# Translation lengths in $Out(F_n)$

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**Abstract**

We prove that all elements of infinite order in $Out(F_n)$ have positive translation lengths; moreover, they are bounded away from zero. As a consequence we get a new proof that solvable subgroups of $Out(F_n)$ are finitely generated and virtually abelian.

## 1 Introduction

In this paper we will study the translation lengths of outer automorphisms of a free group. Following [GS91] we define the translation length $\tau_{X,G}(g)$ of $g \in \Gamma$ to be

$$\lim_{n\to\infty} \frac{\|g^n\|}{n}$$

where $\Gamma$ is a group with finite generating set $X$, and $\|g\|$ denotes the length of $g$ in the word metric on $\Gamma$ associated to $X$.

Farb, Lubotzky and Minsky proved that Dehn twists (more generally, all elements of infinite order) in $Mod(\Sigma_g)$ have positive translation length ([FLM]). We prove

**Theorem 1.1.** *Every infinite order element $\mathcal{O} \in Out(F_n)$ has positive translation length. Furthermore, there exists a positive constant $\varepsilon_n$ such that $\tau(\mathcal{O}) \geq \varepsilon_n, \quad \forall \mathcal{O} \in Out(F_n)$.*

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Once more we can see the strong analogy between mapping class group of a surface, $Mod(\Sigma_g)$, and outer automorphism group of a free group, $Out(F_n)$.

To prove their theorem, Farb, Lubotzky and Minsky found a way to measure how much a Dehn twist is 'twisted' by looking at simple closed curves and their intersection number. Such an approach cannot work in the case of $Out(F_n)$ as we do not have an analogue of the intersection number.

As a consequence of our main result we have

**Corollary 1.2.** *Every abelian subgroup of $Out(F_n)$ is finitely generated.*

**Corollary 1.3.** *Every solvable subgroup of $Out(F_n)$ is finitely generated and virtually abelian.*

Corollary 1.3 was proved in [BFH99a], but Theorem 1.1 offers an alternative proof. Proofs of corollaries 1.2 and 1.3 for Artin groups can be found in [Bes99, 4.2, 4.4]. Artin groups and $Out(F_n)$ share the properties crucial to the aforementioned proofs. In particular they are virtually torsion free, their virtual cohomological dimension is finite, and the translation length restricted to a torsion free abelian subgroup is a norm on that subgroup. The last fact for $Out(F_n)$ follows from the Theorem 1.1.

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# 2 Translation lengths

From the definition of translation length we can see that it depends on the choice of generating set for a group $\Gamma$. We will omit the reference to the generating set, since it will be clear which one we are using.

We list some properties of translation lengths which can be found in [GS91].

**Proposition 2.1.** *Let $X$ be a generating set for a group $\Gamma$.*

1. $0 \leq \tau(g) \leq \|g\|$
2. *For all $x, g \in G$, $\tau(xgx^{-1}) = \tau(g)$.*
3. $\tau(g^n) = n \cdot \tau(g)\ \forall n \in \mathbb{N}$ .

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of generators of a free group $F_n$. Let $Y$ be the set of generators for $Aut(F_n)$ consisting of:

1. permutations ($x_i \mapsto x_j$, $x_j \mapsto x_i$, $x_k \mapsto x_k$ for all $k \neq i, j$),

2. inversions ($x_i \mapsto x_i^{-1}$, $x_j \mapsto x_j$ for all $j \neq i$),

3. Nielsen twists ($x_i \mapsto x_i x_j$, $x_k \mapsto x_k$ for all $k \neq i$).

Let $\tilde{Y}$ denote the generating set for $Out(F_n)$ consisting of equivalence classes of elements of $Y$.

Our goal is to prove that every element of infinite order in $Out(F_n)$ has positive translation length. Since $Aut(F_n)$ embeds into $Out(F_{n+1})$, it will follow that every infinite order element of $Aut(F_n)$ has positive translation length.

We will need the following definition for our proof:

**Definition 2.2.** Define a map $\alpha : F_n \to \mathbb{N}$ by

$$\alpha(w) = \text{the largest } p \geq 0 \text{ such that for some nontrivial reduced word } u \text{ the word } u^p \text{ is a subword of } w,$$

where elements of $F_n$ are regarded as reduced words in the generators and their inverses. We also define

$$\tilde{\alpha}([w]) = \max\{\alpha(u) : u \text{ is a cyclically reduced conjugate of } w\}$$

for the conjugacy class, $[w]$, of $w$.

**Example 2.3.**

$$\alpha(1) = 0$$

$$\alpha(ab^{-7}c) = 7$$

**Lemma 2.4.** *There exists a constant $C > 0$ such that for any $\tilde{g} \in \tilde{Y}$ and any cyclically reduced word $w \in F_n$ we have*

$$\tilde{\alpha}(\tilde{g}([w])) \leq \tilde{\alpha}([w]) + C\,.$$

*Proof.* Let $w \in F_n$ be a cyclically reduced element of length $n$ with $\alpha(w) = p$. Write $w = A\, u^p\, B$, for some $u \in F_n$. Consider

$$g(w) = [[g(A)]]\,[[g(\tilde{w})^p]]\,[[g(B)]]\,,$$

where $[[x]]$ denotes the reduced word obtained from $x$. By the *Bounded Cancellation Lemma* ([Coo87]) there is a constant $C(g)$ such that at most $C(g)$ cancellations occur after concatenation of the words $[[g(A)]]$ and $[[g(\tilde{w})^p]]$. Hence $p$ can decrease by at most $2\,C(g)$ (cancellations may occur at the beginning and at the end of $[[g(\tilde{w})^p]]$). Let $C_g = 2\,\max\{C(g), C(g^{-1})\}$. We now have

$$\begin{aligned}
&\alpha([[g(w)]]) \geq \alpha(w) - C_g \\
&\alpha(w) = \alpha(g^{-1}(g(w))) \geq \alpha([[g(w)]]) - C_g \\
&\alpha([[g(w)]]) \leq \alpha(w) + C_g.
\end{aligned}$$

If we take $C = \max\{C_g : g \in Y\}$, our claim is proved for elements of $Y$.

Let $\tilde{g} \in \tilde{Y}$ and let $g$ be a representative for the equivalence class $\tilde{g}$. The argument in this case differs from the above argument in that after applying $g$ to $w$, $p$ can decrease by at most $3\,C(g)$ (it may happen that $g(w)$ is not cyclically reduced and we can get cancellation at the ends of $g(w)$). We now proceed as above.

□

**Example 2.5.** We illustrate the idea of the proof of Theorem 1.1 with an example of a Nielsen twist. Let $g$ be a Nielsen twist which sends $x_2$ to $x_2 x_1$ and fixes all other generators of $F_n$.

$$\alpha(g^k(x_2)) = \alpha(x_2 x_1^k) = k\,.$$

Write $g^k = g_1 \cdots g_m$ with $g_i \in Y$ and $m = \|g^k\|$. By Lemma 2.4, we have that

$$\begin{aligned}
&k = \alpha(g^k(x_2)) \leq \alpha(x_2) + m\,C = m\,C + 1\,, \\
&\tau(g) = \lim_{k\to\infty} \frac{\|g^k\|}{k} \geq \lim_{k\to\infty} \frac{k-1}{k\,C} = \frac{1}{C} > 0\,.
\end{aligned}$$

So $g$ has positive translation length.

We give a short list of definitions which will be used throughout the rest of the paper, but we suggest that the reader look at [BFH99b].

Every element $\mathcal{O} \in Out(F_n)$ can be represented by a homotopy equivalence $f\colon\ G \to G$ of a graph $G$ whose fundamental group is identified with $F_n$. A map $\sigma\colon\ J \to G$ ($J$ is an interval) is called a *path* if it is either locally injective or a constant map (we also require that the endpoints of $\sigma$ are at vertices). Every map $\sigma\colon\ J \to G$ is homotopic (relative endpoints) to a path $[[\sigma]]$.

If $\sigma = \sigma_1 \dots \sigma_l$ is a decomposition of a path or a circuit $\sigma$ into nontrivial subpaths we say that it is a *k-splitting* if

$$f^k(\sigma) = [[f^k(\sigma_1)]] \dots [[f^k(\sigma_l)]]$$

is a decomposition into subpaths and is a *splitting* if it is a $k$-splitting for all $k > 0$.

We say that a nontrivial path $\sigma \in G$ is a *Nielsen path* for $f : G \to G$ if $[[f(\sigma)]] = \sigma$. The Nielsen path $\sigma$ is *indivisible* if it cannot be written as a concatenation of nontrivial Nielsen paths.

Let $= G_0 \subsetneqq G_1 \subsetneqq \cdots \subsetneqq G_K = G$ be a filtration of $G$ by $f$-invariant subgraphs, and let $H_i = \overline{G_i \backslash G_{i-1}}$. Suppose $H_i$ is a single edge $E_i$ and $f(E_i) = E_i v^l$ for some closed indivisible Nielsen path $v \subset G_{i-1}$ and some $l > 0$. The *exceptional paths* are paths of the form $E_i v^k \overline{E_j}$ or $E_i \overline{v}^k \overline{E_j}$, where $k \geq 0$, $j \leq i$ and $f(E_j) = E_j v^m$, for $m > 0$.

We remind the reader that every element of $Out(F_n)$ of infinite order has either exponential or polynomial growth ([BH92]). A polynomially growing outer automorphism $\mathcal{O} \in Out(F_n)$ is unipotent if its action in $H_1(F_n;\mathbb{Z})$ is unipotent ($UPG$ automorphism).

The following Theorem can be found in [BFH00](page 564).

**Theorem 2.6.** *Suppose that $\mathcal{O} \in Out(F_n)$ is a $UPG$ automorphism. Then there is a topological representative $f\colon\ G \to G$ of $\mathcal{O}$ with the following properties:*

1. *Each $G_i$ is the union of $G_{i-1}$ and a single edge $E_i$ satisfying $f(E_i) = E_i{\cdot}u_i$ for some closed path $u_i$ that crosses only edges in $G_{i-1}$ ( $\cdot$ indicates that the decomposition in question is a splitting).*

2. *If $\sigma$ is any path with endpoints at vertices, then there exists $M = M(\sigma)$ so that for each $m \geq M$, $[[f^m(\sigma)]]$ splits into subpaths that are either single edges or exceptional subpaths.*

$\square$

We need to modify a definition of our map $\alpha$ to the new setting.

**Definition 2.7.** For a path $\gamma$ in a graph $G$ let

$$\alpha(\gamma) = \text{the largest } p \geq 0 \text{ such that for some nontrivial path } \sigma \text{ the path } \sigma^p \text{ is a subpath of } \gamma,$$

The map $\tilde{\alpha}$ is defined on circuits in the exactly same way.

The following example demonstrates the difference between these two seemingly identical maps.

**Example 2.8.** Let $\gamma = abca$.
If $\gamma$ is a path then $\alpha(\gamma) = 1$,
but if it is a circuit, then $\tilde{\alpha}(\gamma) = 2$.

**Lemma 2.9.** *Let $\mathcal{O} \in Out(F_n)$ be a UPG automorphism of infinite order and let $f\colon G \to G$ be its topological representative as in Theorem 2.6. For every path $\gamma$ in $G$ for which $[[f(\gamma)]] \neq \gamma$ there exists $a \in \mathbb{Z}$ such that*

$$\alpha([[f^k(\gamma)]]) \geq k + a\,.$$

*Proof.* We prove our claim by induction on the (minimal) index, $m$, of the filtration element that contains a path $\gamma$.

If $\gamma \subset G_1$ there is nothing to be proved since $G_1$ contains only one edge $E_1$ which is fixed by $f$.

Suppose the claim is true for the subpaths contained in $G_{m-1}$ that satisfy our hypothesis, and let $\gamma$ be a path in $G_m$ for which $[[f(\gamma)]] \neq \gamma$. By Theorem 2.6 for every $m \geq M(\gamma)$, $[[f^m(\gamma)]]$ splits into subpaths that are either single edges or exceptional paths. Denote $[[f^{M(\gamma)}(\gamma)]]$ by $\tilde{\gamma}$, so that $\tilde{\gamma} = \gamma_1 \cdot \ldots \cdot \gamma_p$, where $\gamma_i$ is either a single edge or an exceptional path.

Assume there is an exceptional path $\gamma_t$ which is not fixed by $f$. Without loss of generality we may assume that $\gamma_t = E_i v^r \overline{E_j}$, where $f(E_i) = E_i v^l$ ($l > 0$), $f(E_j) = E_j v^s$ ($s > 0$) and $j \leq i$. Now we have that

$$[[f^k(\gamma_t)]] = E_i v^{k(l-s)+r}\, \overline{E_j}\,,$$

and

$$\alpha([[f^k(\gamma_t)]]) \geq k(l-s) + r, \quad \text{if } l - s > 0\,,$$

$$\alpha([[f^k(\gamma_t)]]) \geq k(s-l)-r, \quad \text{if } l-s<0\,.$$

Since $\gamma_t$ is not fixed, $l$ and $s$ cannot be equal. Therefore

$$\alpha([[f^k(\tilde{\gamma})]]) \geq k-r\,,$$

$$\alpha([[f^k(\gamma)]]) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq k-M(\gamma)-r\,.$$

If all exceptional paths in $\tilde{\gamma}$ are fixed, there exists an edge $\gamma_t = E_i$ which is not fixed by $f$. We know that $f(E_i) = E_i \cdot u_i$, where $u_i$ is a closed path contained in $G_{m-1}$.
If $[[f(u_i)]] = u_i$, our claim is proven since $[[f^k(E_i)]] = E_i u_i^k$ and so

$$\alpha([[f^k(\tilde{\gamma})]]) \geq k\,,$$

$$\alpha([[f^k(\gamma)]]) \geq k-M(\gamma)\,.$$

If $[[f(u_i)]] \neq u_i$, there exists $a \in \mathbb{R}$ such that $\alpha([[f^k(u_i)]]) \geq k+a$. We now have

$$\alpha([[f^k(\gamma)]]) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq \alpha([[f^{k-M(\gamma)}(u_i)]]) \geq k-M(\gamma)+a\,.$$

□

**Lemma 2.10.** *Let $\mathcal{O}$ be a UPG automorphism of $F_n$ of infinite order. There exist a closed path $\sigma$ in $G$, and $b \in \mathbb{R}$ such that*

$$\tilde{\alpha}(\mathcal{O}^k(\sigma)) \geq k+b\,.$$

*Proof.* Let $f\colon G \to G$ be as in Theorem 2.6. Since $\mathcal{O} \neq id$ there is a closed path $\sigma$ which is not fixed by $f$. We know that for every $m \geq M(\sigma)$, $[[f^m(\sigma)]] = \sigma_1 \cdot \ldots \cdot \sigma_p$ splits into subpaths that are either single edges or exceptional paths. Denote $[[f^{M(\sigma)}(\sigma)]]$ by $\tilde{\sigma}$, so that $\tilde{\sigma} = \sigma_1 \cdot \ldots \cdot \sigma_p$.

If there is an exceptional path $\sigma_t$ in this splitting which is not fixed by $f$, we get

$$\tilde{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k-r$$

as in Lemma 2.9.

If all exceptional paths in $\tilde{\sigma}$ are fixed, there exists an edge $\sigma_t = E_i$ such that $f(E_i) = E_i \cdot u_i$, where $u_i$ is a closed path contained in $G_{i-1}$. By Lemma 2.9 there exists $a \in \mathbb{R}$ such that

$$\alpha(f^k(E_i)) \geq k+a\,.$$

Hence, in all the above cases, there is $b \in \mathbb{R}$ such that

$$\tilde{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k+b\,.$$

□

## 3 Proof of Theorem 1.1

We consider the cases of exponentially and polynomially growing outer automorphisms separately.

**Case 1.** Let $\mathcal{O}$ be an exponentially growing outer automorphism of $F_n$. There exist $\lambda > 1$ and a cyclically reduced word $w$ such that $\ell(\mathcal{O}^k([w])) \geq \lambda^k \ell([w])$, for all $k \geq 1$, where $\ell$ denotes the cyclic word length (see [BH92]). Let $\mathcal{O}^k = \tilde{g_1} \dots \tilde{g_m}$, where $\tilde{g_i} \in \tilde{Y}$ and $m = \|\mathcal{O}^k\|$. It is straightforward to show that for all $\tilde{g} \in \tilde{Y}$ and any cyclically reduced word $w$ we have

$$\ell(\tilde{g}([w])) \leq 2\,\ell([w])$$

Using this inequality we obtain:

$$\lambda^k \ell([w]) \leq \ell(\mathcal{O}^k([w])) \leq 2^m \ell([w])$$

Hence

$$m \geq \frac{\log \lambda^k}{\log 2}$$

which implies

$$\tau(\mathcal{O}) \geq \frac{\log \lambda}{\log 2} > 0$$

**Case 2.** Let $\mathcal{O}$ be a $UPG$ automorphism. Again let $\mathcal{O}^k = \tilde{g_1} \dots \tilde{g_m}$, where $\tilde{g_i} \in \tilde{Y}$ and $m = \|\mathcal{O}^k\|$. By Lemma 2.10 there is a closed path $\sigma$ in $G$ such that

$$\tilde{\alpha}(\mathcal{O}^k(\sigma)) \geq k + b$$

Let $u_j = \tilde{g_j} \dots \tilde{g_m}$. Applying Lemma 2.4 we get

$$\tilde{\alpha}(u_i(\sigma)) \leq \tilde{\alpha}(u_{i+1}(\sigma)) + C$$

which yields

$$\begin{aligned} k + b &\leq \tilde{\alpha}(\mathcal{O}^k(\sigma)) \leq mC + \tilde{\alpha}(\sigma) \\ \frac{k + b - \tilde{\alpha}(\sigma)}{C} &\leq m\,. \end{aligned}$$

We have

$$\tau(\mathcal{O}) \geq \lim_{k\to\infty} \frac{k + b - \tilde{\alpha}(\sigma)}{k\,C} = \frac{1}{C}\,.$$

**Case 3.** If $\mathcal{O}$ is any polynomially growing outer automorphism, then there exists $s \geq 1$, bounded above by some $c_2$, ([BFH99b, Definition 3.10, Proposition 3.5]) such that $\mathcal{O}^s$ is a $UPG$ automorphism. Then

$$\tau(\mathcal{O}) = \frac{1}{s}\,\tau(\mathcal{O}^s) \geq \frac{1}{C\,s} > 0\,.$$

In all three cases $\tau(\mathcal{O})$ is bounded away from zero:

*Case 1.* There is a constant $c_1 > 1$ such that $\lambda \geq c_1$ ([BH92]). Therefore $\tau(\mathcal{O})$ is bounded away from zero.

*Case 2.* $\tau(\mathcal{O}) \geq \frac{1}{C}$, for a fixed $C$.

*Case 3.* Since $s$ is bounded by $c_2$, we get $\tau(\mathcal{O}) \geq \frac{1}{C\,c_2} > 0$.

This completes the proof. $\square$

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