Section 4.3.

#3 Do the polynomials $f(t) = 1+2t+9t^2+t^3$; $g(t) = 1+7t+7t^3$, $h(t) = 1+8t+t^2+5t^3$ and $k(t) = 1+8t+4t^2+8t^3$ form a basis of P_3?

Solution:

The quickest way to check is to consider $T: P_3 \to \mathbb{R}^4$, coordinate linear transformation, which we know is an isomorphism. So if $T(f), T(g), T(h)$ & $T(k)$ form a basis of $\mathbb{R}^4$, then f, g, h & k form a basis.

To check that $T(f), T(g), T(h)$ & $T(k)$ form a basis of $\mathbb{R}^4$, it is enough to show that a matrix with those vectors as its columns is invertible, or equivalently that its rank is 4.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 7 & 8 & 8 \\ 9 & 0 & 1 & 4 \\ 1 & 7 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 6 & 6 \\ 0 & -9 & -8 & -5 \\ 0 & 6 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 6/5 & 6/5 \\ 0 & -9 & -8 & -5 \\ 0 & 6 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 6/5 & 6/5 \\ 0 & 0 & 14/5 & 29/5 \\ 0 & 0 & -16/5 & -1/5 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 6/5 & 6/5 \\ 0 & 0 & 1 & 29/14 \\ 0 & 0 & -16 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 6/5 & 6/5 \\ 0 & 0 & 1 & 29/14 \\ 0 & 0 & 0 & \frac{16\cdot 29}{14} - 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 6/5 & 6/5 \\ 0 & 0 & 1 & 29/14 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank is } 4$$

$\Rightarrow$ f, g, h, k form a basis of P_3.

#11 Find matrix of $T: U^{2\times 2} \to U^{2\times 2}$ given by $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ with respect to basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Determine whether T is an isomorphism, find bases for the image & kernel (if it isn't an isomorphism).

$$U = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ U is not invertible & T is not an isomorphism

Basis for $\ker U$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \{1\}$ is a basis for $\ker T$.

Basis for $\operatorname{Im} U$ is $\left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix} \right\} \Rightarrow \{4, 2+8t\}$ is a basis for $\operatorname{Im} T$.

rank $T = 2$.

#29a $T: P_2 \to P_2$, $T(f(t)) = \dfrac{f(t+h) - f(t-h)}{2h}$, $\mathcal{U} = \{1, t, t^2\}$

$T(1) = \dfrac{1-1}{2h} = 0$ $\quad [T(1)]_{\mathcal{U}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(t) = \dfrac{t+h-(t-h)}{2h} = \dfrac{2h}{2h} = 1$ $\quad [T(t)]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(t^2) = \dfrac{(t+h)^2 - (t-h)^2}{2h} = \dfrac{\cancel{t^2} + 2th + \cancel{h^2} - \cancel{t^2} + 2th - \cancel{h^2}}{2h} = 2t$ $\quad [T(t^2)]_{\mathcal{U}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$$\Rightarrow U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\ker U = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \Rightarrow \ker T = \operatorname{Span}(1)$

$\operatorname{Im} U = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) \Rightarrow \operatorname{Im} T = \operatorname{Span}(1, 2t)$, $\operatorname{rank}(T) = 2$.

$T(f(t))$ is a slope of the line through points $(t-h, f(t-h))$ & $(t+h, f(t+h))$.

#43: $\mathcal{B} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$; $\mathcal{U} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

a) Find $S_{\mathcal{B} \to \mathcal{U}}$

$$\left[\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{U}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \left[\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right]_{\mathcal{U}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} ; \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{U}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$S_{\mathcal{B} \to \mathcal{U}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

b) Verify the formula $SB = AS$ for matrices you found in #10 & #11.

We have $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ from #11.

We need U from #10.

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & -6 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

My diagram is

$$\begin{tikzcd}
{[f]_{\mathcal{B}}} \arrow[r, "B"] \arrow[d, "S_{\mathcal{B}\to\mathcal{U}}"'] & {[T(f)]_{\mathcal{B}}} \arrow[d, "S_{\mathcal{B}\to\mathcal{U}}"] \\
{[f]_{\mathcal{U}}} \arrow[r, "U"] & {[T(f)]_{\mathcal{U}}}
\end{tikzcd}$$

$$US = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$SB = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$US = SB$

c) Find $S_{\mathcal{U}\to\mathcal{B}}$.

One way is to find $S^{-1}_{\mathcal{B}\to\mathcal{U}}$. The other is by hand

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$S_{\mathcal{U}\to\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

#57 $V: x_1 + 2x_2 + 3x_3 = 0$ in $\mathbb{R}^3$ with basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$. Find $\mathcal{B}$-matrix of $T: V \to V$ given by $T(\vec{x}) = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$

$$T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad B = \begin{bmatrix} -1 & 3 \\ -1 & 0 \end{bmatrix}.$$

Section 5.1:

#16: Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_3 \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^4$. Can you find a vector $\vec{u}_4$ in $\mathbb{R}^4$ st the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ & $\vec{u}_4$ are orthonormal? If so, how many such vectors can you find?

Solution:

$$\|\vec{u}_1\| = \sqrt{4 \cdot \frac{1}{4}} = 1 \quad ; \quad \|\vec{u}_2\| = \sqrt{2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4}} = 1 \quad , \quad \|\vec{u}_3\| = 1$$

$$\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$$

So we just need one vector that is perpendicular to all these $\vec{u}_1, \vec{u}_2$ & $\vec{u}_3$ and which is a unit vector. Equivalently, we want a unit vector in the kernel of the orthogonal projection onto the subspace W spanned by $\vec{u}_1, \vec{u}_2, \vec{u}_3$.

$$\mathrm{proj}_W(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2 + (\vec{x} \cdot \vec{u}_3) \cdot \vec{u}_3 = 0 \iff$$

$$\left.\begin{array}{l} \vec{x} \cdot \vec{u}_1 = 0 \\ \vec{x} \cdot \vec{u}_2 = 0 \\ \vec{x} \cdot \vec{u}_3 = 0 \end{array}\right\} \iff \left.\begin{array}{l} \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 0 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \\ \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \end{array}\right\} \iff$$

$$\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} x_1 = x_4 \\ x_2 = -x_4 \\ x_3 = -x_4 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathrm{Ker}(\mathrm{proj}_W) \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ -x_4 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

We need a unit vector $\begin{bmatrix} x_4 \\ -x_4 \\ -x_4 \\ x_4 \end{bmatrix} = \vec{x}$

$$\|\vec{x}\| = \sqrt{x_4^2 + (-x_4)^2 + (-x_4)^2 + x_4^2} = \sqrt{4x_4^2} = 2|x_4|$$

$$\vec{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \quad \text{or} \quad \vec{u}_4 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

#17. Find a basis for $W^\perp$, where

$$W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right)$$

Solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in W^\perp \iff \vec{x} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0 \quad \& \quad \vec{x} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0 \iff$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$
$$5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$x_1 = x_3 + 2x_4$$
$$x_2 = -2x_3 - 3x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $W^\perp$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

#23. Prove that $(V^\perp)^\perp = V$ for any subspace V of $\mathbb{R}^n$.

Solution:

$$(V^\perp)^\perp = \{ \vec{w} \in \mathbb{R}^n \mid \vec{w} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V^\perp \}$$
$$V^\perp = \{ \vec{u} \in \mathbb{R}^n \mid \vec{u} \cdot \vec{x} = 0 \text{ for every } \vec{x} \in V \}$$

$\Rightarrow \quad V \subseteq (V^\perp)^\perp$ (since if $\vec{x} \in V$, then $\vec{x} \cdot \vec{u} = 0$ for $\vec{u} \in V^\perp$ by definition of $V^\perp$ $\Rightarrow$ $\Rightarrow \vec{x} \in (V^\perp)^\perp$)

also, if we consider orthogonal projection onto $V^\perp$, then

$$\ker(\text{proj}_{V^\perp}) = (V^\perp)^\perp \quad \& \quad \text{Im}(\text{proj}_{V^\perp}) = V^\perp$$

$$\Rightarrow \quad n = \dim \mathbb{R}^n = \dim (V^\perp)^\perp + \dim V^\perp \qquad (1)$$

Further,

$$\ker(\text{proj}_V) = V^\perp \quad \& \quad \text{Im}(\text{proj}_V) = V \Rightarrow$$

$$\Rightarrow \quad n = \dim \mathbb{R}^n = \dim V^\perp + \dim V \qquad (2)$$

(1) & (2) give $\dim(V^\perp)^\perp + \dim V^\perp = \dim V^\perp + \dim V \Rightarrow$

$$\Rightarrow \dim(V^\perp)^\perp = \dim V$$

Now we have

$$\left.\begin{array}{l} V \subseteq (V^\perp)^\perp \\ \dim V = \dim (V^\perp)^\perp \end{array}\right\} \Rightarrow V = (V^\perp)^\perp.$$

#28. Find the orthogonal projection of $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ onto the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$.

Solution:

We need an orthonormal basis for the subspace onto which we want to orthogonally project. These three vectors are already perpendicular, so we just need to make them unit. Each has length 2, so we divide each by 2:

$$W = \text{Span}\left(\underbrace{\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}}_{= \vec{u}_1}, \underbrace{\begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}}_{= \vec{u}_2}, \underbrace{\begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}}_{= \vec{u}_3} \right)$$

$$\text{proj}_W(\vec{x}) = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\cdot\vec{u}_2 + (\vec{x}\cdot\vec{u}_3)\cdot\vec{u}_3$$

$$\text{proj}_W\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \frac{1}{2}\cdot\vec{u}_1 + \frac{1}{2}\cdot\vec{u}_2 + \frac{1}{2}\cdot\vec{u}_3 = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}3/4\\1/4\\-1/4\\1/4\end{bmatrix}$$

Section 5.5

#4: In $\mathbb{R}^{n\times m}$ consider the inner product $\langle A, B\rangle = \text{trace}(A^T B)$.

a) Find a formula for this inner product in $\mathbb{R}^{n\times 1} = \mathbb{R}^n$

b) Find a formula for this inner product in $\mathbb{R}^{1\times m}$.

Solution:

a) Let $A, B \in \mathbb{R}^{n\times 1} \Rightarrow A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}$

$$\Rightarrow \langle A, B\rangle = \text{trace}\left(\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \right) = \text{trace}\left(\begin{bmatrix} a_{11}b_{11} + a_{21}b_{21} + \cdots + a_{n1}b_{n1} \end{bmatrix} \right)$$

$$= a_{11}b_{11} + a_{21}b_{21} + \cdots + a_{n1}b_{n1}$$

If we think of A & B as vectors in $\mathbb{R}^n$, then $\langle A, B\rangle = A \cdot B$
dot product

b) Let $A, B \in \mathbb{R}^{1\times m} \Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \end{bmatrix}$; $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \end{bmatrix}$

$$\langle A, B\rangle = \text{trace}\left(\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \end{bmatrix} \right) =$$

$$= \text{trace}\left(\begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1m} \\ a_{12}b_{11} & a_{12}b_{12} & \cdots & a_{12}b_{1m} \\ \vdots & & & \\ a_{1m}b_{11} & \cdots & \cdots & a_{1m}b_{1m} \end{bmatrix} \right) = a_{11}b_{11} + a_{12}b_{12} + \cdots + a_{1m}b_{1m}$$

#9: $f: \mathbb{R} \to \mathbb{R}$ is called even if $f(-t) = f(t), \forall t$

odd if $f(-t) = -f(t), \forall t$

Show that f is an odd function & g is an even function (both continuous), then f is orthogonal to g in the space $C[-1,1]$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$

Solution

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt = \int_{-1}^{0} f(t)g(t)\,dt + \int_{0}^{1} f(t)g(t)\,dt = \int_{1}^{0} f(-t)g(t)\,dt +$$

$$+ \int_0^1 f(t)\,g(t)\,dt = -\int_0^1 f(t)g(t)\,dt + \int_0^1 f(t)g(t)\,dt = 0.$$

#14. Which of the following is an inner product on P_2?

a) $\langle f,g \rangle = f(1)g(1) + f(2)g(2)$

b) $\langle\langle f,g \rangle\rangle = f(1)g(1) + f(2)g(2) + f(3)g(3)$

Solution.

If these two maps are to be inner products, then they should satisfy

$$\langle f,f \rangle > 0 \text{ if } f \neq 0 \quad \& \quad \langle\langle f,f \rangle\rangle > 0 \text{ if } f \neq 0.$$

a) $\langle f,f \rangle = f(1)f(1) + f(2)f(2) = (f(1))^2 + (f(2))^2 = 0 \iff f(1) = 0 \;\&\; f(2) = 0$

But this is true for a nonzero polynomial $f(t) = (t-1)(t-2)$

hence this is not an inner product

b) $\langle\langle f,f \rangle\rangle = (f(1))^2 + (f(2))^2 + (f(3))^2 = 0 \iff f(1) = f(2) = f(3) = 0$, but if a polynomial of degree 2 has 3 zeros, then the polynomial must be identically $0 \Rightarrow \langle\langle f,f \rangle\rangle > 0$ for each $f \neq 0$.

#15. For which values of b, c, d is the following an inner product on $\mathbb{R}^2$?

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = x_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2$$

Solution:

If this is an inner product then all 4 conditions need to be satisfied. So we check them.

1) $$\left.\begin{aligned} \langle x,y \rangle &= x_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2 \\ \langle y,x \rangle &= y_1x_1 + by_1x_2 + cy_2x_1 + dy_2x_2 \end{aligned}\right\} = \Rightarrow bx_1y_2 + cx_2y_1 = by_1x_2 + cy_2x_1$$

since this holds for all $x_1, x_2, y_1, y_2 \Rightarrow \underline{\underline{b = c}}$

2) $\langle x,x \rangle = x_1^2 + 2bx_1x_2 + dx_2^2 = x_1^2 + 2bx_1x_2 + b^2x_2^2 - b^2x_2^2 + dx_2^2 =$

$= (x_1 + bx_2)^2 + (d - b^2)x_2^2 > 0$ for all pairs (x_1, x_2), unless they are both $0. \Rightarrow d - b^2 > 0 \Rightarrow \underline{\underline{d > b^2}}$

The requirements $\langle x+y, z \rangle = \langle x,z \rangle + \langle y,z \rangle$

$\langle kx, y \rangle = k\langle x,y \rangle$

are easily checked.