Math 431 Homework 2 Solutions

**1.** Two or more lines are said to be *concurrent* if there is exactly one point that lies on all of them. Prove that in incidence geometry, there exist three distinct lines that are not concurrent. This is Proposition 2.2 from the textbook.

*Proof.* By I-3 there exist three distinct points $P, Q, R$ that do not all lie on the same line. By I-1 these points determine three lines $\overleftrightarrow{PQ}, \overleftrightarrow{PR}, \overleftrightarrow{QR}$. We claim that these three lines are not concurrent. Suppose to the contrary that they are concurrent; that is, there is a point $S$ that lies on all three lines. Consider the two cases $S = P$ or $S \neq P$. If $S = P$ then $P$ lies on all three lines. In particular, $P$ lies on $\overleftrightarrow{QR}$. Thus $\overleftrightarrow{QR}$ passes through all three points $P, Q, R$, contradicting the fact that $P, Q, R$ do not all lie on the same line. The assumption $S = P$ led to a contradiction, so the assumption was false and we move on to the next case. Suppose that $S \neq P$. We know that both $S$ and $P$ lie on the line $\overleftrightarrow{PQ}$ as well as the line $\overleftrightarrow{PR}$. By the uniqueness part of I-1, we must have $\overleftrightarrow{PQ} = \overleftrightarrow{PR}$. Thus the three points $P, Q, R$ all lie on $\overleftrightarrow{PQ}$ (and $\overleftrightarrow{PR}$), again a contradiction.

Both assumptions $S = P$ and $S \neq P$ led to contradictions, so our earlier assumption that the three lines $\overleftrightarrow{PQ}, \overleftrightarrow{PR}, \overleftrightarrow{QR}$ are concurrent must be false. Therefore these three lines are not concurrent, as claimed. $\square$

**2.** This problem was group dependent and I can not provide solutions.

**3.** In class we found an interpretation of the undefined terms of incidence geometry—point, line, and lie on—in which Axioms I-2 and I-3 are satisfied and Axiom I-1 is not satisfied.

(a) Find an interpretation of the undefined terms of incidence geometry in which Axioms I-1 and I-2 are satisfied and Axiom I-3 is not satisfied.

**Solution:** Our interpretation has no points and no lines.

I-1 and I-2 are trivially satisfied in this interpretation. I-3 certainly is not satisfied, since there do not exist any points at all, let alone three distinct points.

(Let us explain what "trivially satisfied" means in the case of I-1. Note that I-1 can be rephrased as follows: "If $P$ and $Q$ are two distinct points

then there exists a unique line through both of them". The hypothesis of this statement is always false in our interpretation (because there do not exist any points, let alone two distinct points). Thus, according the truth table for an if-then statement, I-1 is true in this interpretation. Similar reasoning applies to I-2.)

Here are some other interpretations that satisfy I-1 and I-2 but not I-3:

**Solution 2:** Our interpretation has no lines and only one point.

**Solution 3:** Our interpretation has only one line, say $l$, and two points that both lie on $l$. (In fact you can let there be any number of points, as long as there are at least two and they all lie on $l$).

(b) Find an interpretation of the undefined terms of incidence geometry in which Axioms I-1 and I-3 are satisfied and Axiom I-2 is not satisfied.

**Solution:** Our interpretation has three points $A, B, C$ and four lines $\{A, B\}$, $\{A, C\}$, $\{B, C\}$, $l$. "Incidence" means set membership in the case of the first three lines; the last line $l$ passes through none of the three points.

I-1 is satisfied, by inspection. Since $A, B, C$ are three distinct points with no line passing through all of three of them, I-3 is satisfied. I-2 is not satisfied because there are no points lying on $l$.