Section 2.3

#20:
$$x_1+3x_2+3x_3=y_1$$
$$x_1+4x_2+8x_3=y_2$$
$$2x_1+7x_2+12x_3=y_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 8 & 0 & 1 & 0 \\ 2 & 7 & 12 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -12 & 4 & -3 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & -15 & 12 \\ 0 & 1 & 0 & 4 & 6 & -5 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$x_1 = -8y_1 - 15y_2 + 12y_3$$
$$x_2 = 4y_1 + 6y_2 - 5y_3$$
$$x_3 = -y_1 - y_2 + y_3$$

claim:

#36: To determine whether a square matrix A is invertible, reduce it to triangular form (upper or lower), using elementary row operations. A is invertible if and only if all entries on the diagonal of this triangular form are nonzero.

Proof:

There is a sequence of elementary row operations that take A into uppertriangular matrix U (or lowertriangular matrix L). There is another sequence of elementary row operations that take U (L) to rref U (rref L) = rref A.

$\Leftarrow$

1) all diagonal entries u_{ii} of an uppertriangular matrix are nonzero. Divide row 1 by u_{11} to get a leading 1, everything else in Column 1 is already 0, so we move to row 2. $u_{22}\neq 0$, divide row 2 by u_{22} and get a leading 1. Use it to make $u_{12}=0$. Continue. Since each $u_{ii}\neq 0$, we will get a leading 1 in every row $\Rightarrow$ rref U = rref A = I_n $\Rightarrow$ A is invertible.

2) All diagonal entries l_{ii} of a lower triangular matrix L are nonzer by dividing row 1 by l_{11} we obtain a leading 1 in row 1 and use it to, via el. row operations, obtain 0s as other entries in the first colum. Notice that since all other entries in row 1 are zeros, no other column had changed, that is the new matrix still has all diagona

entries not equal to zero. We move to l_{22} and continue the algorithm, until we divided the last row by l_{nn} which gave us $\operatorname{rref} L = \operatorname{rref} A = I_n \Rightarrow A$ is invertible.

$\rightarrow$

1) We know that we can use the sequence of opposite el. row operations get from $I_n = \operatorname{rref} A = \operatorname{rref} U \xrightarrow[S_1]{\text{to}} U \xrightarrow{\text{to}} A$.

To get an uppertriangular matrix from I_n by a seq. of el. row operations we will always be adding multiples of rows only to the row above it (never to the ones below it). Further, doing so will not affect diagonal entries further to the left (since given row has 0 in those columns) nor the ones to the right (since those are in the rows below it). Hence the only way to change the diagonal entries during this sequence will be to multiply a given row by a nonzero constant $\Rightarrow U$ has nonzero diagonal entries.

2) Similar to 1) except now we are always adding multiples of a row only to the rows below it, hence the diagonal entries to the right of the one under consideration will not be affected, since our row has zeros in columns that contain those diagonal entries (nor will be the ones to the left, since they lie in the rows above our row). Proceed as in 1).

Section 2.4.

#15. Compute the matrix product

$$\begin{bmatrix} 1 & -2 & -5 \\ -2 & 5 & 11 \end{bmatrix}\begin{bmatrix} 8 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Explain why the solution does not contradict Fact 2.4.9

Solution:

$$\begin{bmatrix} 1 & -2 & -5 \\ -2 & 5 & 11 \end{bmatrix}\begin{bmatrix} 8 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In fact 2.4.9 both matrices are required to be $n \times n$, which these two are not

#18. If A is an $n \times n$ invertible matrix is the following necessarily true?

A^2 is invertible and $(A^2)^{-1} = (A^{-1})^2$

Yes. We know that a product of two invertible $n \times n$ matrices A & B is itself invertible. Further $(AB)^{-1} = B^{-1}A^{-1}$ Analogously

$A \cdot A = A^2$ is invertible & $(A^2)^{-1} = (A \cdot A)^{-1} = A^{-1} \cdot A^{-1} = (A^{-1})^2$.

#19: $A+B$ is invertible & $(A+B)^{-1} = A^{-1} + B^{-1}$.

No. If $(A+B)^{-1} = A^{-1} + B^{-1}$ then $(A+B)(A+B)^{-1} = I \iff$

$$(A+B)(A^{-1}+B^{-1}) = I \iff$$

$$\underbrace{A \cdot A^{-1}}_{=I} + AB^{-1} + BA^{-1} + \underbrace{BB^{-1}}_{=I} = I$$

$$2I + AB^{-1} + BA^{-1} = I$$

$$I + AB^{-1} + BA^{-1} = 0$$

which certainly does not have to be true for any pair of matrices (e.g. $A = B = I$ would not satisfy this equation).

#20: $(A-B)(A+B) = A^2 - B^2$ false because

$(A-B)(A+B) = A^2 + AB - BA - B^2$, but we know that multiplication of matrices is not commutative so $AB \neq BA$ necessarily. Hence, in general this formula is false

#30: If A is a noninvertible $n\times n$ matrix, can you always find a nonzero $n\times n$ matrix B such that $AB=0$?

$$B=[\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$$

$$AB=[A\vec{v}_1\ A\vec{v}_2\ \cdots\ A\vec{v}_n]$$

If $AB=0$, then $A\vec{v}_i=0$, $\forall i=1,\dots,n$. So our question becomes whether there exists a nonzero solution of the equation $A\vec{x}=\vec{0}$. Since A is noninvertible we know that its rank is smaller than # of its columns $\Rightarrow$ there are free variables $\Rightarrow$ $A\vec{x}=\vec{0}$ has infinitely many solutions. Let $\vec{v}$ be one of them, i.e. $A\vec{v}=\vec{0}$. Then the matrix

$$B=[\vec{v}\ \vec{v}\ \cdots\ \vec{v}]$$

will do.

#35: Consider an $m\times n$ matrix B & an $n\times m$ matrix A s.t $BA=I_m$.

a) Find all solutions of the system $A\vec{x}=\vec{0}$.
b) Show that the linear system $B\vec{x}=\vec{b}$ is consistent for all vectors $\vec{b}\in\mathbb{R}^m$
c) What can you say about rank(A)? What about rank(B)?
d) Explain why $m\le n$.

Solution:

a) Suppose $\vec{a}$ is a solution of $A\vec{x}=\vec{0}$, i.e. $A\vec{a}=\vec{0}$. Then

$$B(A\vec{a})=B\vec{0}=\vec{0}$$
$$\parallel$$
$$(BA)\vec{a}=I_m\vec{a}=\vec{a}$$

$\Rightarrow$ $\vec{a}=\vec{0}$ is the only solution of $A\vec{x}=\vec{0}$.

b) Let $\vec{b}$ be any vector $\in\mathbb{R}^m$. Let $\vec{x}=A\vec{b}$. Then

$$B\vec{x}=B(A\vec{b})=(BA)\vec{b}=I_m\vec{b}=\vec{b}$$

hence $A\vec{b}$ is a solution to $B\vec{x}=\vec{b}$.

c) From a) we know that $A\vec{x}=\vec{0}$ has a unique solution $\Rightarrow$ rref A has no free variables $\Rightarrow$ rank A = # columns $=m$.

From b) we know that $B\vec{x}=\vec{b}$ is consistent for all $\vec{b}$ $\Rightarrow$ rref B has no 0 rows $\Rightarrow$ rank B = # rows $=m$.

d) $\text{rank } A\le \underset{m}{\underset{\parallel}{\#\text{columns}}}$ & $\text{rank } A\le \underset{n}{\underset{\parallel}{\#\text{rows}}}$ & rank $A=m$ (from c) $\Rightarrow$ $\Rightarrow m\le n$

Section 3.1.

#10: Find vectors that span $\ker A$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{matrix} x_1 = x_3 \\ x_2 = -2x_3 \\ x_4 = 0 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\ker A = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right).$$

#14. Find vectors that span the image of A

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

Solution

$\operatorname{Im} A$ = Span of its column vectors $= \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right)$

#18: Describe the image of the transformation $T(\vec{x}) = A\vec{x}$ geometrically.

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$$

Solution:

$\operatorname{Im} A = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \end{bmatrix} \right)$. But notice that $\begin{bmatrix} 4 \\ 12 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$,

So if $\vec{v} = a \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \cdot \begin{bmatrix} 4 \\ 12 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \cdot b \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (a+4b) \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

is a linear combination of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ 12 \end{bmatrix}$ then it is a linear combination of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ only,

i.e. if $\vec{v} \in \operatorname{Im} A$, then $\vec{v}$ is a multiple of the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$,

so $\operatorname{Im} A$ is a line.

#24: Describe the image & kernel of the orthogonal projection onto the plane $x+2y+3z=0$ geometrically.

Solution: The image is the whole plane $x+2y+3z=0$, while the kernel is the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, line perpendicular to that plane.

#35: Consider a nonzero vector $\vec{v}$ in $\mathbb{R}^3$. Arguing geometrically, describe the image & the kernel of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}$ given by

$$T(\vec{x}) = \vec{v} \cdot \vec{x}$$

Solution: $\ker T = \{\vec{x} \in \mathbb{R}^3 : T(\vec{x}) = \vec{0}\} = \{\vec{x} \in \mathbb{R}^3 : \vec{x} \cdot \vec{v} = 0\} =$

$=$ set of all vectors in $\mathbb{R}^3$ that are perpendicular to $\vec{v}$

$=$ plane perpendicular to $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \ldots\ v_1 x + v_2 y + v_3 z = 0$

$\operatorname{Im} T = \{T(\vec{x}) : \vec{x} \in \mathbb{R}^3\} = \{\vec{v} \cdot \vec{x} : \vec{x} \in \mathbb{R}^3\}$.

We know $\operatorname{Im} T \subset \mathbb{R}$. In fact it will be the whole $\mathbb{R}$ because the dot product $\vec{v} \cdot \vec{x}$ represents the length of the orthogonal projection of vector $\vec{x}$ onto line spanned by $\vec{v}$. If we just take scalar multiples of vector $\vec{v}$ then their images under orthogonal projection will have covered the whole line spanned by $\vec{v}$

$\Rightarrow \operatorname{Im} T = \mathbb{R}$.

or different approach: Let $y \in \mathbb{R}$. Then

$$\vec{v} \cdot \left(\frac{y \cdot \vec{v}}{\|v\|^2} \right) = y \cdot \frac{\|\vec{v}\|^2}{\|\vec{v}\|^2} = y$$

So any $y \in \mathbb{R}$ is $T\left(\frac{y \vec{v}}{\|\vec{v}\|^2}\right)$.

#38: Consider a square matrix A.

a) What is the relationship between $\ker A$ & $\ker A^2$? Are they necessarily equal? Is one of them necessarily contained in the other? More generally what can you say about $\ker A, \ker A^2, \ker A^3, \ldots$?

b) What can you say about $\operatorname{Im} A, \operatorname{Im} A^2, \operatorname{Im} A^3, \ldots$?

Solution:

a) Let $\vec{x} \in \ker A$. Then $A^2\vec{x} = A(A\vec{x}) = A(\vec{0}) = \vec{0} \Rightarrow \vec{x} \in \ker A^2 \Rightarrow \ker A \subseteq \ker A^2$.

Let $\vec{x} \in \ker A^2 \Rightarrow \vec{0} = A^2 x = A(A\vec{x}) \Rightarrow A\vec{x} \in \ker A$, but we can't say anything about $\vec{x}$.

Similarly, $\vec{x} \in \ker A^k$ then $A^{k+1}\vec{x} = A(A^k\vec{x}) = A\vec{0} = \vec{0} \Rightarrow \ker A^k \subseteq \ker A^{k+1}$

$$\ker A \subseteq \ker A^2 \subseteq \ker A^3 \subseteq \ker A^4 \subseteq \ldots$$

b) Let $\vec{y} \in \operatorname{Im} A^2$. $\exists \vec{x}$ st $\vec{y} = A^2 x$. Then $\vec{y} = A(A\vec{x})$, so if we set $\vec{w} = A\vec{x}$ then $\vec{y} = A\vec{w} \Rightarrow \vec{y} \in \operatorname{Im} A \Rightarrow \operatorname{Im} A^2 \subseteq \operatorname{Im} A$.

More generally, if $\vec{y} \in \operatorname{Im} A^{k+1}$ then $\exists \vec{x}$ st $\vec{y} = A^{k+1}\vec{x} = A^k(A\vec{x})$

Setting $\vec{w} = A\vec{x}$ we get $\vec{y} = A^k\vec{w} \Rightarrow \vec{y} \in \operatorname{Im} A^k \Rightarrow \operatorname{Im} A^{k+1} \subseteq \operatorname{Im} A^k$

$$\operatorname{Im} A \supseteq \operatorname{Im} A^2 \supseteq \operatorname{Im} A^3 \supseteq \ldots \supseteq \operatorname{Im} A^k \supseteq \operatorname{Im} A^{k+1} \supseteq \ldots$$

If, on the other hand, $\vec{y} \in \mathrm{Im}\, A^k$ Can we say that $\vec{y} \in \mathrm{Im}\, A^{k+1}$?
$\vec{y} = A^k \vec{x}$, for some $\vec{x}$. If $\vec{x} = A\vec{z}$ for some $\vec{z}$, then
$\vec{y} = A^k \vec{x} = A^k (A\vec{z}) = A^{k+1} \vec{z}$, so $\vec{y}$ would be in the image of A^{k+1}. However, there is nothing that would guarantee the existence of such $\vec{z}$, hence we do not know whether $\mathrm{Im}\, A^k \subseteq \mathrm{Im}\, A^{k+1}$.

Remark: Notice the following:

$$\mathrm{Ker}\, A \subseteq \mathrm{Ker}\, A^2 \subseteq \mathrm{Ker}\, A^3 \subseteq \dots \subseteq \mathrm{Ker}\, A^n \subseteq \dots \subseteq \mathbb{R}^n$$

&

$$\mathbb{R}^n \supseteq \mathrm{Im}\, A \supseteq \mathrm{Im}\, A^2 \supseteq \mathrm{Im}\, A^3 \supseteq \dots \supseteq \mathrm{Im}\, A^n \supseteq \dots \supseteq \vec{0}$$

Can the elements of the first sequence increase all the time (without ever becoming equal to each other).
Similarly, can the elements of the second sequence decrease all the time (without becoming the same set)?