.8. Given a smooth proper variety X over K, a *simple normal crossing model* (*snc model*, for short) of X over R is a projective morphism $f\colon \mathcal{X} \to \operatorname{Spec} R$ such that $\mathcal{X}$

[3] Equivalently, one can restrict to the formal neighborhood of C at 0, hence take $R = \mathbb{C}[[t]]$, where t is a local parameter of C at 0, and $\mathbb{C} = k((t))$, as it is done in [KT19].

is a regular scheme, the generic fiber $\mathcal{X}_e$ is isomorphic to X, and the closed fiber $\mathcal{X}_0$ is supported on a simple normal crossing divisor

$$D = \sum_{i=1}^{r} D_i.$$

Here, the D_i are smooth irreducible divisors intersecting transversally, and we assume that all intersections $D_I := \bigcap_{i\in I} D_i$, for $\emptyset \neq I \subset \{1,\dots,r\}$, are irreducible. We will tacitly restrict index sets I to those for which $D_I \neq \emptyset$. For every $I \neq \emptyset$, we denote by η_I the generic point of D_I and let L_I be a purely transcendental extension of degree $|I|-1$ of the residue field $k(\eta_I)$:

$$L_I := k(\eta_I)(x_1,\dots,x_{|I|-1}).$$

Note that L_I is an extension of $\mathbb{C}$ of transcendence degree n. We call L_I the field *associated to* the stratum D_I. We denote by $\mathcal{D}(D)$ the *dual complex* of D. This is a CW-complex with a vertex v_i for every component D_i and a face σ_I of dimension $|I|$ for every nonempty stratum D_I, with glueing dictated in the obvious way by inclusions among strata.

Proof of Theorem 7.7. We define ρ_n on any $[L/K]$, where L is a finitely generated extension of K, and extend by linearity.

Given L, we fix a smooth proper variety X over K with function field L and a snc model $\mathcal{X}$ of X over R with central fiber supported on the divisor $D = \sum_{i=1}^{r} D_i$. We then set

$$\rho_n([L]) := \sum_{I\neq\emptyset} (-1)^{|I|-1}[L_I/\mathbb{C}].$$

The resulting map ρ_n clearly satisfies the required property stated in the theorem. All we need to do is to check that ρ_n is well defined. Namely, we need to check that it does not depend on the choice of X nor, for any given X, on the choice of snc model $\mathcal{X}$.

Let us show first that, given a smooth proper model X over K with function field L, the definition of ρ_n is independent of the choice of snc model $\mathcal{X}$ over R. If $\mathcal{X}'$ is another snc model of X over R, then there is a birational map $\mathcal{X} \dashrightarrow \mathcal{X}'$ defined over R which is an isomorphism away from the central fibers. By the weak factorization theorem [AKMW02], ϕ decomposes as the composition of blow-ups and blow-downs with smooth centers that are contained in the central fibers and are transversal to their strata. More precisely, there is a sequence of snc models $\mathcal{X}^{(i)}$ of X over R, with reduced central fibers $D^{(i)} = (\mathcal{X}^{(i)})_{\mathrm{red}}$, and birational maps $\phi_i\colon \mathcal{X}^{(i-1)} \dashrightarrow \mathcal{X}^{(i)}$, such that

(1) $\mathcal{X}^{(0)} = \mathcal{X}$ and $\mathcal{X}^{(m)} = \mathcal{X}'$, and
(2) either ϕ_i is the blow-up of $\mathcal{X}^{(i)}$ along a smooth subvariety contained in the central fiber and intersecting transversally every stratum of $D^{(i)}$, or ϕ_i^{-1} is the blow-up of $\mathcal{X}^{(i-1)}$ along a smooth subvariety contained in the central fiber and intersecting transversally every stratum of $D^{(i-1)}$.[4]

We therefore reduce to verify that the definition of $\rho_n([L])$ does not change if we replace $\mathcal{X}$ with a snc model $\mathcal{X}'$ that is obtained by blowing up a smooth subvariety $Z \subset \mathcal{X}$ of codimension ≥ 2 that is contained in the central fiber and intersects transversally every stratum of D. The reduced central fiber of $\mathcal{X}'$ supports the divisor $D' = \sum_{i=0}^{r} D'_i$ where D'_0 is the exceptional divisor of ϕ and, for $i > 0$, D'_i is the proper transform of D_i. We

[4]A subvariety C of a regular scheme $\mathcal{X}$ is said to intersect every stratum of a snc divisor D *transversally* if in the formal neighborhood at any closed point, C and D are locally defined by monomial equations in the same system of coordinates.

denote by η'_J the generic point of a stratum D'_J of D', let $L'_J := k(\eta'_I)(x_1, \dots, x_{|J|-1})$, and write

$$\rho'_n([L]) = \sum_{J \neq \emptyset} (-1)^{|J|-1} [L'_J/\mathbb{C}]$$

for the element of $\mathrm{Burn}(\mathbb{C})$ obtained using $\mathcal{X}'$ in place of $\mathcal{X}$ in the definition of the specialization map. One way to see how $\rho'_n([L])$ matches $\rho_n([L])$ is by looking at the dual complex $\mathcal{D}(D)$ of D and comparing it to the dual complex $\mathcal{D}(D')$ of D'.

Suppose first that Z is equal to a stratum of D_{I_0} of D. In this case, $\mathcal{D}(D')$ is a *stellar subdivision* of $\mathcal{D}(D)$: the two complexes have the same underlying topological space,[5] one vertex v'_0 (corresponding to D'_0) is added inside the face σ_{I_0}, and $\mathcal{D}(D')$ is the minimal subdivision of $\mathcal{D}(D)$ that includes this new vertex. The support of every new face σ'_J added in this process is contained in the support of a face σ_I, and the corresponding stratum D'_J has associated field $L'_J \simeq L_I$. Let us write $J \succeq I$ whenever $|\sigma'_J| \subset |\sigma_I|$. The sign in which the corresponding term $[L'_J/k]$ appears in $\rho'_n([L])$ depends on $|J|$, and needs to be compared with the sign of $[L_I/k]$ in $\rho_n([L])$, which depends on $|I|$. Then, to verify that $\rho'_n([L]) = \rho_n([L])$, one needs to check that

$$(-1)^{|I|-1} = \sum_{J \succeq I} (-1)^{|J|-1}.$$

This can be check easily, or just deduced from the fact that computing Euler characteristics is independent of the triangulation.

Suppose now that Z is not equal to a stratum of D. The set $S = \{I \mid Z \cap D_I \neq \emptyset\}$ parameterizes a sub-CW-complex $\mathcal{D}_S(D)$ of $\mathcal{D}(D)$, and $\mathcal{D}(D')$ is the complex obtained from $\mathcal{D}(D)$ by adding an additional vertex v'_0 (again, corresponding to D'_0), and taking the cone with vertex v'_0 over $\mathcal{D}_S(D)$. For every J such that σ'_J is a face of the open cone $\mathcal{D}(D') \setminus \mathcal{D}(D)$, let D_I be the minimal stratum containing $\phi(D'_J)$, and write $J \succeq I$. We have that $L'_J = L_I$, and a similar argument as before shows this time that

$$\sum_{J \succeq I} (-1)^{|J|-1} = 0.$$

Therefore we have $\rho'_n([L]) = \rho_n([L])$, and this shows that, given X, the the definition of ρ_n is independent of the choice of snc model $\mathcal{X}$.

It remains to check that the definition is independent of the choice of X. If X' is any other smooth proper model, then it is birationally equivalent to X By the weak factorization theorem, we reduce to the case where X' is the blow-up of X along a smooth center W. By the existence of resolutions of marked ideals [Wło05], we can find a snc model $\mathcal{X}$ of X over R such that the closure $\overline{W}$ of W in $\mathcal{X}$ intersects transversally the central fiber $\mathcal{X}_0$ and does not contain any of its strata in its support. Then the blow-up of $\mathcal{X}$ along $\overline{W}$ produces a snc model $\mathcal{X}'$ of X', and the map $\mathcal{X}' \to \mathcal{X}$ induces birational maps from the strata of $\mathcal{X}'_0$ to the strata of $\mathcal{X}_0$. Therefore the definition of $\rho_n([L])$ remain unchanged. This finishes the proof of the theorem. ☐

Proof of Theorem 7.3. Let $f\colon X \to C$ is a smooth family of projective varieties of relative dimension n over a smooth curve, and assume that the generic fiber X_η is rational. By hypothesis, X_η is birational (over $K = k(\eta)$) to $\mathbb{P}^n_K$. This implies that X is birational (over

[5] There is a natural way to identify these spaces once we think of the points of each dual complex as valuations on L.

C) to $\mathbb{P}^n_C$. Fix any closed point $0 \in C$ and let $R = \mathcal{O}_{C,0}$ and $\mathcal{X} = X \times_C \operatorname{Spec} R$. Using the model $\mathcal{X}$ over R to compute the image of $K(X_\eta)$ via $\rho_n \colon \operatorname{Burn}_n(K) \to \operatorname{Burn}_n(\mathbb{C})$, we get

$$\rho_n([K(X_\eta)/K]) = [\mathbb{C}(X_0)/\mathbb{C}].$$

Using the model $\mathbb{P}^n_R$ to do the same computation, we get

$$\rho_n([K(X_\eta)/K]) = [\mathbb{C}(\mathbb{P}^n)/\mathbb{C}].$$

This implies that X_0 is rational. □

Proof of Theorem 7.1. Let $f \colon X \to T$ be a smooth family of projective varieties. By Proposition 7.2, the rational locus $\operatorname{Rat}(t) \subset T$ is a countable union of locally closed subsets of T. To prove that it is a countable union of closed subsets of T, we need to show that if $S \subset \operatorname{Rat}(f)$ is any set that is locally closed in T and $\overline{S}$ is its closure in T, then $\overline{S} \subset \operatorname{Rat}(f)$. It suffices in fact to check that every closed point of $\overline{S}$ is in $\operatorname{Rat}(t)$. Let therefore $p \in \overline{S}$ be an arbitrary closed point, and let $B \subset \overline{S}$ be a general complete intersection curve; we assume that the generic point ξ of B is contained in S. By hypothesis, X_ξ is geometrically rational. Then we can find a covering map $C \to B$, where C is a smooth curve, such that if $\eta \in C$ is the generic point then the base change X_η is $k(\eta)$-rational. It follows then by Theorem 7.3 that the closed fibers of $X \times_T C$ are all rational, hence X_p is rational. This implies that $\overline{S} \subset \operatorname{Rat}(t)$. □

Theorem 7.3 was inspired by an analogous result of [NS19] which, in the same setting, establishes the existence of a *motivic reduction map*

$$\mathrm{MR} \colon K_0(\mathrm{Var}_K) \to K_0(\mathrm{Var}_\mathbb{C})$$

with the property that if $\mathcal{X}$ is smooth over R then $\mathrm{MR}([\mathcal{X}_e]) = [\mathcal{X}_0]$ modulo $[\mathbb{A}^1_k]$. Here $K_0(\mathrm{Var}_k)$ denotes the Grothendieck ring of varieties over a field k. We recall that this ring is additively generated by isomorphism classes of varieties modulo the *scissor relations* $[X] = [Y] + [X \setminus Y]$ holding whenever X is a variety and $Y \subset X$ is a closed subvariety; multiplication it defined by taking products of varieties.

It was proved in [LL03] that two k-varieties X and X' defines the same class in the quotient ring $K_0(\mathrm{Var}_k)/[\mathbb{A}^1_k]$ if and only if they are *stably birational equivalent*, which means that there is a positive integers m, n such that $X \times \mathbb{P}^m_k$ is birationally equivalent to $X' \times \mathbb{P}^n_k$. In particular, the aforementioned result of [NS19] implies that smooth specializations of stably rational varieties (i.e., varieties that are stably birational to a projective space) are themselves stably rational.

8. Lecture 8: Singular degenerations of rational varieties

The theorem of Konstevich and Tschinkel does no longer hold if we allow singular fibers, even in low dimensions. For example, consider the case of smooth cubic surfaces degenerating to a cone over an elliptic curve: after a base change, we can assume that the generic fiber is rational, but the special fiber is not.

Nonetheless, positive results still hold in good situations.

8.1. **B-rational singularities.**

The proof of Theorem 7.3 can be extended to deal with singular families in the following way. We follow closely [KT19].

The first step is to extend the definition of Burnside group.

Definition 8.1. Given a field k and a separated scheme S of finite type over k, we denote by $\mathrm{Burn}(S/k)$ the Grothendieck group generated by finitely generated field extensions L/k endowed with a map $\operatorname{Spec} L \to S$, modulo isomorphisms over S. We denote the element defined by L by $[L/S]$ (granting the ground field k is clear from the context). Just like the Burnside group of a field, there is a natural grading

$$\mathrm{Burn}(S/k) = \bigoplus_{n\geq 0} \mathrm{Burn}_n(S/k)$$

given by transcendence degree over k.

A morphism $g\colon S' \to S$ of separated schemes of finite type over k induces a group homomorphism

$$g_*\colon \mathrm{Burn}(S'/k) \to \mathrm{Burn}(S/k),$$

preserving grading, and this correspondence is functorial.

Theorem 8.2 ([KT19])**.** *To any n-dimensional variety X and any proper closed subscheme $Z \subset X$, there is a unique way of associating an element*

$$\partial_Z(X) \in \mathrm{Burn}_{n-1}(X/\mathbb{C})$$

such that the following properties are satisfied:

1. *If X is smooth and $Z = \sum D_i$ is a snc divisor, then*
$$\partial_Z(X) = \sum_{I\neq\emptyset} (-1)^{|I|-1}[L_I/Z]$$
where, as before, we denote by η_I the generic point of the stratum $D_I = \bigcap_{i\in I} D_i$ and set $L_I = k(\eta_I)(x_1, \dots, x_{|I|-1})$.
2. *If $g\colon X' \to X$ is a log resolution of (X, Z) and $Z' = f^{-1}(Z)$, then*
$$\partial_Z(X) = (g|_{Z'})_*(\partial_{Z'}(X')).$$

By resolution of singularities, the properties listed in the theorem provide a definition of $\partial_Z(X)$, and the proof that the definition is independent of a choice of resolution is similar to the proof of Theorem 7.3. It is clear from the theorem that the element $\partial_Z(X)$ only depends on the reduced subscheme Z_{red}.

Definition 8.3. Let X be a variety and $Z \subset X$ be a subvariety of codimension 1. The pair (X, Z) has *B-rational singularities* if

$$\partial_Z(X) = [\mathbb{C}(Z)/Z].$$

Example 8.4. Suppose X is smooth and Z is a subvariety of codimension 1 with an isolated singularity $p \in Z$. Assume that the tangent cone $C_p(Z)$ of Z at p is smooth away from the vertex (we say that p is an *ordinary multiple point* (or *quasi-homogeneous singularity*) of Z). Note that this means that the blow-up $\mathbb{Z} = \mathrm{Bl}_p Z \to Z$ of Z at p is smooth and intersects transversally the exceptional divisor $E \cong \mathbb{P}^{n-1}$ of $\widetilde{X} = \mathrm{Bl}_p X \to X$ along its own exceptional divisor F, which is therefore embedded in E as a smooth hypersurface. Using the model $(\widetilde{X}, \widetilde{Z} + E)$, which provides a log resolution, we see that

$$\partial_Z(X) = [\mathbb{C}(Z)/Z] + [\mathbb{C}(E)/Z] - [\mathbb{C}(F)(x)/Z].$$

Note that both maps $\operatorname{Spec}\mathbb{C}(E) \to Z$ and $\operatorname{Spec}\mathbb{C}(F) \to Z$ factor through p. We conclude that (X, Z) has B-rational singularities if and only if $F \times \mathbb{P}^1$ is rational. This happens, for instance, if Z has an ordinary double point.

Example 8.5. Let X be a smooth surface and Z an irreducible curve. Then (X, Z) has B-rational singularities if and only if Z is unibranched at its singular points.

We can now generalize Theorem 7.3 to singular families as follows. The definition of B-rational singularities is tailored for the purpose.

Theorem 8.6 ([KT19])**.** *Let $f\colon X \to C$ is a proper flat morphism from a variety X to a smooth curve, and assume that generic fiber is rational. Then for every closed point $0 \in C$ such that the pair (X, X_0) has B-rational singularities, the fiber X_0 is rational.*

The proof of this theorem follows the same strategy of the proof of Theorem 7.3, by passing to a log resolution of the pair and using the condition of B-rationality to conclude that X_0 is rational.

Example 8.7. Going back to the example of smooth cubic surfaces degenerating to a cone over an elliptic curve, we can realize such a degeneration in a family $f\colon X \to C$ whose total space of deformation is smooth (e.g., by the equation $x^3 + y^3 + z^3 + tw^3 = 0$ in $\mathbb{P}^3 \times \mathbb{A}^1$, where t is the coordinate in $\mathbb{A}^1$). The above theorem does not apply to this example because (X, X_0) in this case does not have B-rational singularities (cf. Example 8.4).

8.2. **Rationality in families of threefolds.** A key feature of the example of smooth cubic surfaces degenerating to a cone over an elliptic curve is that in this case the central fiber is not rationally connected. We recall that a projective variety V is *rationally connected* if two very general points $p, q \in V$ can be join by a rational curve in V, that is, there is a morphism $h\colon \mathbb{P}^1 \to V$ such that $h(0) = p$ and $h(\infty) = q$.[6] We should stress that in the definition of rational connectedness, we require that p and q are connected by a single rational curve, and not by a chain of rational curves (which would correspond to the notion of *rationally chain connected*). Note indeed that the cone over an elliptic curve is not rationally connected but it is rationally chain connected, since any point is connected by a rational curve to the vertex of the cone.

As rational projective varieties are rationally connected, asking that the fibers are (geometrically) rationally connected is the minimal requirement if one wants to extend Theorem 7.1 to singular families. It turns out that in relative dimension 3, this is in fact enough.

Let us first define the *rationally connected locus* of a family of projective varieties $f\colon X \to T$ to be

$$\mathrm{RC}(f) := \{t \in T \mid X_t \text{ is geometrically rationally connected}\}.$$

Remark 8.8. In characteristic zero (which is our setting), if f is a smooth morphism then $\mathrm{RC}(f)$ is both open and closed in T by [Kol96, Theorem 3.11]. It follows by resolution of singularities, generic smoothness, and Noetherian induction that in general, for any family f, the rationally connected locus $\mathrm{RC}(f)$ is a constructible subset of T.

As we already mentioned, there is an inclusion $\mathrm{Rat}(t) \subset \mathrm{RC}(f)$. In relative dimension 3, we have the following generalization of Theorem 7.1.

Theorem 8.9 ([dFF13])**.** *For every family $f\colon X \to T$ of projective varieties of dimension $n \le 3$, the rational locus $\mathrm{Rat}(f)$ is a countable union of closed subsets of the rationally connected locus $\mathrm{RC}(f)$.*

[6]In positive characteristics, the correct notion for our purposes is that of *separably rational connectedness*, where the map $h\colon \mathbb{P}^1 \to V$ is required to be a separable morphism.

We deduce this result from Proposition 7.2 and the following result (cf. [dFF13, Theorem 3.1]).

Theorem 8.10. *Let $f\colon X \to C$ be a flat projective morphism of relative dimension $n \le 3$ from a variety to a smooth curve. Assume that the generic fiber X_η is rational. Then for every closed point $t \in C$, every rationally connected component D of X_t that intersects the nonsingular locus of X is rational. In particular, if X is regular in codimension 1 (e.g., normal), then every rationally connected component D of X_t is rational.*

Proof. The assumption that the generic fiber X_η is rational implies that there is birational map $\phi\colon X \dashrightarrow \mathbb{P}^n_T$ defined over T. Since D has codimension 1 in X and the generic point of D is contained in X_{reg}, the local ring $\mathcal{O}_{X,D}$ is a discrete valuation ring. That is, D defines a divisorial valuation val_D on the function field $\mathbb{C}(X)$ of X. As the latter is birational to $\mathbb{P}^n_T$, we can regard val_D as a divisorial valuation of the function field of $\mathbb{P}^n_T$. Using that the birational map ϕ is defined over T and $\mathbb{P}^n_T$ is proper over T, we see that val_D has center in $\mathbb{P}^n_T$ which is contained in the fiber of $\mathbb{P}^n_T$ over t. If this center is equal to the whole fiber, then ϕ maps D birationally to the fiber and we are done. Assume otherwise that the center is a proper subset of the fiber of $\mathbb{P}^n_T$.

We consider sequence of blow-ups

$$\cdots \to Y_i \xrightarrow{g_i} Y_{i-1} \to \cdots \to Y_1 \xrightarrow{g_1} Y_0 = \mathbb{P}^n_T$$

defined inductively by taking g_i to be the blow-up of Y_{i-1} along the center $W_{i-1} = c_{Y_{i-1}}(\mathrm{val}_D)$ of val_D in there. Here we regard the center of the valuation as a closed subvariety, and its blow-up is the blow-up of its defining ideal sheaf. Note that the models Y_i and the center W_i may be singular, and the exceptional divisors of the blow-ups g_i may have several irreducible components. However, one can prove by induction that for every i the both variety Y_{i-1} and the center W_{i-1} are smooth at the generic point of W_{i-1} the exceptional divisor of g_i contains a unique irreducible component E_i dominating the center W_{i-1}. In fact, after restricting over an open neighborhood of the generic point of W_{i-1}, g_i is locally the blow-up of a smooth variety along a smooth subvariety, and therefore the projection $E_i \to W_{i-1}$ is generically a projective bundle. Note also that W_i is a subvariety of E_i dominating W_{i-1}.

By a lemma of Zariski [KM98, Lemma 2.45], after finitely many steps we reach a point where $W_i = E_i$ and $\mathcal{O}_{Y_i,E_i} = \mathcal{O}_{X,D}$ via the identification $\mathbb{C}(X) \simeq \mathbb{C}(Y_i)$. For any such i the birational map $Y_i \dashrightarrow X$ restricts to a birational map $E_i \dashrightarrow D$, and if we pick i to be the least index with the above property then W_{i-1} is a proper subvariety of E_{i-1}, and hence $\dim W_{i-1} < \dim E_i$.

Fix i as above. Since E_i is birational to D, it is rationally connected. Therefore W_{i-1} is rationally connected, and since its dimension is at most 2 (as $n \le 3$), it follows that W_{i-1} is rational. Using then that $E_i \to W_{i-1}$ is generically a projective bundle, we deduce that E_i is rational. This proves that D is rational. □

Remark 8.11. One can reformulate the proof in a more transparent way using resolution of singularities and the weak factorization theorem. Writing the proof as we did has the advantage that the argument is characteristic-free. In fact, the same result holds over algebraically closed fields of arbitrary characteristics as long as one requires separably rational connectedness in place of rational connectedness.

8.3. **A higher dimensional example.** We close with the discussion of an example showing that a result like Theorem 8.9 cannot hold in arbitrary dimensions. The example, which is in fact quite simple, was originally constructed in [dFF13] conditionally to a result on the failure of stable rationality of some Fano hypersurfaces that was not available at the time. The example was revisited and included in [Tot16a] where the necessary result on stable rationality is established. More examples were later obtained in [Tot16b, Per17]. Notably, the example in [Per17] only relies on Voisin's theorem on the failure of stable rationality of certain double covers of $\mathbb{P}^4$ [Voi15], which is the first result of this kind and the only such result that was available at the time of the writing of [dFF13].[7]

The idea of the example is to construct an elementary transformation

$$\begin{tikzcd} & \mathrm{Bl}_V(\mathbb{P}^1 \times \mathbb{P}^n) \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 \times \mathbb{P}^n \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

by blowing up a smooth hypersurface V of degree n in a fiber $\{0\} \times \mathbb{P}^n$ of g and contract the proper transform of the fiber to a point. The resulting fiber X_0 of f is a cone over V, hence it is rationally connected. However, if V is not stably rational then X_0 is not rational, hence $\mathrm{Rat}(f)$ is locally closed but not closed in $\mathrm{RC}(f)$.

To actually realize the transformation, consider the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(n)$ on $\mathbb{P}^1 \times \mathbb{P}^n$, and let $\mathcal{I}_V$ be the ideal sheaf of W in $\mathbb{P}^1 \times \mathbb{P}^n$. The sheaf $\mathcal{L} \otimes \mathcal{I}_V$ is globally generated, and thus the linear system $|\mathcal{L} \otimes \mathcal{I}_V|$ defines a rational map

$$\psi \colon \mathbb{P}^1 \times \mathbb{P}^n \dashrightarrow X \subset \mathbb{P}H^0(\mathcal{L} \otimes \mathcal{I}_V)$$

which is resolved by the blow-up $Y := \mathrm{Bl}_V(\mathbb{P}^1 \times \mathbb{P}^n)$ of $\mathcal{I}_V$. Here X denotes the closure of the image of the map. The map ψ is defined over $\mathbb{P}^1$, and thus there is a morphism $f \colon X \to \mathbb{P}^1$. Furthermore, ψ induced an isomorphism away from the fibers over 0, so that $X_t \cong \mathbb{P}^n$ for $t \neq 0$. On the other hand the induced morphism $Y \to X$ contracts the proper transform of $\{0\} \times \mathbb{P}^n$ to a point and maps the exceptional divisor of the blow-up birationally to the fiber X_0, which is thus isomorphic to the cone over V.

Now, if $n \geq 4$, then V is not stably rational by [Tot16a], hence X_0 is not rational. This shows that the hypothesis on dimension in Theorem 8.9 is optimal.

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[7] As a side remark, it is interesting to point out that Voisin's strategy of proof, which was picked up in many subsequent papers on the subject, uses a degeneration argument to singular varieties (a variety with an ordinary double point) to compute certain stably birational invariants.

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