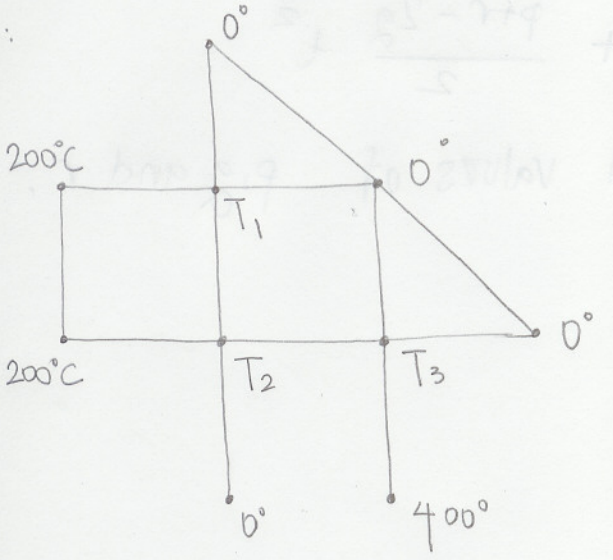
HOMEWORK 1

Section 1.1

28:




$$T_1 = \frac{200+0+0+T_2}{4}$$

$$T_2 = \frac{200+T_1+T_3+0}{4}$$

$$T_3 = \frac{T_2+0+0+400}{4}$$

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$$4T_1 - T_2 = 200$$
$$-T_1+4T_2-T_3 = 200 \quad /\cdot 4 \;\; +$$
$$-T_2+4T_3 = 400$$

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$$15T_2 - 4T_3 = 1000 \quad +$$
$$-T_2+4T_3 = 400$$

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$$14T_2 = 1400$$

$$T_2 = 100$$

$$T_3 = \frac{400+T_2}{4} = \frac{500}{4} = 125$$

$$T_1 = \frac{200+T_2}{4} = \frac{300}{4} = 75$$

30: $f(t) = a+bt+ct^2$

$$\left.\begin{array}{lll} f(1)=p & \Rightarrow & a+b+c=p \\ f(2)=q & \Rightarrow & a+2b+4c=q \quad -1^{st} \\ f(3)=r & \Rightarrow & a+3b+9c=r \quad -2^{nd} \end{array}\right\} \Rightarrow \left.\begin{array}{l} b+3c=q-p \\ b+5c=r-q \quad -1^{st} \end{array}\right\} =$$

$$\Rightarrow 2c = r-q-q+p \qquad 2c = p+r-2q \qquad c = \frac{p+r-2q}{2}$$

$$b = q-p-3c = \frac{2q-2p-3p-3r+6q}{2} = \frac{-5p-3r+8q}{2}$$

$$a = p - b - c = \frac{2p+5p+3r-8q-p-r+2q}{2} = \frac{6p+2r-6q}{2} = 3p+r-3q$$

$$f(t) = (3p+r-3q) + \frac{-5p-3r+8q}{2}\,t + \frac{p+r-2q}{2}\,t^2$$

Yes, this polynomial is defined for all values of $p, q$ and $r$.

Section 1.2

12:

$$\begin{aligned}
2x_1 \quad -3x_3 \quad +7x_5+7x_6 &= 0\\
-2x_1+x_2+6x_3 \quad -6x_5-12x_6 &= 0\\
x_2-3x_3 \quad +x_5+5x_6 &= 0\\
-2x_2 \quad +x_4+x_5+x_6 &= 0\\
2x_1+x_2-3x_3 \quad +8x_5+7x_6 &= 0
\end{aligned}$$

$$\begin{bmatrix} 2 & 0 & -3 & 0 & 7 & 7\\ -2 & 1 & 6 & 0 & -6 & -12\\ 0 & 1 & -3 & 0 & 1 & 5\\ 0 & -2 & 0 & 1 & 1 & 1\\ 2 & 1 & -3 & 0 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/2 & 0 & 7/2 & 7/2\\ 0 & 1 & 3 & 0 & 1 & -5\\ 0 & 1 & -3 & 0 & 1 & 5\\ 0 & -2 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & -3/2 & 0 & 7/2 & 7/2\\ 0 & 1 & 3 & 0 & 1 & -5\\ 0 & 0 & -6 & 0 & 0 & 10\\ 0 & 0 & 6 & 1 & 3 & -9\\ 0 & 0 & -3 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/2 & 0 & 7/2 & 7/2\\ 0 & 1 & 3 & 0 & 1 & -5\\ 0 & 0 & 1 & 0 & 0 & -10/6\\ 0 & 0 & 6 & 1 & 3 & -9\\ 0 & 0 & -3 & 0 & 0 & 5 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 7/2 & 1\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & -10/6\\ 0 & 0 & 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{aligned} &\longrightarrow x_1 = -7/2\,x_5 - x_6\\ &\longrightarrow x_2 = -x_5\\ &\longrightarrow x_3 = 5/3\,x_6\\ &\longrightarrow x_4 = -3x_5 - x_6 \end{aligned}$$

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{bmatrix} = \begin{bmatrix} -7/2\\ -1\\ 0\\ -3\\ 1\\ 0 \end{bmatrix} x_5 + \begin{bmatrix} -1\\ 0\\ 5/3\\ -1\\ 0\\ 1 \end{bmatrix} x_6 \quad , \; x_5, x_6 \in \mathbb{R}$$

26: There is a segrence of elementary row operations that transforms A into rref A; call it $E_1, E_2, \ldots, E_K$. If every elementary row operation E had its opposite elementary row operation, $E^{op}$, then by performing a segrence

$$E_K^{op}, E_{K-1}^{op}, \ldots, E_2^{op}, E_1^{op}$$

of elementary row operations we would transform rref A into A. We need to show that for each elementary row operation E there is an elementary operation that "undoes" E, that is if $A \xrightarrow{E} B$, then $B \xrightarrow{E^{op}} A$.

| E | | $E^{op}$ |
|---|---|---|
| 1) switches $i^{th}$ and $j^{th}$ row | $\leftrightarrow$ | switches $i^{th}$ and $j^{th}$ row |
| 2) multiplies $i^{th}$ row by $k \in \mathbb{R} \setminus \{0\}$ | $\leftrightarrow$ | multiplies $i^{th}$ row by $\frac{1}{k} \in \mathbb{R} \setminus \{0\}$ |
| 3) adds $k \cdot (i^{th}$ row$)$, $k \in \mathbb{R} \setminus \{0\}$, to $j^{th}$ row. | $\leftrightarrow$ | adds $(-\frac{1}{k}) \cdot (i^{th}$ row$)$ to $j^{th}$ row |

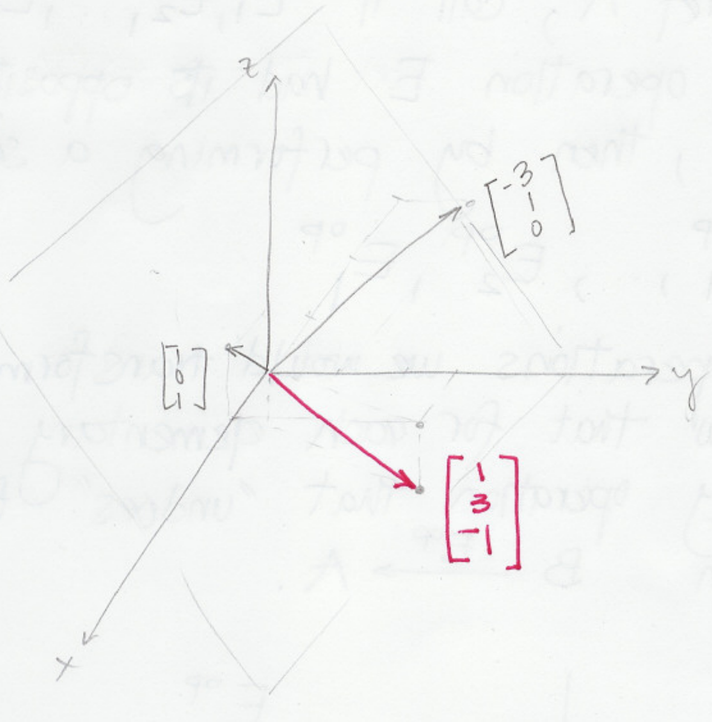
34: Find all vectors in $\mathbb{R}^3$ perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$. Draw a sketch.

We are looking for $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ st $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 0$, i.e.

$$x_1 + 3x_2 - x_3 = 0$$

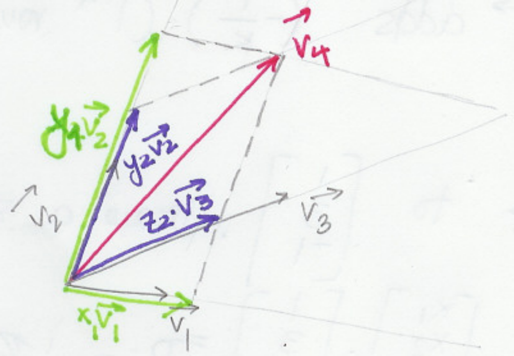
$$\Rightarrow x_1 = -3x_2 + x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}.$$




Section 1.3.

8: $x\vec{v_1} + y\vec{v_2} + z\vec{v_3} = \vec{v_4}$




We know that any two non collinear vectors span $\mathbb{R}^2$, so we can write $\vec{v_4}$ as a linear combination of the remaining vectors.

One solution can be : $x_1\vec{v_1} + y_1\vec{v_2} + 0\cdot\vec{v_3} = \vec{v_4}$ (in green)

another one : $0\cdot\vec{v_1} + y_2\vec{v_2} + z_2\cdot\vec{v_3} = \vec{v_4}$ (in blue)

Obvious explanation of why this system has infinitely many solutions is to remember that any system can have:

0 solutions; exactly one solution or infinitely many solutions. We have already found 2, so our system must have infinitely many solutions.

One may also argue geometrically. Take any linear combination of vectors $\vec{v_1}$ and $\vec{v_2}$, say (any pair of $\vec{v_1}, \vec{v_2}$ and $\vec{v_3}$ would work equally well) $a\vec{v_1} + b\vec{v_2}$, where $a, b \in \mathbb{R}$, but make sure that $a\vec{v_1} + b\vec{v_2}$ and $\vec{v_3}$ are not colinear. Since they are not, any other vector in $\mathbb{R}^2$ can be written as their linear combination. Hence there are numbers $p$ and $r$ such that

$$p(a\vec{v_1} + b\vec{v_2}) + r\vec{v_3} = \vec{v_4} \Rightarrow$$

$$\Rightarrow pa\vec{v_1} + pb\vec{v_2} + r\vec{v_3} = \vec{v_4}$$

Since there are infinitely many choices for $a$ & $b$ (actually there is only 1 pair of numbers that would not work), we have infinitely many solutions for our initial system.

20: a) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 7 & 6 \\ 6 & 6 \end{bmatrix}$

b) $9\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 9 & -9 & 18 \\ 27 & 36 & 45 \end{bmatrix}$

24: $A \in M_{4,4}$ ; $\vec{b}, \vec{c} \in \mathbb{R}^4$

$A\vec{x} = \vec{b}$ has a unique solution $\Rightarrow$ rank $A = 4$ $\Rightarrow$

$\Rightarrow$ every row of rref $A$ has a leading one, as does every column $\Rightarrow$ rref$[A|\vec{c}]$ will not have a row of the form $[0 \ 0 \ \ldots \ 0|1]$ nor will it have free variables $\Rightarrow$ $A\vec{x} = \vec{c}$ has unique solution.

48: $\vec{x}_1$ is a solution to $A\vec{x}=\vec{b}$

a) $\vec{x}_h$ is a solution of $A\vec{x}=\vec{0}$. Then

$$A(\vec{x}_1+\vec{x}_h) = \underbrace{A\vec{x}_1}_{=\vec{b}} + \underbrace{A\vec{x}_h}_{=\vec{0}} = \vec{b}+\vec{0}=\vec{b}$$

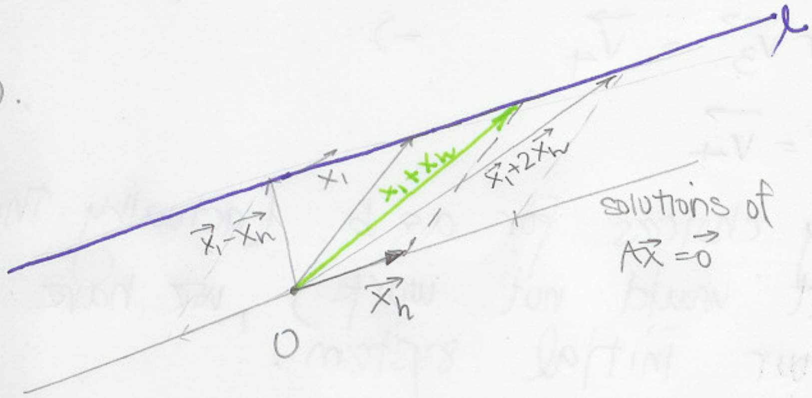
$\Rightarrow \vec{x}_1+\vec{x}_h$ is a solution of $A\vec{x}=\vec{b}$

b) $\vec{x}_2$ is a solution of $A\vec{x}=\vec{b}$. Then

$$A(\vec{x}_2-\vec{x}_1) = A\vec{x}_2 - A\vec{x}_1 = \vec{b}-\vec{b} = \vec{0}$$

$\Rightarrow \vec{x}_2-\vec{x}_1$ is a solution of $A\vec{x}=\vec{0}$.

c).




$A\vec{x}_1=\vec{b}$. Draw a line consisting of all solutions to $A\vec{x}=\vec{b}$, given a line of solutions $A\vec{x}=\vec{0}$.

Pick a vector $\vec{x}_h$ on this line (any vector). From a) we know that $\vec{x}_1+\vec{x}_h$ (green vector) is also a solution to $A\vec{x}=\vec{b}$. Let $k\in\mathbb{R}$. Then

$$A(\vec{x}_1+k\vec{x}_h) = \underbrace{A\vec{x}_1}_{=\vec{b}} + k\cdot\underbrace{A\vec{x}_h}_{=\vec{0}} = \vec{b}$$

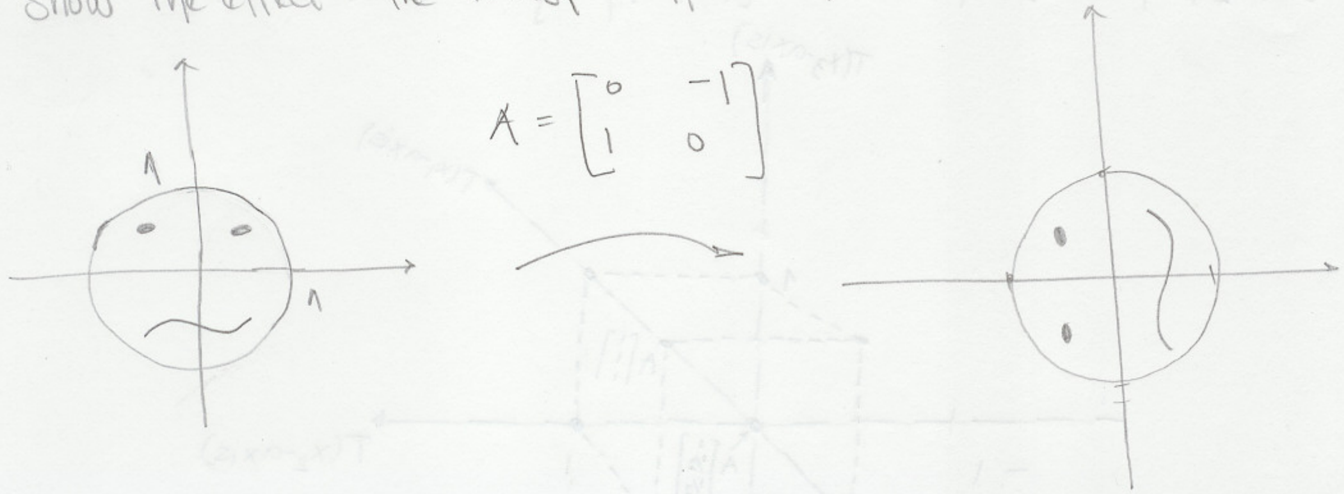
so $\vec{x}_1+k\vec{x}_h$ is a solution to $A\vec{x}=\vec{b}$, for every real number $k$. By varying $k$, we get a line $l$.

Homework #2 solution set

Section 2.1:

#24: Show the effect the transformation $T(\vec{x}) = A\vec{x}$ has on

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

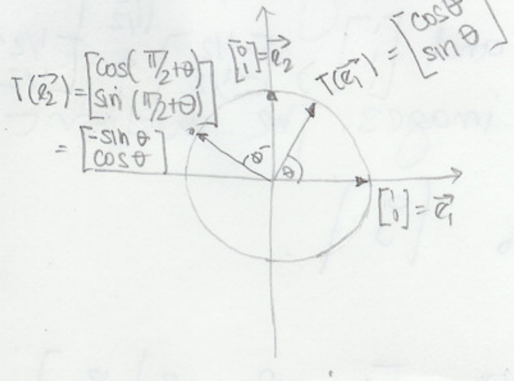


#34: Consider the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each vector $\vec{x} \in \mathbb{R}^2$ through a given angle $\theta$ in the counterclockwise direction. You are told $T$ is linear. Find the matrix of $T$ in terms of $\theta$.

We have already shown that if $T$ is a linear transformation then its matrix is given by

$$\begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{bmatrix}$$

So we only need to find out the effect of $T$ on the coordinate vectors.



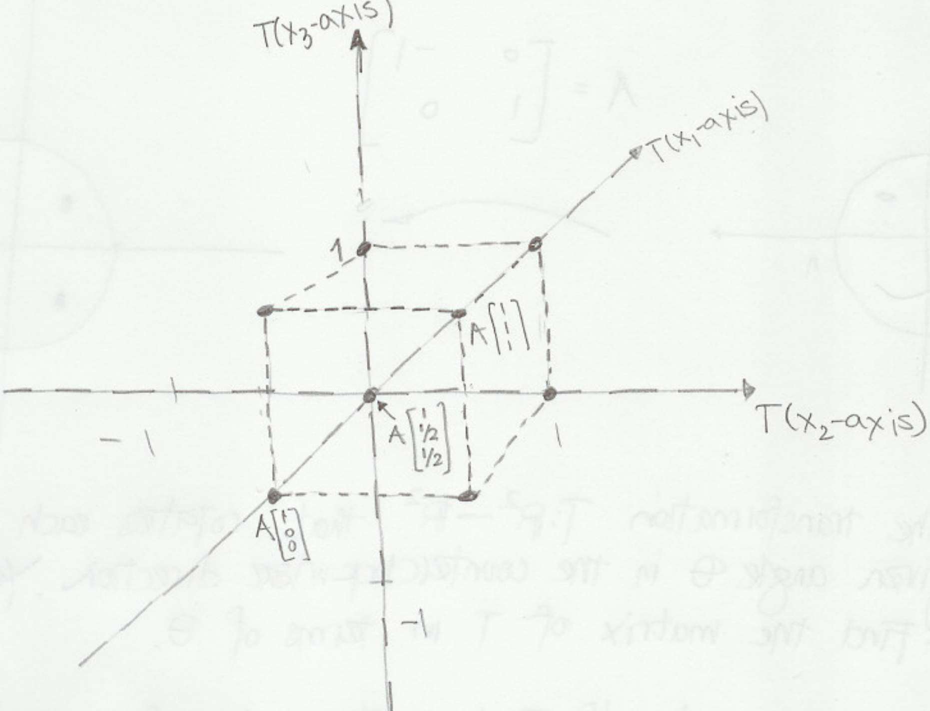

$$\Rightarrow A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

#42: When representing a 3-d object in the plane one needs to transform spatial coordinates $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ into plane coordinates $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ using, for example, a linear transformation given by the matrix

$$A = \begin{bmatrix} -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

a) $A\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ ; $A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}-1/2\\-1/2\end{bmatrix}$ ; $A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ ; $A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$ ,

$A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1/2\\-1/2\end{bmatrix}$ , $A\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ ; $A\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}-1/2\\1/2\end{bmatrix}$ , $A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1/2\\1/2\end{bmatrix}$




b) Represent the image of $\begin{bmatrix}1\\1/2\\1/2\end{bmatrix}$ in your figure in a)

The point $\begin{bmatrix}1\\1/2\\1/2\end{bmatrix}$ lies half way between points $\begin{bmatrix}1\\0\\0\end{bmatrix}$ & $\begin{bmatrix}1\\1\\1\end{bmatrix}$ on the line determined by those two points. Hence the image of $\begin{bmatrix}1\\1/2\\1/2\end{bmatrix}$ under $T$ will lie half way between images of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1/2\\-1/2\end{bmatrix}$ & $\begin{bmatrix}1/2\\1/2\end{bmatrix}$, respectively, on the line spanned by those images. The midpoint of this segment is $\begin{bmatrix}0\\0\end{bmatrix}$.

c) Find all points in $\mathbb{R}^3$ that are mapped to $\begin{bmatrix}0\\0\end{bmatrix}$.
We have to solve

$$A\vec{x}=\vec{0}$$

$$\begin{bmatrix}-1/2 & 1 & 0\\-1/2 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \iff \left[\begin{array}{ccc|c}-1/2 & 1 & 0 & 0\\-1/2 & 0 & 1 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & -2 & 0 & 0\\0 & -1 & 1 & 0\end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c}1 & -2 & 0 & 0\\0 & 1 & -1 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\0 & 1 & -1 & 0\end{array}\right] \Rightarrow \begin{array}{l}x_1=2x_3\\x_2=x_3\end{array} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}2\\1\\1\end{bmatrix},\ x_3\in\mathbb{R}$$

#28:

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projection

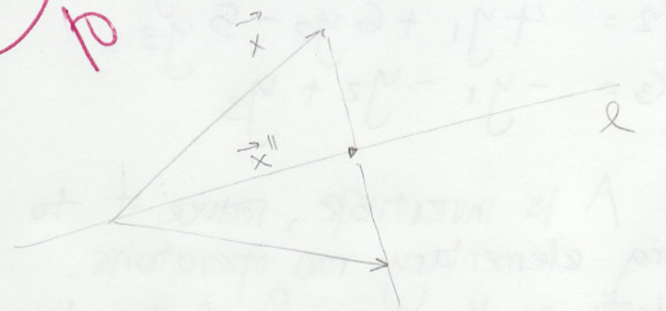
$C = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ rotation

$F = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ reflection

$D = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ scaling

$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ shear

#34: p




$\text{proj}_L(\vec{x}) = A\vec{x} = \begin{bmatrix} \text{proj}_L(\vec{e}_1) & \text{proj}_L(\vec{e}_2) & \text{proj}_L(\vec{e}_3) \end{bmatrix}$

all these columns will be parallel, since they are all projections onto L. The only matrix among offered with parallel columps is:

$$B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\text{refl}_L(\vec{x}) = B\vec{x} = \begin{bmatrix} \text{refl}_L(\vec{e}_1) & \text{refl}_L(\vec{e}_2) & \text{refl}_L(\vec{e}_3) \end{bmatrix}\vec{x}$

We've found that reflection preserves angles, so all column vectors have to be pairwise orthogonal. That only happens in matrix

$$F = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

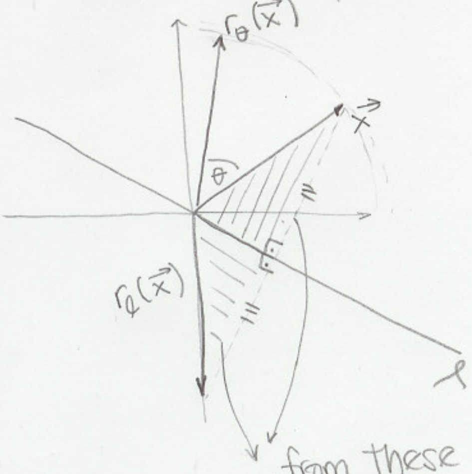
Section 2.2:

#24: 12

Rotations & reflections preserve lengths & angles.

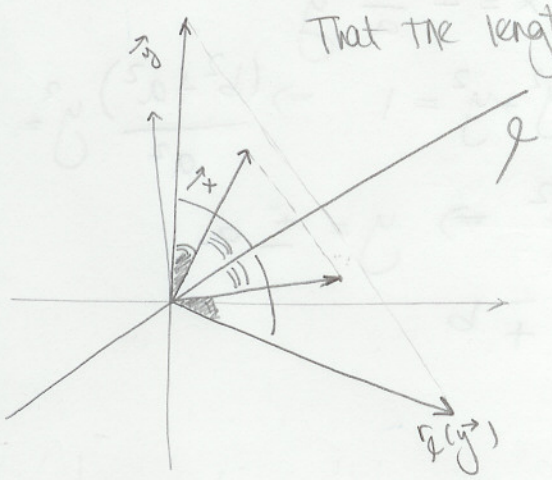
Rotation by $\theta$ : $r_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

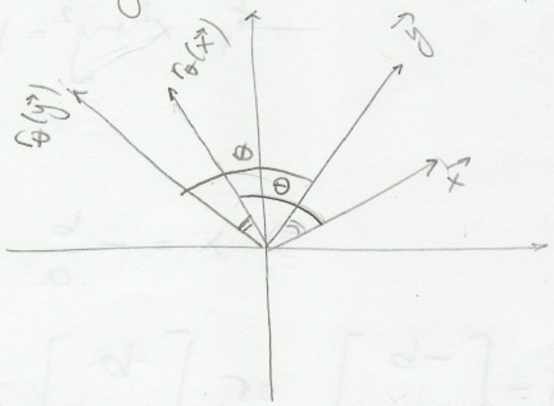
Reflection about $\ell$ : $r_\ell : \mathbb{R}^2 \to \mathbb{R}^2$




from these 2 triangles we see that length$(\vec{x})$ = length$(r_\ell(\vec{x}))$

That the length is preserved by rotations follows from def.




a) $T(\vec{x}) = A\vec{x} = [\vec{v} \ \ \vec{w}]\vec{x}$ preserves length & angles.

Claim: $\vec{v}$ & $\vec{w}$ must be perpendicular unit vectors.

$T(\vec{e_1}) = \vec{v}$

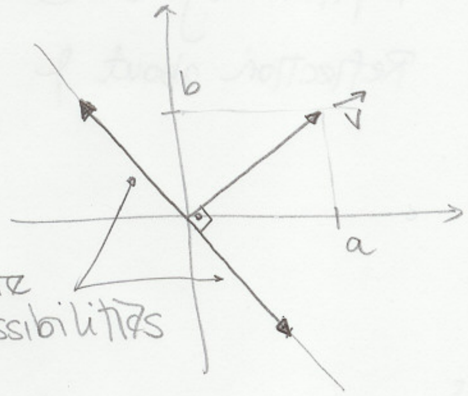
$T(\vec{e_2}) = \vec{w}$

Since $\vec{e_1}$ & $\vec{e_2}$ are unit vectors & $T$ preserves length, it follows that $T(\vec{e_1})$ & $T(\vec{e_2})$ are unit vectors, i.e. $\vec{v}$ & $\vec{w}$ are unit vectors.

Since $T$ preserves angles and $\vec{e_1}$ & $\vec{e_2}$ are perpendicular, so are $T(\vec{e_1}) = \vec{v}$ & $T(\vec{e_2}) = \vec{w}$.

b) $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ & $a^2+b^2=1$

From a) we know that $\vec{w}$ will be perpendicular to $\vec{v}$ and will also be a unit vector:




these are the two possibilities for $\vec{w}$.

Let $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $ax + by = 0 \Rightarrow x = -\frac{b}{a} y$

& $x^2+y^2=1 \Rightarrow \frac{b^2}{a^2} y^2 + y^2 = 1 \Rightarrow \frac{(b^2+a^2)}{a^2} y^2 = 1$

$\Rightarrow y^2 = a^2 \Rightarrow y = \pm a$

$\Rightarrow x = -\frac{b}{a}(\pm a) = \mp b$

$\Rightarrow \vec{w} = \begin{bmatrix} -b \\ a \end{bmatrix}$ or $\begin{bmatrix} b \\ -a \end{bmatrix}$

c) From b) we know that A is either

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ or $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ & $a^2+b^2=1$

rotation

reflection