Section 7.2

#10. $A = \begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$

$\det(A-\lambda I) = \det\begin{bmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{bmatrix} = (-3-\lambda)(3-\lambda)(-1-\lambda)+8(-1-\lambda) =$

$= -(1+\lambda)(-9+3\cancel{\lambda}-\cancel{3\lambda}+\lambda^2+8) = -(1+\lambda)(\lambda^2-1) = -(1+\lambda)(\lambda-1)(1+\lambda) = 0$

$\Rightarrow$ $\lambda_1 = 1$ alg mul. is 1

$\lambda_1 = -1$ alg. mul. is -1

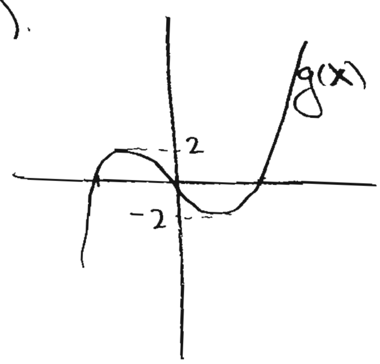
#29: $A = (a_{ij})$ & $\sum_{j=1}^{n} a_{ij} = 1$ for every i, then

$A \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} \cdot 1 \\ \sum_{j=1}^{n} a_{2j} \cdot 1 \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector with associated eigenvalue 1.

#32: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k & 3 & 0 \end{bmatrix}$

$0 = \det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ k & 3 & -\lambda \end{bmatrix} = -\lambda^3 + k + 3\lambda \Rightarrow$

$\lambda^3 - 3\lambda - k = 0$ needs to have 3 real solutions. The graph of the function $f(\lambda) = \lambda^3 - 3\lambda - k$ is obtained from that of the graph of $g(\lambda) = \lambda^3 - 3\lambda$ by shifting it in the vertical direction by k (up if k is negative, down if k is positive).



If we translate graph of g by 2 units (or more) then the new function will have 2 (or 1) zero, so

$-2 < k < 2.$

#40. $tr(AB) = tr BA$

$A = (a_{ij}) \quad B = (b_{ij})$

$$tr(AB) = (AB)_{11} + (AB)_{22} + \cdots + (AB)_{nn} =$$

$$\begin{aligned} = & \; a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \\ + & \; a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} \\ & \vdots \\ + & \; a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn} = \end{aligned}$$

$$(BA)_{11} + (BA)_{22} + \cdots + (BA)_{nn} = tr(BA)$$

#43. $AB - BA = I_n$ would mean

$$tr(AB - BA) = tr\, I_n = n$$

$$\| $$

$$tr\,AB - tr\,BA$$

$$\|$$

$$0$$

So, no, there are no such matrices.

Section 7.2

#10: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad 0 = \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda(1-\lambda)^2$

$\Rightarrow \lambda_1 = 0$ & $\lambda_2 = 1$

$E_0 = \ker A$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker A \iff \begin{matrix} x=0 \\ y=0 \end{matrix} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad$ Basis for E_0 is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$E_1 = \ker(A - I) = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_1 \iff \begin{matrix} y=0 \\ z=0 \end{matrix} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad$ Basis for $E_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow$ There is no eigenbasis.

#13 $A = \begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$ $0 = \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & -2 \\ -7 & -\lambda & 4 \\ 4 & 0 & -3-\lambda \end{bmatrix} =$

$= (3-\lambda)(3+\lambda)\lambda - 8\lambda = \lambda(9 - \lambda^2 - 8) = \lambda(1-\lambda^2) =$
$= \lambda(1-\lambda)(1+\lambda)$

$\Rightarrow \lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1$

$E_0 = \ker(A)$ $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_0 \iff x = z = 0 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

$E_1 = \ker(A - I) = \ker \begin{bmatrix} 2 & 0 & -2 \\ -7 & -1 & 4 \\ 4 & 0 & -4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ -7 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & +1 & +3 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -3z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$

$E_{-1} = \ker(A + I) = \ker \begin{bmatrix} 4 & 0 & -2 \\ -7 & 1 & 4 \\ 4 & 0 & -2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1/2 \\ -7 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z/2 \\ -z/2 \\ z \end{bmatrix} = \frac{z}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$

Eigenbasis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}.$

#17 $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} =$

$= \lambda^2(1-\lambda)^2 \qquad \lambda_1 = 0$
$\lambda_2 = 1$

$E_0 = \ker A$ $\quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \ker A \iff \begin{matrix} y = -z \\ w = 0 \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ -z \\ z \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ Basis for E_0 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$

$$E_1 = \ker\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in E_1 \iff \begin{matrix} x=0 \\ z=0 \end{matrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \\ w \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$ Basis for E_1 is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Eigenbasis is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

#19

- $a=b=c=0$ $\quad E_1 = \ker(A-I) = \ker 0 \Rightarrow \dim E_1 = 3$.

- $a=0$ & either b or $c \neq 0$ $\quad E_1 = \ker(A-I) = \ker\begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_1 \iff z=0 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$$

$\dim E_1 = 2$

- $a \neq 0$, $c \neq 0$ $\quad E_1 = \ker\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \ker\begin{bmatrix} 0 & 1 & b/a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$z=0$, $y=0$ $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \dim E_1 = 1$

- $a \neq 0$, $c = 0$ $\quad E_1 = \ker\begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad y = -\frac{b}{a} z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -b/a\, z \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -b/a \\ 1 \end{bmatrix}.$$

$\dim E_1 = 2$.

- $a \neq 0$, $b=c=0$ $\quad E_1 = \ker\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y=0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ $\dim E_1 = 2$.

21. Find A st $E_1 = \begin{bmatrix}1\\2\end{bmatrix}$; $E_2 = \begin{bmatrix}2\\3\end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \qquad \begin{array}{l} a+2b=1 \\ c+2d=2 \end{array}$$

$$A\begin{bmatrix}2\\3\end{bmatrix} = 2\begin{bmatrix}2\\3\end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \qquad \begin{array}{l} 2a+3b=4 \\ 2c+3d=6 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & +1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & +1 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$. This is the only such matrix

#33 A is similar to $B \Rightarrow \exists$ invertible S st $A = SBS^{-1}$

$A - \lambda I = SBS^{-1} - \lambda SIS^{-1} = S(BS^{-1} - \lambda IS^{-1}) = S(B-\lambda I)S^{-1} \Rightarrow$

$\Rightarrow A - \lambda I$ & $B - \lambda I$ are similar.

#35 The matrix $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ since they have different determinants.

Section 7.4

#18 $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$0 = \det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3 - (1-\lambda) =$$

$$= (1-\lambda)\left((1-\lambda)^2 - 1\right) = (1-\lambda)(\cancel{1} - 2\lambda + \lambda^2 - \cancel{1}) =$$

$$= (1-\lambda)\,\lambda(\lambda-2)$$

It is a 3x3 matrix with 3 distinct eigenvalues $\Rightarrow$ It is diagonalizable

It will be similar to

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$E_0 = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cong \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = -z \\ y = 0 \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$E_1 = \ker \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow x = z = 0 \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$E_2 = \ker \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = z \\ y = 0 \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

#20 $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3 - (1-\lambda) = \\ = \lambda(1-\lambda)(\lambda-2)$$

$$E_0 = \ker \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cong \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = -z \\ y = 0 \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$E_1 = \ker \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cong \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = 0 \\ z = 0 \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$E_2 = \ker \begin{bmatrix} +1 & 0 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & +1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = z \\ y = 2z \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \& \quad S = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

#28: $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{bmatrix}$ $\quad 0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & a \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 + a\lambda = -\lambda(\lambda^2 - a)$

$= -\lambda(\lambda - \sqrt{a})(\lambda + \sqrt{a})$

If $a \neq 0$ then A has 3 distinct eigenvalues $\Rightarrow$ A is diagonalizable.

If $a = 0$ then $\lambda = 0$ is the only eigenvalue & its geometric multiplicity is $\dim \ker A = 1$ since $\operatorname{rank} A = 2$, and hence A is not diagonalizable.

#33 $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$0 = \det(A - \lambda I) = \det \begin{vmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 6 = 6 - \lambda - 6\lambda + \lambda^2 - 6 =$

$= \lambda^2 - 7\lambda = \lambda(\lambda - 7)$

$E_0 = \ker A = \ker \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad x = -2y \quad \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$

$E_7 = \ker(A - 7I) = \ker \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} \sim \ker \begin{bmatrix} 0 & 0 \\ 3 & -1 \end{bmatrix} \quad y = 3x \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} x$

$D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}; \quad S = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \quad S^{-1} = -\frac{1}{7} \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}$

$A = SDS^{-1} \quad A^t = SD^tS^{-1} = -\frac{1}{7} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 7^t \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} =$

$= -\frac{1}{7} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -7^t & -2 \cdot 7^t \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -7^t & -2 \cdot 7^t \\ -3 \cdot 7^t & -6 \cdot 7^t \end{bmatrix} =$

$= \frac{7^t}{7} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = 7^{t-1} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$

#47. $T: P_2 \to P_2 \quad T(f(x)) = f(-x)$

$f(x) = a + bx + cx^2$

$f(-x) = a - bx + cx^2$

$T(f(x)) = \lambda f(x)$

$a + bx + cx^2 = \lambda a - \lambda bx + \lambda cx^2$

$$a = \lambda a$$
$$b = -\lambda b \qquad\qquad \lambda = -1 \text{ or } b = 0$$
$$c = \lambda c \Rightarrow \lambda = 1 \text{ or } c = 0$$

- $\lambda = 1$

$$b = -b \Rightarrow b = 0$$
$$f(x) = a + cx^2 \qquad \Rightarrow E_1 = \text{span}(1, x^2).$$

- $\lambda = -1$

$$f(x) = bx \qquad \Rightarrow E_{-1} = \text{span}(x)$$

$\Rightarrow$ T is diagonalizable as there is an eigenbasis $\{1, x, x^2\}$.

#51 $T : P \to P \quad T(f) = f'$

$$f \in P \Rightarrow f(x) = a_0 + a_1 x + \dots + a_n x^n, \quad a_n \neq 0$$
$$f'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1}$$

$$T(f) = \lambda f \iff a_1 + 2a_2 x + \dots + n a_n x^{n-1} = \lambda a_0 + \lambda a_1 x + \dots + \lambda a_n x^n$$

$$\iff \begin{aligned} a_1 &= \lambda a_0 \\ 2a_2 &= \lambda a_1 \\ &\vdots \\ n a_n &= \lambda a_{n-1} \\ 0 &= \lambda a_n \end{aligned} \Rightarrow \lambda = 0 \text{ or } a_n = 0 \ ↯$$

$$\Rightarrow \lambda = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$$
$$\Rightarrow f(x) = a_0$$

T is not diagonalizable.