Fractals

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Iterated Function Systems

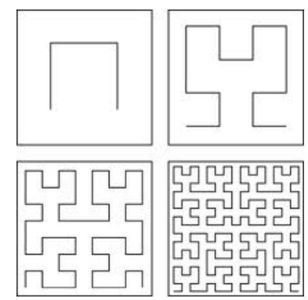
The fractals are constructed using a fixed geometric replacement rule: Cantor set, Sierpinski carpet or gasket, Peano curve, Koch snowflake, Menger sponge.

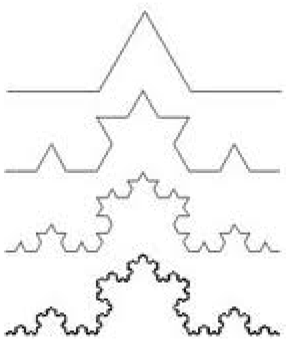
Karl Weierstrass (1872): Nondifferentiable function

Georg Cantor (1883): Cantor set

Giuseppe Peano (1890), David Hilbert (1891):

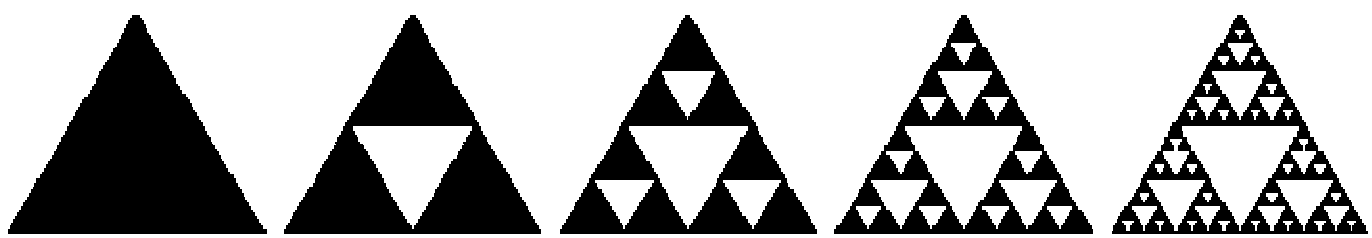
Space filling curves





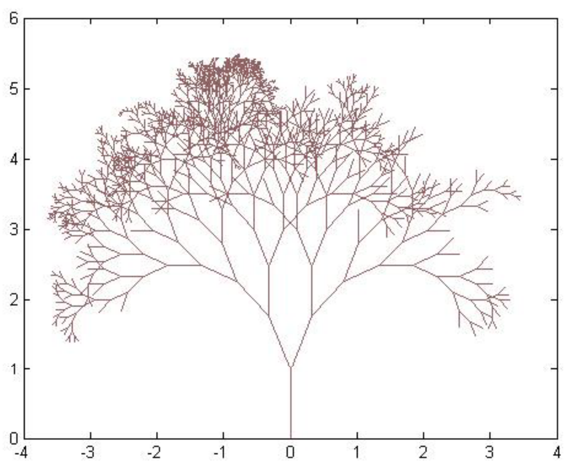
Helge Von Koch (1904): Koch snowflake

Waclaw Sierpinski (1915): Sierpinski triangle and carpet



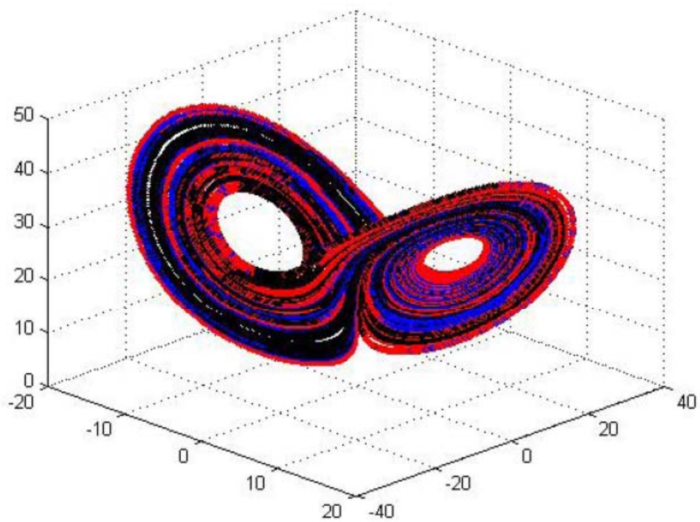
Random Fractals

Random fractals can be generated by stochastic rather than deterministic processes, for example, trajectories of the Brownian motion, fractal landscapes and random trees.

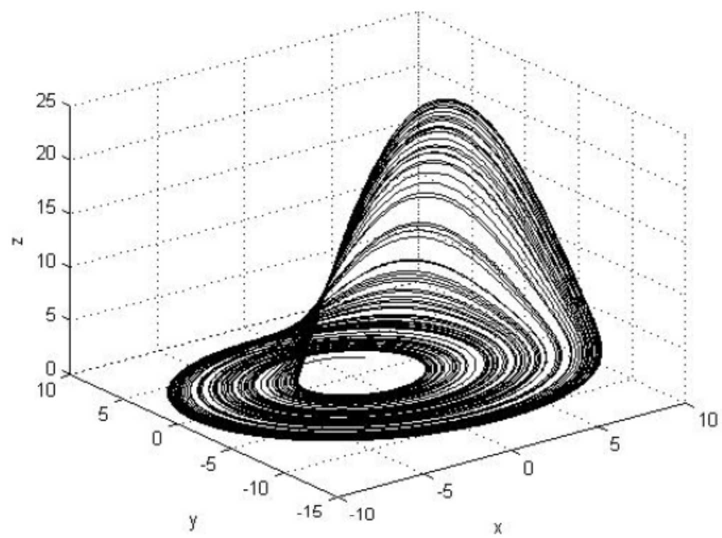


Fractals as Attractors of Nonlinear Dynamical Systems

Fractals can be generated as strange attractors of Nonlinear Dynamical Systems, for example, attractor of trajectories of the Lorenz dynamical system, Rossler attractor, attractor of Ueda system.



Lorenz attractor

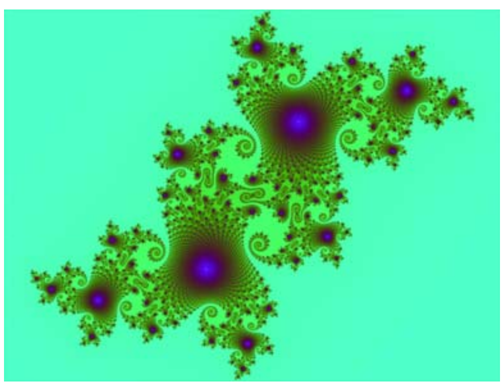


Rossler attractor

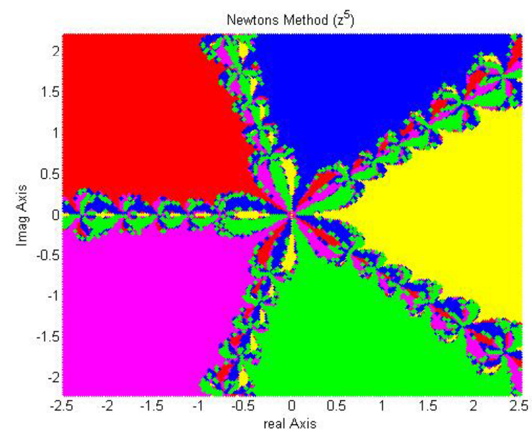
Escape-time fractals

Escape-time fractals — These are based on sensitive dependence of the trajectories on the starting point or on initial conditions.

Examples of this type are the Julia and Mandelbrot sets (Gaston Julia, Pierre Fatou, Benoit Mandelbrot), and Newton fractal.



Julia set

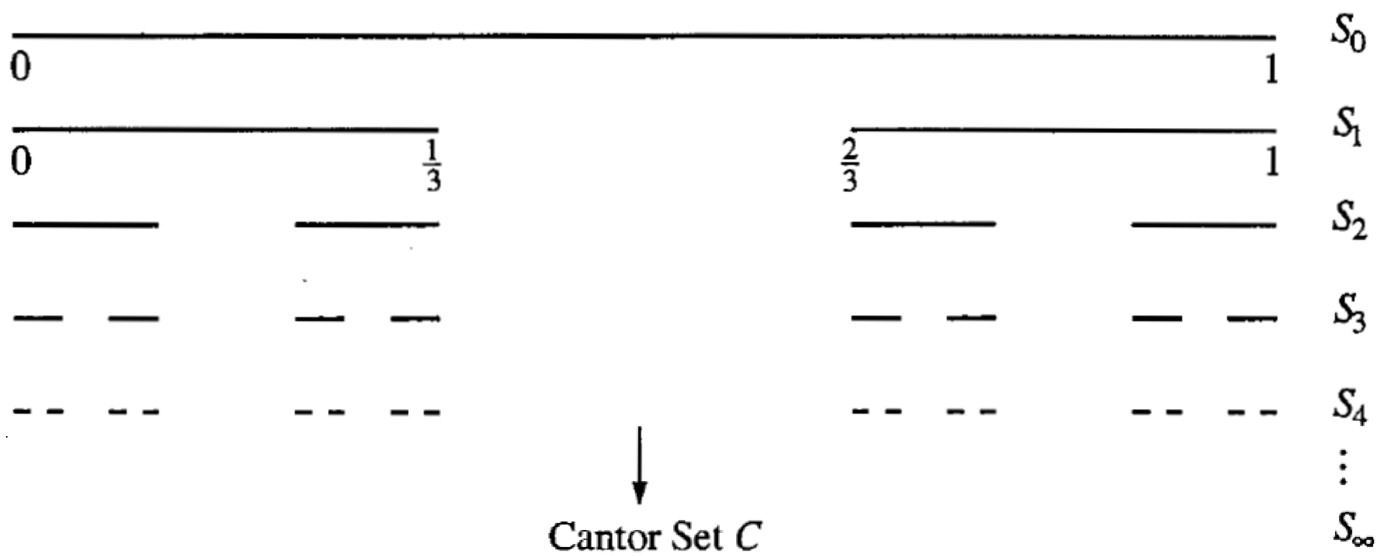


Newton fractal

Forthcoming Book: Benoit Mandelbrot, A Life in Many Dimensions

- **Contents:**
- Introduction — Benoit Mandelbrot: Nor Does Lightning Travel in a Straight Line *(M Frame)*
- Fractals in Mathematics — Chapters by Michael Barnsley, Julien Barral, Kenneth Falconer, Hillel Furstenberg, Stephane Jaffard, Michael Lapidus, Jacques Peyriere & Murad Taqqu
- Fractals in Physics — Chapters by Amon Aharony, Bernard Sapoval, Michael Shlesinger, Katepalli Sreenivasan & Bruce West
- Fractals in Computer Science — Chapters by Henry Kaufman & Ken Musgrave
- Fractals in Engineering — Chapters by Nathan Cohen & Marc-Olivier Coppens
- Fractals in Finance — Chapters by Martin Shubik & Nassim Taleb
- Fractals in Art — Chapters by Javier Barrallo, Ron Eglash & Rhonda Roland Shearer
- Fractals in History — Chapter by John Gaddis
- Fractals in Architecture — Chapter by Emer O'Daly
- Fractals in Physiology — Chapter by Ewald Weibel
- Fractals in Education — Chapters by Harlan Brothers & Nial Neger
- Fractals in Music — Chapter by Charles Wuorinen
- Fractals in Film — Chapter by Nigel Lesmoir-Gordon
- Fractals in Comedy — Chapter by Demetri Martin

Cantor Set



On each iteration step, delete middle third of each interval.

Properties:

C has structure at arbitrary small scales;

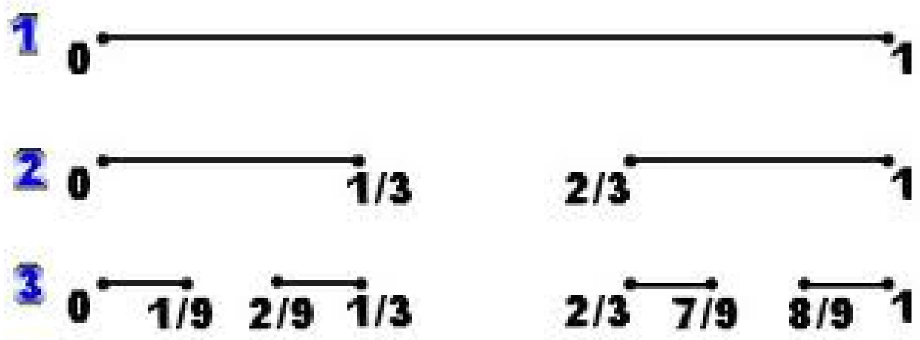
C is self-similar;

The dimension of C is not integer;

C has measure zero;

C consists of uncountably many points.

Cantor Set: Measure zero



Sum up lengths of the deleted sets:

$$\frac{1}{3}+2\frac{1}{3^2}+4\frac{1}{3^3}+\ldots \quad = \quad \frac{1}{3}\left(1+\frac{2}{3^2}+\frac{2^2}{3^3}+\ldots\right)$$

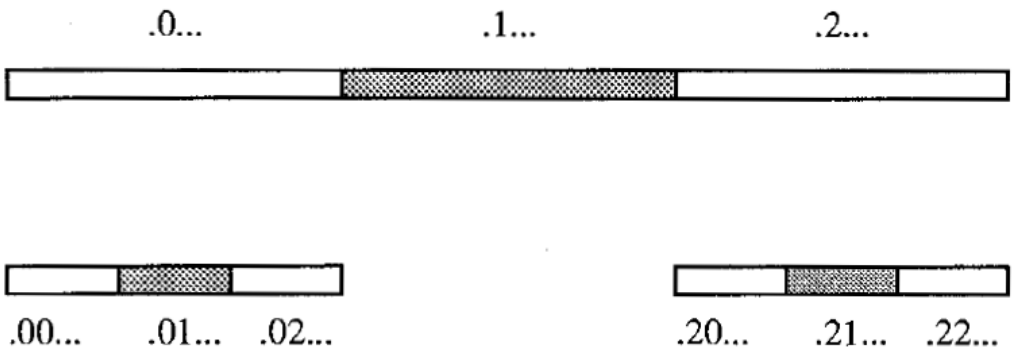
$$= \quad \frac{1}{3}\;\frac{1}{1-2/3} \quad =1$$

Measure (length) of the deleted set = 1

Measure of C is zero.

Cantor Set: Continuum of points

Expand x in base-3: $x \in [0,1]$, $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \ldots$, $a_k \in \{0, 1, 2\}$



Points in the Cantor set do not have 1 in the base-3 representation

One-to-one correspondence with base-2 representation of the points in the unit interval

→ Cardinality of Cantor set is continuum !

Cantor set

Cantor set can be generated iteratively using two transformations:

$$f_1(x) = \frac{1}{3}x\,, \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}$$

Construct a sequence of closed nested intervals:

$$I_0 \supset I_1 \supset I_2 \supset .... \supset I_n \supset ...$$

$$I_0 = [0,1]$$

$$I_1 = f_1(I_0) \cup f_2(I_0) = I_{00} \cup I_{01}$$

$$I_2 = f_1(I_1) \cup f_2(I_1) = f_1(I_{00} \cup I_{01}) \cup f_2(I_{00} \cup I_{01})$$

$$= I_{000} \cup I_{010} \cup I_{001} \cup I_{011}$$

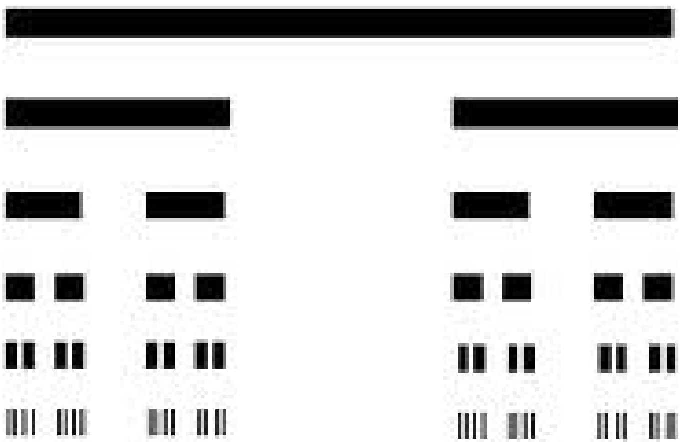
...

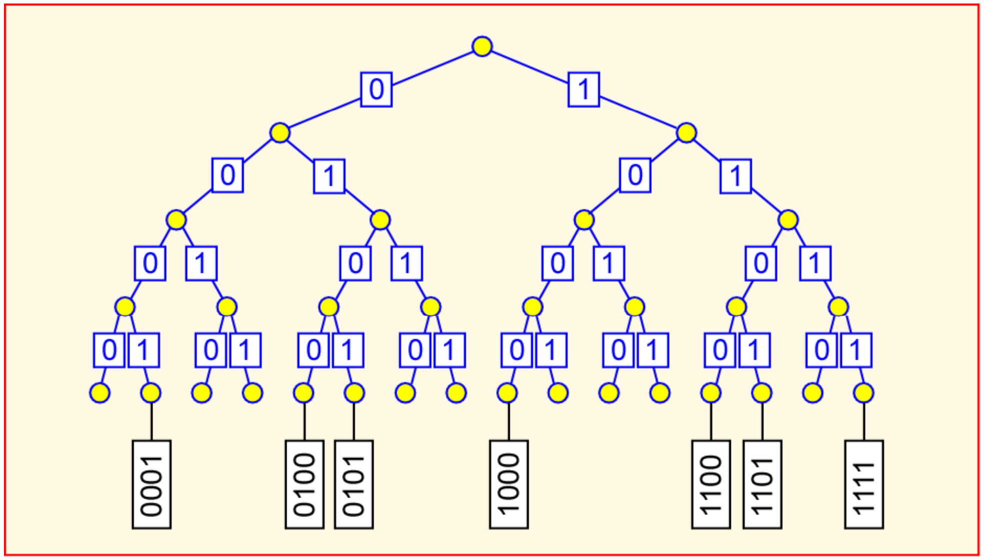
Affine transformations in R^1:

$$f(x) = a\,x + b,$$

a is scaling coeff., b is translation or shift

Cantor Set: Continuum of points





Cantor set is equivalent to the set of all possible sequences of 0 and 1

Affine transformations in 2D

2D affine transformation has the form:

$$w(x) = w\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = A\,x + t$$

Matrix A can be written as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r_1 \cos\theta_1 & -r_2 \sin\theta_2 \\ r_1 \sin\theta_1 & r_2 \cos\theta_2 \end{pmatrix}$$

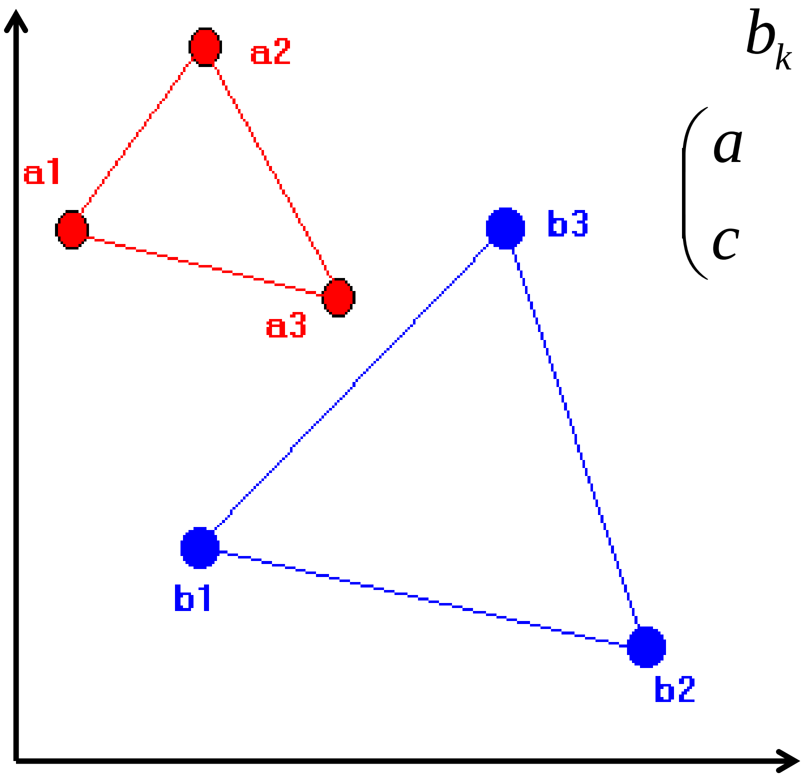
Examples: Scaling, shift, rotation, reflection.

Affine transformation consists of a linear transformation A followed by a translation t.

Affine transformations in 2D

How to find w ?

Use: w(Red_triangle) = Blue_triangle



$$b_k = w(a_k), \quad k = 1, 2, 3$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_{kx} \\ a_{ky} \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} b_{kx} \\ b_{ky} \end{pmatrix}, \quad k = 1, 2, 3$$

Solve for *a, b, c, d, e, f.*

Metric Space

A metric space *(X, d)* is a space *X* together with a real-valued function *d: X x X -> R* which measures the distance between pairs of pts *x* and *y* ∈ *X*.

A metric space *X* is complete if every Cauchy sequence has a limit in *X*.

Forward iterates of *f* are transformations

$f^{\circ n} : X \to X$ defined by

$$f^{\circ 0}(x) = x,$$

$$f^{\circ 1}(x) = f(x),$$

$$f^{\circ(n+1)}(x) = f \circ f^{\circ n}(x) = f(f^{\circ n}(x)), \quad n = 0, 1, 2, \ldots$$

Contraction Mapping

A transformation $f : X \to X$ on a metric space (X, d) is a contraction mapping if there is a constant

$$0 \le s < 1, \quad \text{such that} \quad d(f(x), f(y)) \le s\, d(x, y)$$

s is contractivity factor for f.

The Contraction Mapping Thm:

Let f be a contraction mapping on a complete metric space (X, d). Then f has exactly one fixed point x_f, and for any x, the sequence of iterates $\{ f^{\circ n}(x) : n = 0, 1, 2, \ldots \}$ converges to x_f:

$$\{ f^{\circ n}(x) \} \to x_f \text{ as } n \to \infty$$

Contraction Mapping on the Space of Fractals

Let (X, d) be a metric space, and let $(H(X), h(d))$ be the corresponding space of nonempty compact subsets of X with Hausdorff metric $h(d)$.

Let $w: X \to X$ be a contraction mapping on the metric space (X, d) with contractivity factor s.
Then, $w: H(X) \to H(X)$ defined by

$$w(B) = \{w(x) : x \in B\}$$

is a contraction mapping on $(H(X), h(d))$ with contractivity factor s.

Iterated Function System

IFS: An Iterated Function System consists of

a complete metric space (X, d)

together with a finite set of contraction mappings w_n :

$\{X;\ w_n,\ n = 1, \ldots N\}$

with contractivity factor s, $\quad s = \max\{s_n,\ n = 1, \ldots N\}$

$$W(B) = w_1(B) \cup w_2(B) \ldots \cup w_N(B)$$

is a contraction mapping on the space H.

Its unique fixed point satisfies

$A = W(A) = w_1(A) \cup w_2(A) \ldots \cup w_N(A),$

$A = \lim W^{\circ n}(B)$ as $n \to \infty$ for any $B \in H$.

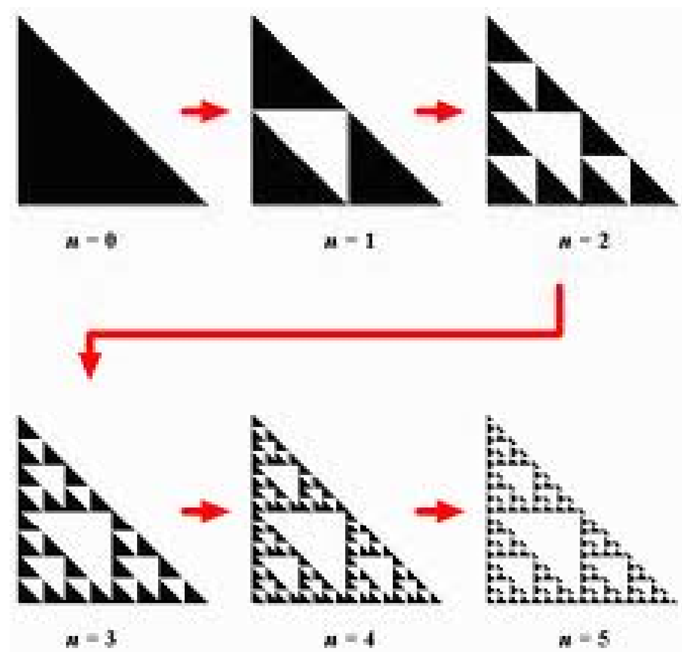
A is attractor of IFS.

Example: Sierpinski Triangle

$$W(B) = w_1(B) \cup w_2(B) \cup w_3(B)$$

Calculate iterations of W :

$$A_n = W^{\circ n}(A_0)\,, \quad n = 1,2,\ldots$$



$$w_1(x) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$w_2(x) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$$

$$w_3(x) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$$

Deterministic and Random Algorithms

Deterministic fractal: $IFS : \{X ; w_1, w_2, \ldots w_N\}$

$W(B) = w_1(B) \cup w_2(B) \cup w_3(B)$

Choose a compact set A_0. Compute iteratively

$A_n = W^{\circ n}(A_0)\,, \quad n = 1, 2, \ldots$

Sequence of iterates converges to the attractor of IFS -- deterministic fractal.

Random Iteration Algorithm: "Apply w_i with probability p_i"

Start with $x_0 \in X$;

Choose recursively $x_{n+1} \in \{w_1(x_n), w_2(x_n), \ldots w_N(x_n)\}$ with probability p_i.

3D IFSs and 3D Fern

- Instead of a 2x2 real matrix A and a column vector t (*,*), we have a 3x3 real matrix A and a column vector t (*,*,*) for a 3D IFS. Again, it can be expressed as $w(x)= Ax+t$.

- As an example of 3D, we introduce a 3D Fern, which is the attractor of an IFS of affine maps in 3D.

3D Fern

The IFS for the 3D Fern

$$w_1(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.18 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$w_2(x) = \begin{bmatrix} 0.85 & 0 & 0 \\ 0 & 0.85 & 0.1 \\ 0 & -0.1 & 0.85 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1.6 \\ 0 \end{bmatrix},$$

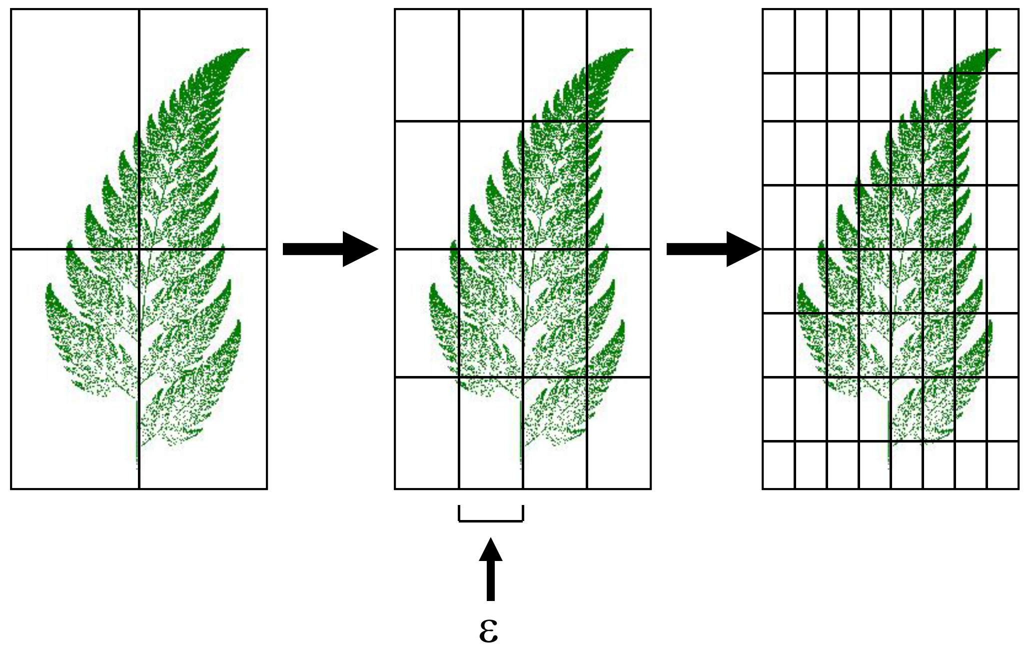
$$w_3(x) = \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.2 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.8 \\ 0 \end{bmatrix},$$

$$w_4(x) = \begin{bmatrix} -0.2 & 0.2 & 0 \\ 0.2 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.8 \\ 0 \end{bmatrix}$$

Fractal Dimension

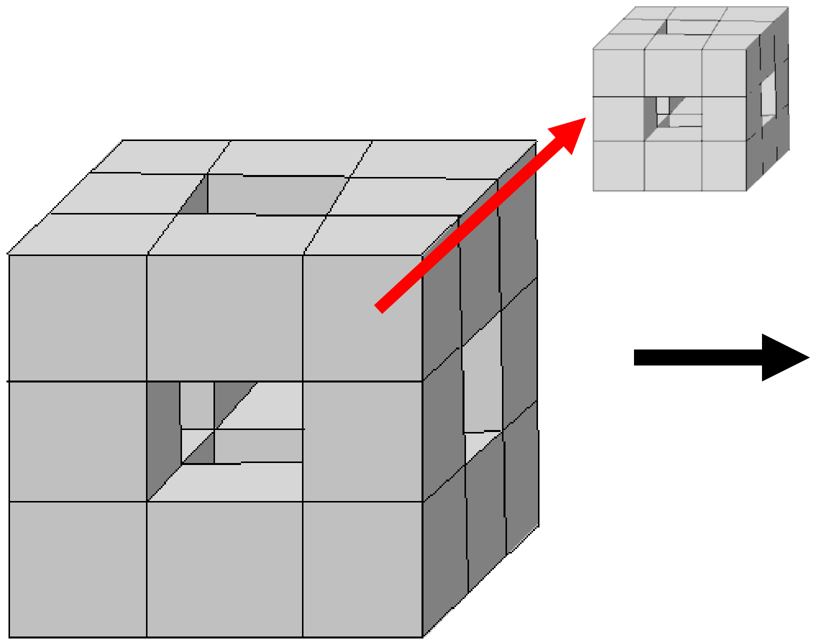
Box dimension:

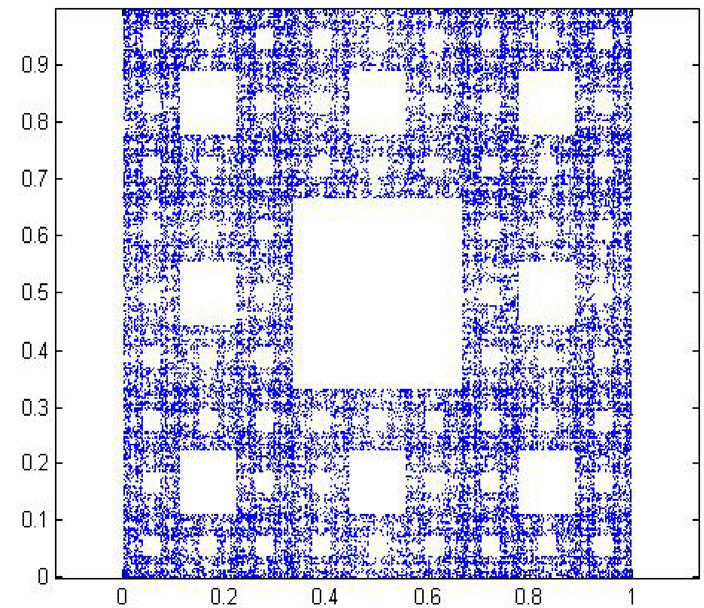
$$D = \lim_{\varepsilon \to 0} \frac{\log(N)}{\log\left(1/\varepsilon\right)}$$



3D IFS fractals: Menger sponge

$$w_1(x)=\begin{bmatrix}1/3 & 0 & 0\\ 0 & 1/3 & 0\\ 0 & 0 & 1/3\end{bmatrix}x+\begin{bmatrix}2/3\\ 0\\ 0\end{bmatrix},\qquad w_2(x)=\begin{bmatrix}1/3 & 0 & 0\\ 0 & 1/3 & 0\\ 0 & 0 & 1/3\end{bmatrix}x+\begin{bmatrix}0\\ 0\\ 2/3\end{bmatrix},\ldots$$

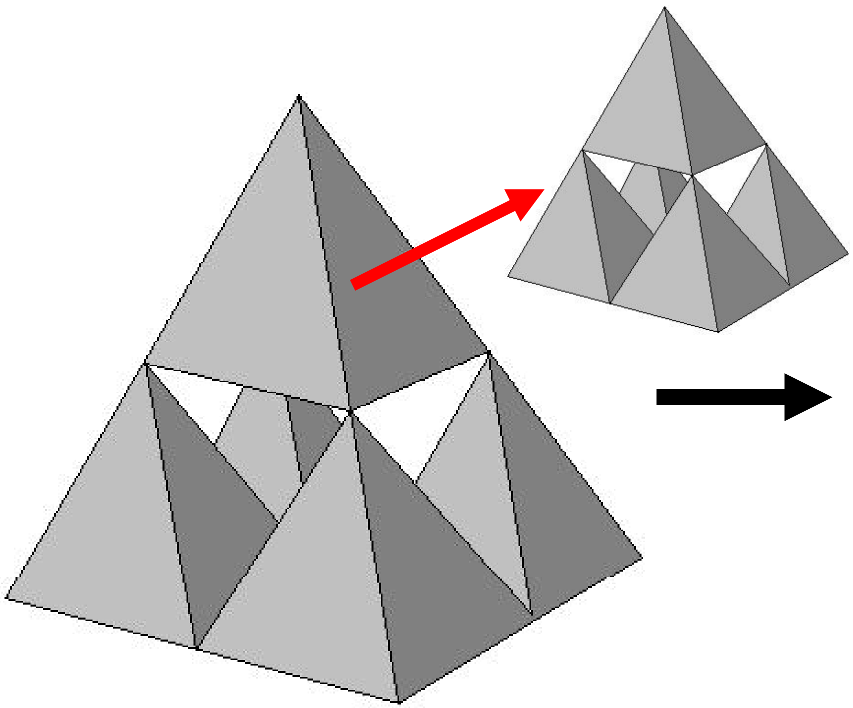


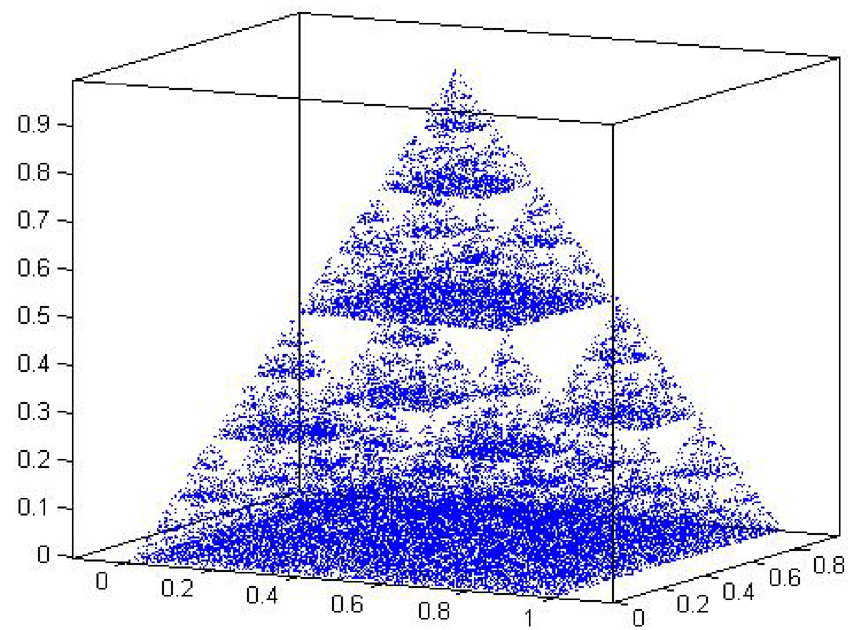


$$D=\frac{\log(20)}{\log(3)}\cong 2.73$$

Sierpinski pyramid

$$D = \frac{\log(5)}{\log(2)} \cong 2.32$$



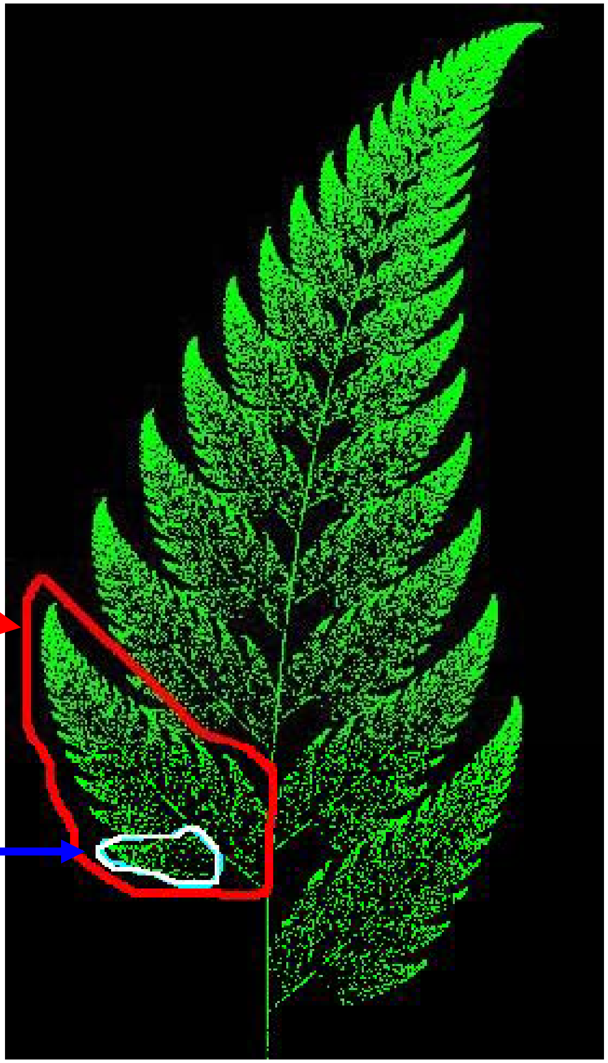


Self similar fractals

Looks the same no matter how close you get.

A smaller copy of the fern

A copy within the copy



Continuous Dependence on Parameters

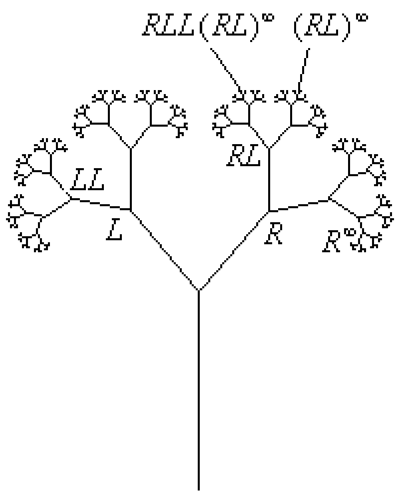
If the contraction *w* continuously depends on a parameter *p*, then the fixed point depends continuously on *p*.

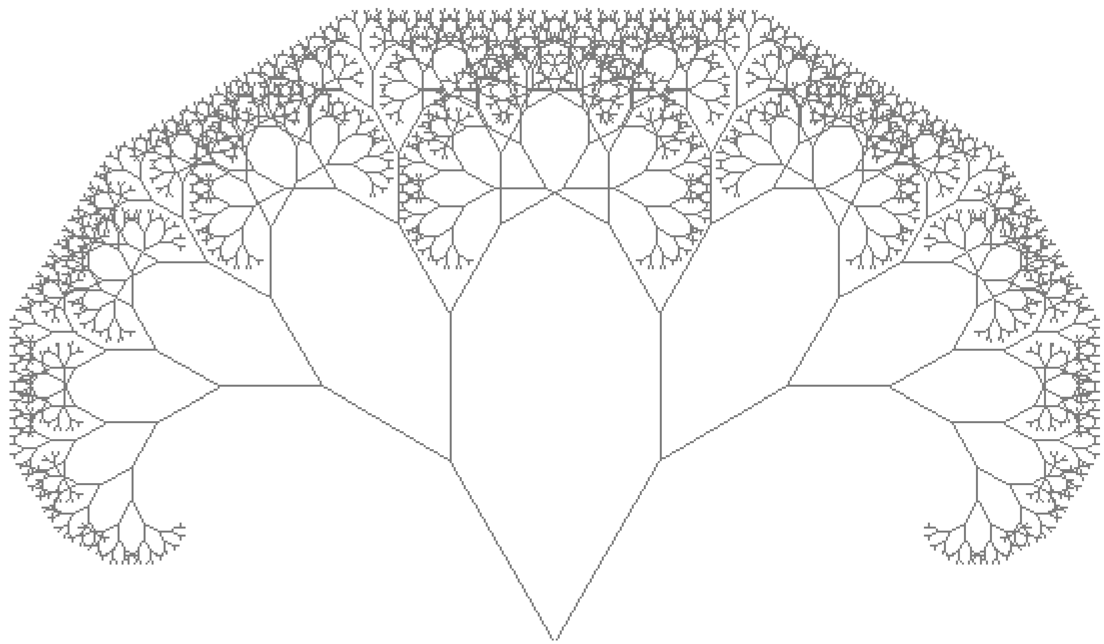
The attractor changes continuously as you change the parameters.

Animation : Dancing fern

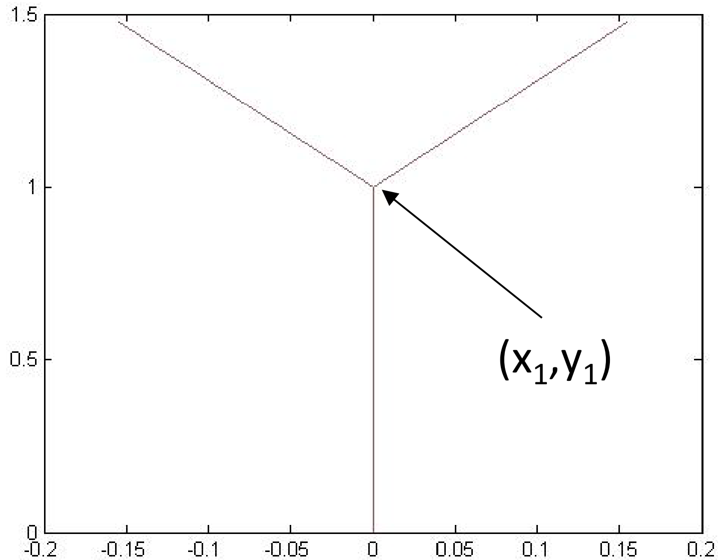
Simulations by Eric Heisler

Deterministic and random trees





Tree Fractals: Transformations



$$w_1(\vec{v}) = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}\vec{v} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

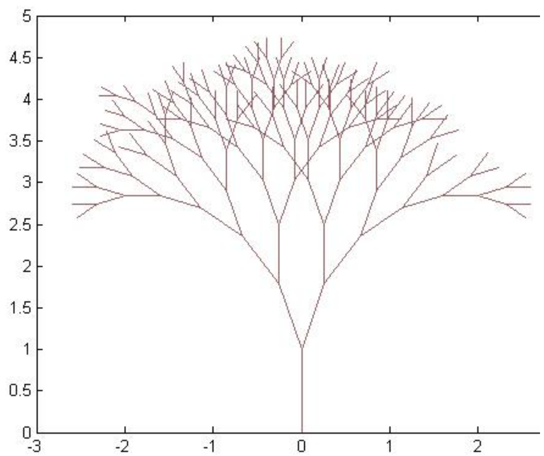
$$w_2(\vec{v}) = \begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}\vec{v} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

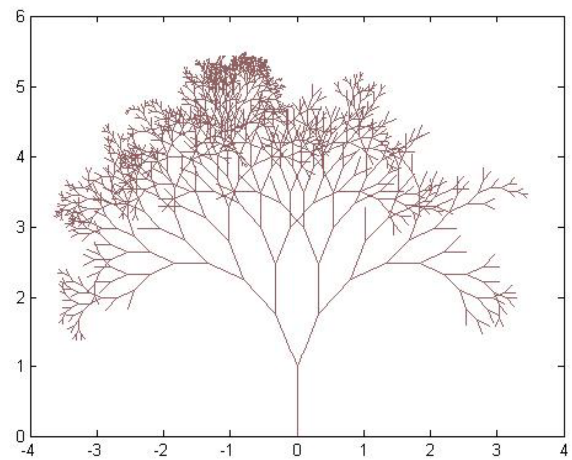
Each iteration takes a line segment and creates two branches.

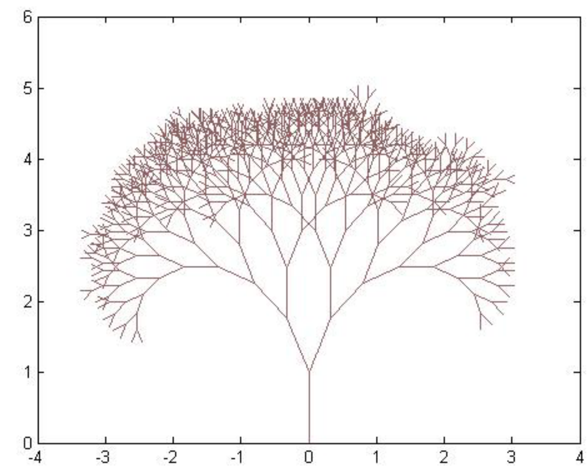
Transformations w rotate trunk by ϑ or $-\vartheta$,

shrink by r, and move to new position

Tree Fractals: IFS with condensation







Let C be the trunk of the tree: $w_0(B) = C, \quad B \in H$

$$W = w_0 \cup w_1 \cup w_2$$

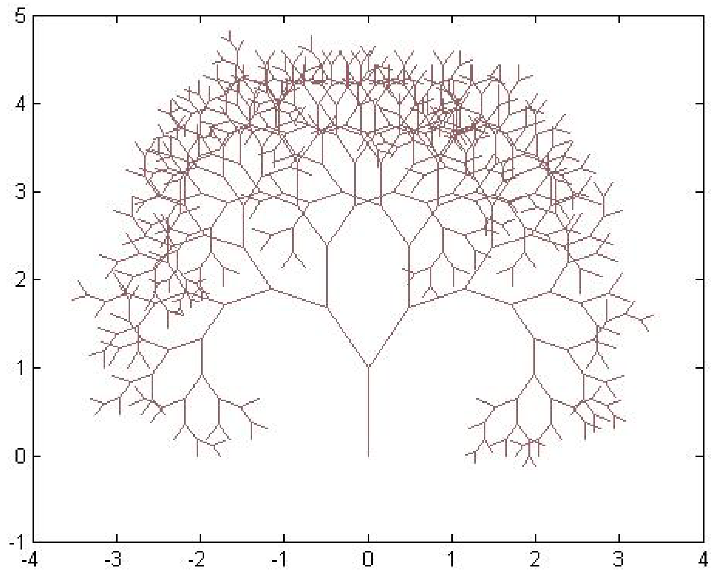
$A_0 = C\,, \quad C$ is a condensation set.

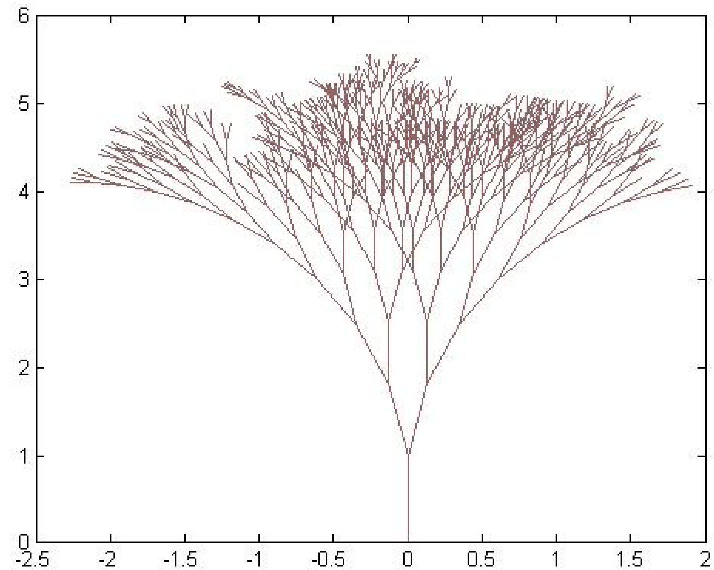
$$A_1 = W(A_0) = C \cup w_1(C) \cup w_2(C)$$

$$A_2 = W(A_1) = C \cup w_1(C) \cup w_2(C) \cup (w_1 \cup w_2)(w_1(C) \cup w_2(C))$$

$$A_3 = W(A_2) = C \cup w_1(C) \cup w_2(C) \cup \ldots$$

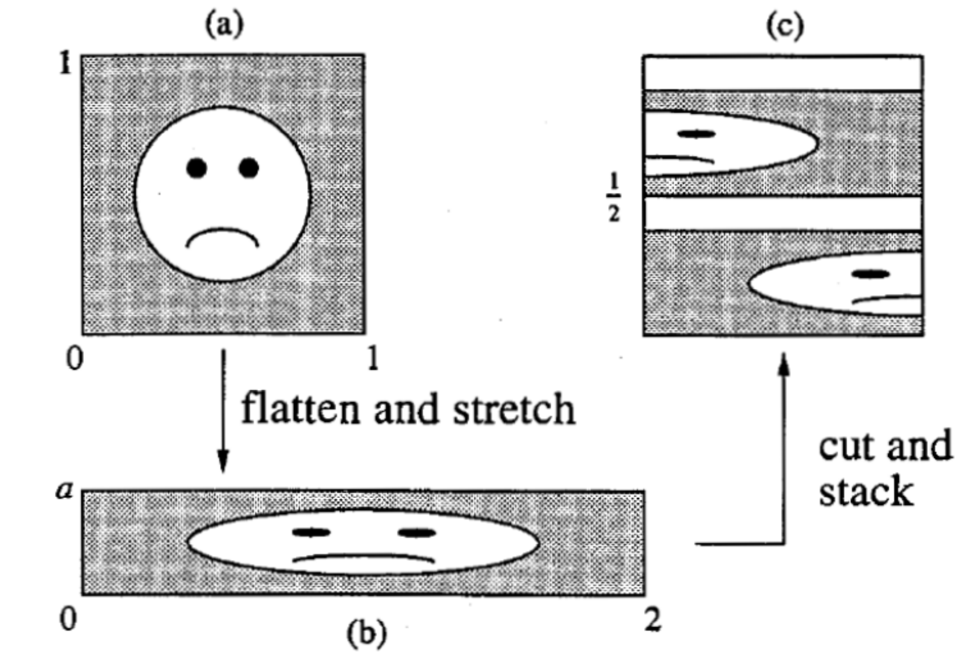
Random Trees





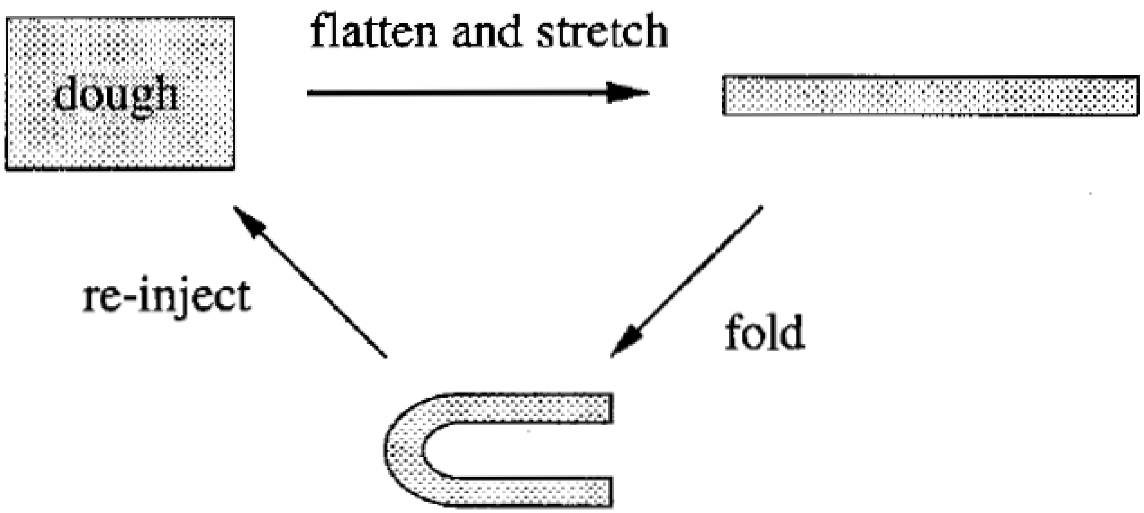
Examples of random trees calculated with different parameters of the contraction (different angles)

Baker’s Map



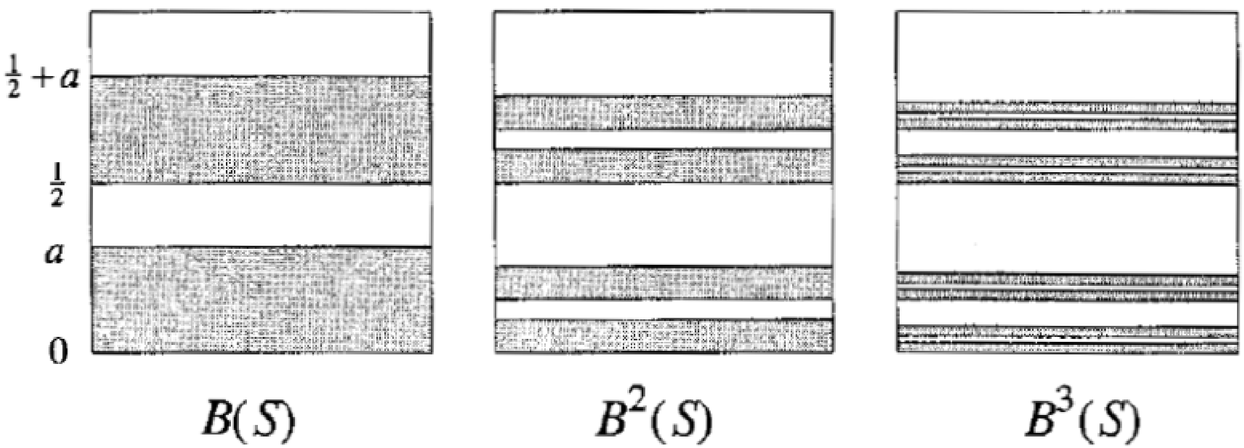
$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n,\ ay_n) & \text{for } 0 \le x_n < \frac{1}{2} \\ (2x_n - 1,\ ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases}$$

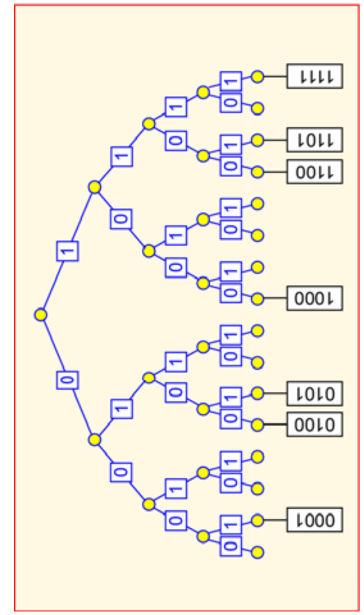
Baker's map: Attractor



Stretching and folding are two main mechanisms of forming an attractor

Baker's map: Attractor





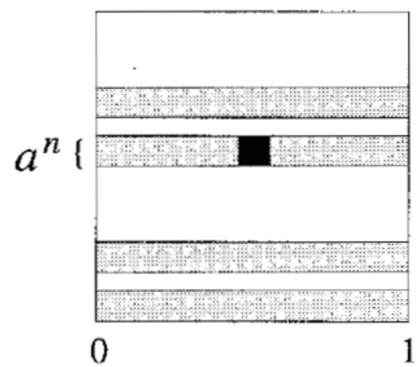
At the crossection:

topological Cantor set!

(all possible sequences of 0 and 1)

Baker's map: Fractal Dimension

Let $a < 1/2$. The attractor A is approximated by $B^{on}(S)$, which consists of 2^n strips of height a^n and unit length. One can cover A with $2^n a^{-n}$ squares of side $\varepsilon = a^n$



Then, $N = (a/2)^{-n}$, and box dimension can be calculated as limit when ε goes to zero:

$$d = \lim \frac{\ln(N)}{\ln(1/\varepsilon)} = 1 + \frac{\ln(1/2)}{\ln(a)} < 2$$

Ueda Attractor

$$\dot{x} = y$$
$$\dot{y} = -x^3 - ky + B\cos(z)$$
$$\dot{z} = 1$$

$$k = 0.05, \; B = 7.5$$

Start with a patch of initial conditions

which experiences stretching and folding

Animation of forming an attractor

Simulations by Quishi Wang

The Escape Time Algorithm: Julia Set

Suppose, $f: C \to C$ is a polynomial.

Start with $z_0 = z,$

$z_1 = f(z_0) = f^{o1}(z),$

$z_2 = f(z_1) = f^{o2}(z),$

...

$z_n = f(z_{n-1}) = f(f(...(z)...)) = f^{on}(z),$

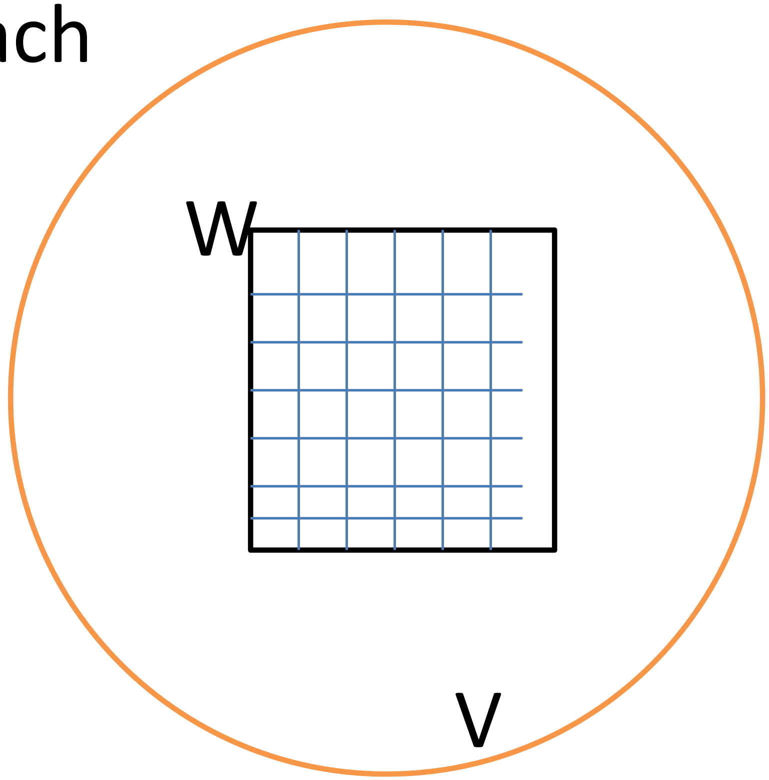
...

Let F_f be the set of points in C whose orbits do not converge to ∞

$$F_f = \left\{ z \in C : \left\{ | f^{on}(z) | \right\}_{n=0}^{\infty} \text{ is bounded} \right\}$$

Then F_f is a Julia set, its boundary J_f is Julia set of f.

The Escape Time Algorithm: Numerical Approach



Discretize computational domain W.

For each $z = z_{p,q}$, iterate $f(z)$:

$z_0, z_1, z_2, \ldots$

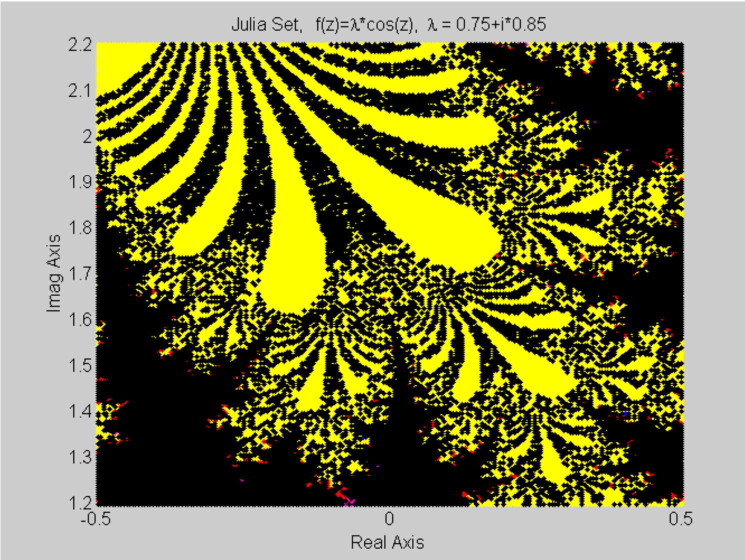
What is the number of iterations needed to escape from V ?

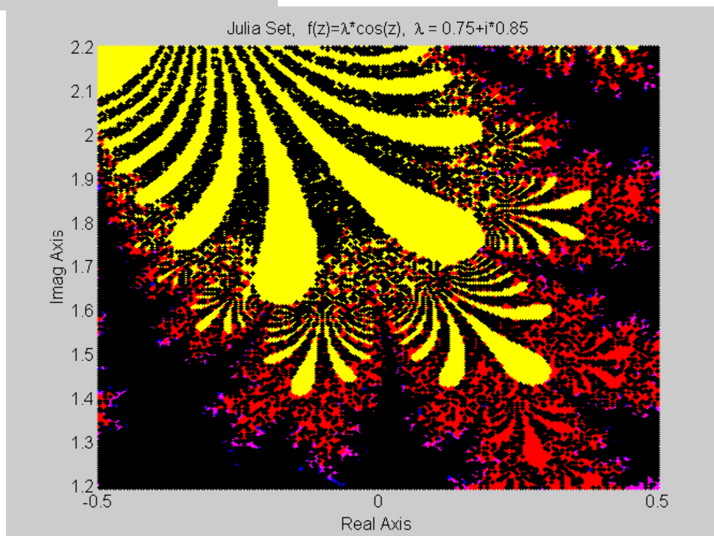
Color points in V according to the number of iterations needed

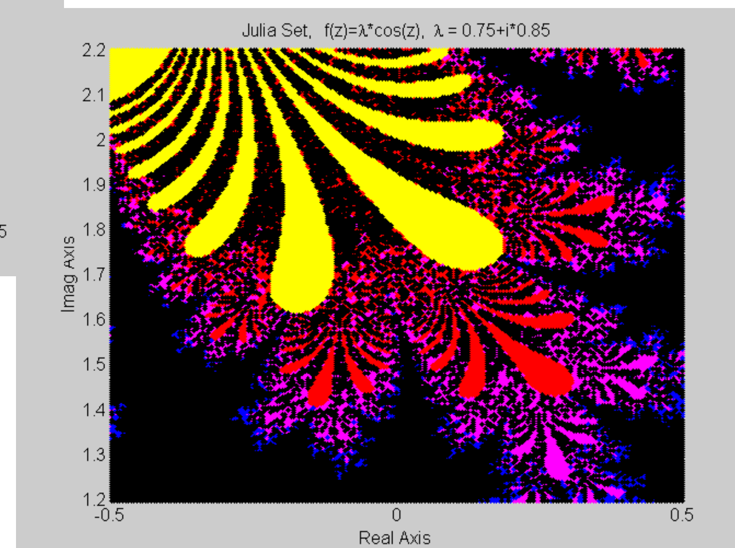
==> Julia Set

Julia Set

Color points in W according to the number of iterations needed for an orbit starting at point z, to escape.







Here,

$f(z) = \lambda \cos(z)$, $\lambda = 0.75+i*0.85$

Simulations by Brandon Olson

Newton fractal

Newton's method of solution $f(x)=0$ fast (quadraticaly) converges when starting point is close to the solution.

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=g(x_n)$$

For solution of the equation: $z^4-1=0$

the Newton's method gives:

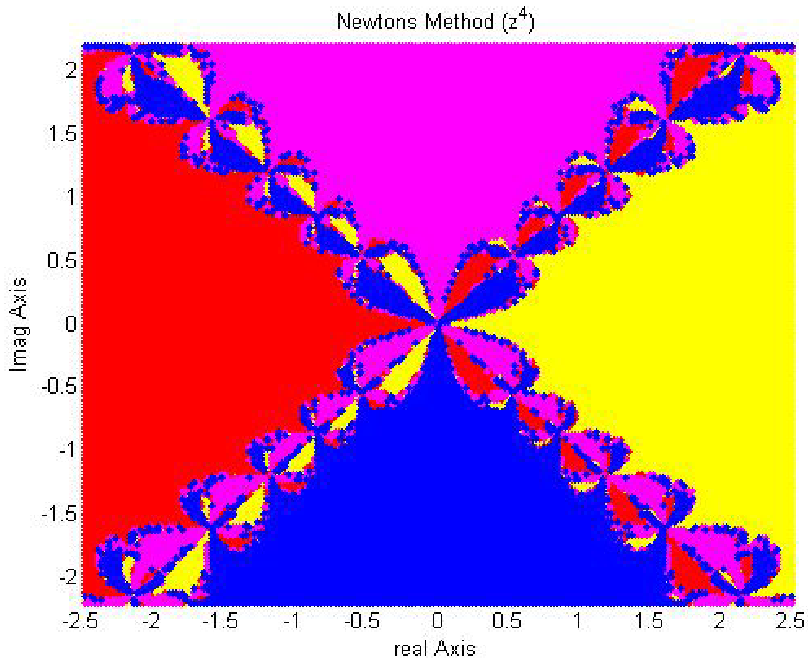
$$g(z)=z-\frac{z^4-1}{4z^3}$$

The roots are 1, -1, *i,* and *–i,* there are 4 attracting points.

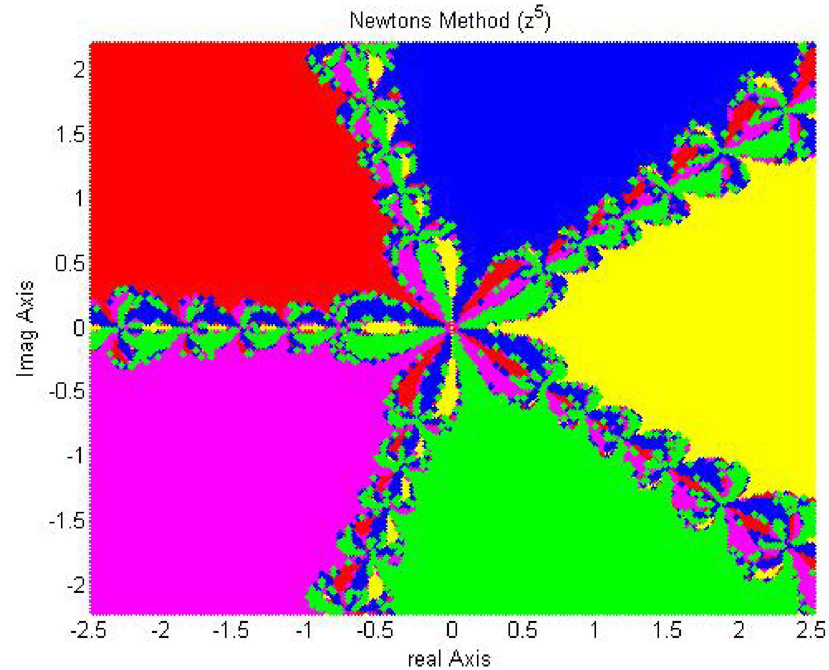
Points of the complex plane are colored by a different color, depending on the root to which the Newton method converges.

Newton fractal:
Sensitive dependence on initial conditions

Outside the region of quadratic convergence the Newton's method can be very sensitive to the choice of starting point.



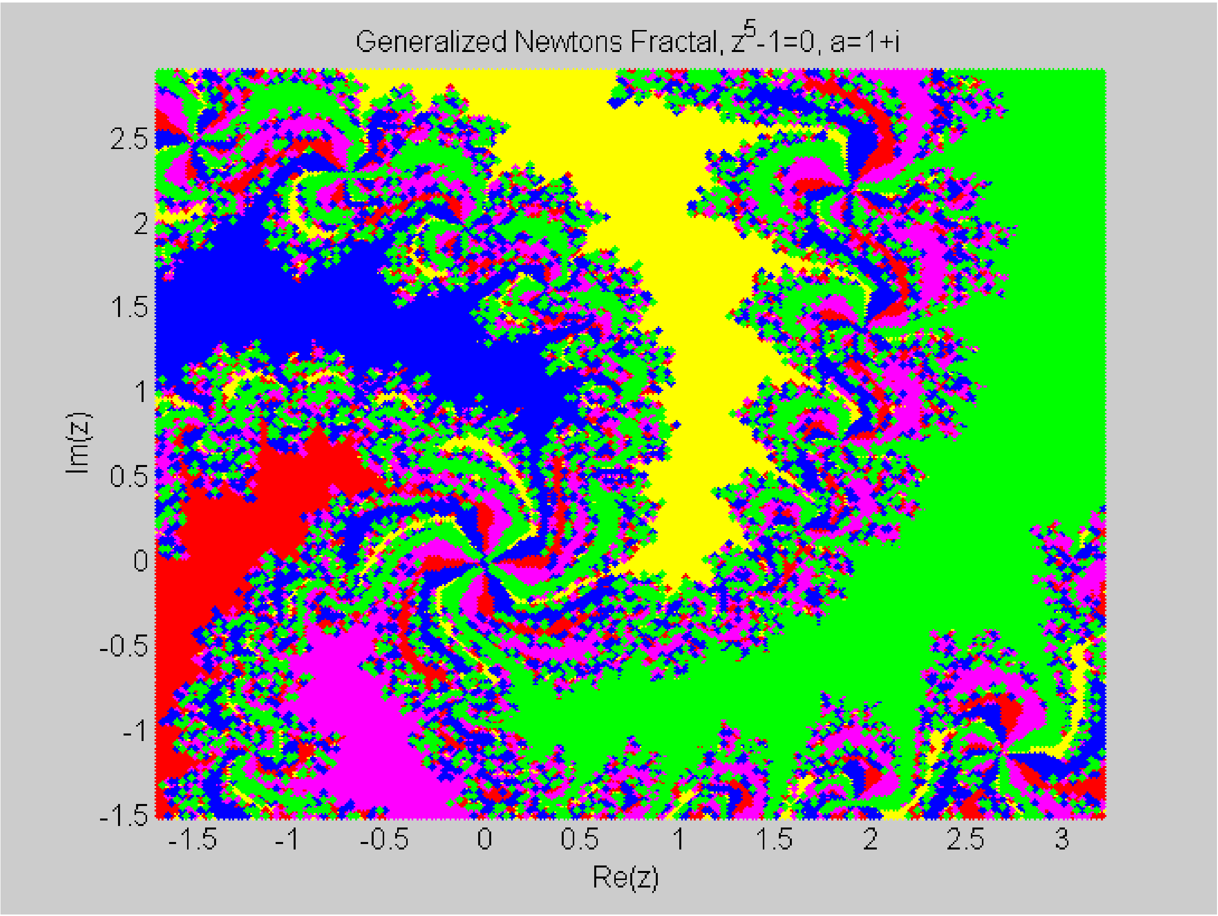
$$z^4 - 1 = 0$$



$$z^5 - 1 = 0$$

Simulations of Aryn Roth

Generalized Newton fractal



$$z_{n+1} = g(z_n)$$

$$g(z) = z - a\frac{z^4 - 1}{4z^3}$$

$$a = (1+i)$$

Participants

- Brandon Olson
- Roxanne Brinkerhoff
- Bill Clark
- Gregory Danner
- Eric Heisler
- Masaki Iino
- Jordan Judkins
- Carl Tams
- Liz Doman
- Aryn Roth
- Quishi Wang