Math 431 Solutions to Homework 3

1. Show that the following familiar interpretation is a model for incidence geometry. Note that in class we already showed that Axiom I-1 is satisfied, so you only need to show that I-2 and I-3 are satisfied.

- The "points" are all ordered pairs (x, y) of real numbers.
- A "line" is specified by an ordered triple (a, b, c) of real numbers such that either $a \neq 0$ or $b \neq 0$; it is defined as the set of all "points" (x, y) that satisfy the equation $ax + by + c = 0$.
- "Incidence" is defined as set membership.

Proof that I-1 is satisfied. This is an expanded version of what was done in class. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points. We need to show that there is a unique line passing through both of them.

Case 1: Suppose that $x_1 = x_2$. In this case the line given by the equation

$$x = x_1 \tag{1}$$

passes through P and Q (since (x_1, y_1) and (x_2, y_2) satisfy this equation). To show that this line is unique, suppose we are given any line l passing through P and Q. Let $ax + by + c = 0$ be an equation for l. Since P and Q satisfy this equation,

$$ax_1 + by_1 + c = 0, \qquad \text{and} \qquad ax_2 + by_2 + c = 0.$$

Now we have

$$\begin{aligned} & ax_1 + by_1 + c = ax_2 + by_2 + c \\ \iff\ & ax_1 + by_1 = ax_2 + by_2 \\ \iff\ & ax_1 - ax_2 = by_2 - by_1 \\ \iff\ & -a(x_2 - x_1) = b(y_2 - y_1). \end{aligned}$$

We know that $y_1 \neq y_2$ (because $x_1 = x_2$ and $P \neq Q$). Thus $b(y_2 - y_1) = 0$. Thus $b = 0$. Thus $a \neq 0$. Thus the equation for l can be written as $x = -c/a$. Since l passes through P, we must have $-c/a = x_1$ (substitute x_1 for x and

y_1 for y). We started with an arbitrary line l passing through P and Q and concluded that in fact l is given by Equation (**??**). This proves the uniqueness of l.

Case 2: Suppose that $x_1 \neq x_2$. One can check by substitution that the line given by the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1) \tag{2}$$

passes through P and Q. To prove uniqueness, suppose that l is any line passing through P and Q. Let $ax + by + c = 0$ be an equation for l. The same calculation as in Case 1 gives us

$$-a(x_2 - x_1) = b(y_2 - y_1).$$

Since $x_1 \neq x_2$, we have $b \neq 0$ (if $b = 0$ then $a = 0$, which is not allowed by our definition of a line). Thus

$$-\frac{a}{b} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Now the equation for l can be written as

$$y = \frac{y_2 - y_1}{x_2 - x_1} \cdot x - \frac{c}{b}. \tag{3}$$

Plugging in (x_1, y_1) gives

$$y_1 = \frac{y_2 - y_1}{x_2 - x_1} x_1 - \frac{c}{b} \qquad \text{or equivalently} \qquad -\frac{c}{b} = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1.$$

Substituting this expression for $-c/b$ into Equation (**??**) and rearranging terms shows that the equation for l is equivalent to Equation (**??**). This proves the uniqueness of l. □

Proof that I-2 is satisfied. Let l be any line in $\mathbb{R}^2$ having equation $ax + by + c = 0$. We will find two points on l.

Case 1: Suppose that $b = 0$. Then the equation for l may be written as $x = -c/a$. Two points lying on l are $(-c/a, 0), (-c/a, 1)$. (Any point of the form $(-c/a, y)$ would work.)

Case 2: Suppose that $b \neq 0$. Then the equation for l may be written as $y = -a/bx - c/b$. Two points lying on this line are $(0, -c/b), (1, -a/b - c/b)$. (Any point of the form $(x, -a/b \cdot x - c/b)$ would work.) □

Proof that I-3 is satisfied. We claim that the three points $(0,0), (0,1), (1,0)$ do not all lie on the same line. In verifying I-1 we saw that there is only one line passing through $(0,0)$ and $(1,0)$, and its equation is $y = 0$. This line does not pass through $(0,1)$ since $(0,1)$ does not satisfy the equation $y = 0$. Thus, no line can pass through all three of these points. $\square$

Recall that a statement in an axiomatic system is *independent* if it is not a theorem and its negation is not a theorem. To conclude that a statement S is independent, you need to find a model that satisfies S and a model that does not satisfy S. In problems 1, 2, and 3, you are given a statement S. For each problem, decide if (a) S is a theorem in incidence geometry; (b) not-S is a theorem in incidence geometry; or (c) S is an independent statement in incidence geometry. Provide justification (give a proof or supply the appropriate models). You may quote results that were proved in class or on homework. Remember that when giving a model, you should check that it satisfies the axioms.

2. There exist four distinct lines.

Solution: This statement is independent. As justification, we provide two models: one that satisfies the statement and another that does not.

<u>Model 1:</u> There are exactly four "points" A, B, C, D. The "lines" are all sets of two points, $\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{B,C\}$, $\{B,D\}$, $\{C,D\}$. "Lie on" means set membership.

This is the four-point model discussed in class and in the textbook on pages 54-55. We claim that this is a model that satisfies the statement "there exist four distinct lines".

To see that our interpretation is actually a model for incidence geometry, we have to check that the incidence axioms are satisfied. I-1 is satisfied because for any two distinct points $X, Y \in \{A,B,C,D\}$, according to our model the only line passing through them is $\{X,Y\}$. I-2 is satisfied because every line $\{X,Y\}$ by definition has two points X and Y lying on it. I-3 is satisfied because there is no two element set that contains all three points A, B, C, so these three points are not all on the same line.

We also have to verify that the statement "there exist four distinct lines" is true in our model. Well, here are four distinct lines: $\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{B,C\}$. Thus the statement is satisfied.

Remark 1: Since I-3 is a "there exists" statement, you only need to find *three particular points*, say A, B, C, that are not collinear; you should not try to show that *every three points* are not collinear. By contrast, you need to check that in your model, I-1 is true "for every two points" and I-2 is true "for every line".

Remark 2: "There exist four distinct lines" means the same thing as "there exist at least four lines," so any model with four or more lines, such as the Cartesian plane, satisfies this statement.

Model 2: There are exactly three "points" A, B, C. The "lines" are the sets of two points $\{A, B\}$, $\{A, C\}$, $\{B, C\}$. "Lie on" means set membership.

This is the three point model discussed in class and in the textbook on page 52, Example 1. The verification of the axioms is carried out in the book, so this really is a model. There are only three lines in this model, so the statement "there exist four distinct lines" is not satisfied.

3. There exists a point such that at most one line passes through it.

Solution: The negation of this statement is a theorem. The negation is

For any point there are at least two lines passing through it.

This is Proposition 2.5 from the textbook. If we haven't given a proof already, let me know and I will write one up.

4. For any three distinct lines l, m, and n, if $l \parallel m$ and $n \nparallel l$, then $n \nparallel m$.

Solution: This statement is independent. As justification we give two models: one that satisfies the statement and another that does not.

Model 1: The standard three-point model (discussed in the previous problem) satisfies the statement. This is because there are no parallel lines in the three-point model, so the "if" part of the statement is always false, making the "if-then" statement always true.

Model 1 (alternative): Use the four-point model from the previous problem. We already know it is a model, so we won't verify the axioms again.

It remains to show that the statement given in the problem is satisfied. Suppose we have three (distinct) lines l, m, and n in our model. To be more specific, let us say $l = \{P, Q\}$ and $m = \{R, S\}$ where $P, Q, R, S \in \{A, B, C, D\}$ (we used new names P, Q, R, S because we do not know which pair of points is on l and which pair is on m; these new names allow for the possibility that $P = A$ or $P = B$ or $P = C$ or $P = D$, for example). Assume further that $l \parallel m$ and $n \nparallel l$. We want to show that $n \nparallel m$. Since $l \parallel m$, these two lines have no point in common, so P, Q, R, S must be four distinct points. Since $n \nparallel l$, n has a point in common in l by definition of parallel. We also know by Proposition 2.1 that this point is unique. Without loss of generality, say that this common point is P. (We actually do not know that this point is P; it could possibly be Q. By saying "without loss of generality", we are indicating that the argument we are about to give will apply just as well to Q.) Now that we know P is one of the points on n, the other point on n must be either R or S (it cannot be Q because then by the uniqueness part of I-1, n would equal l, contrary to our assumption that the three lines are distinct). In either case, n will have a point in common with the line $m = \{R, S\}$, so n is not parallel to m.

Remark: Students were not expected to give a proof with this much detail. When reasoning with models in this course, we will generally not be able to give proofs from first principles, because the underlying assumptions have not been completely laid out. For example, we have not discussed axioms for the real numbers, so our proofs involving real numbers, detailed as they seem, will not be as complete as our proofs in axiomatic geometry. On the other hand, the proof we just gave merely involved a set of four points; there is less to take on faith in trying to comprehend what a set of four points is compared to, say, what the real numbers are.

Model 2: There are exactly five points in our model: A, B, C, D, E. The lines are the sets consisting of exactly two points (there are 10 lines total). "Lie on" will mean set membership.

We wish to show that our model does not satisfy the statement in the problem; that is, our model satisfies the negation of the statement:

> There exist three lines l, m, and n such that $l \parallel m$, $l \nparallel n$, and $n \parallel m$.

It suffices to find three lines that have the properties specified in the statement. Here are three such lines: $l = \{A, B\}, m = \{C, D\}, n = \{A, E\}$.

5. An *affine plane* is a model for incidence geometry having the Euclidean parallel property: For every line l and every point P not on l, there exists exactly one line through P that is parallel to l.

Prove that in an affine plane, for any three distinct lines l, m, and n, if $l \parallel m$ and $n \nparallel l$, then $n \nparallel m$. (Thus, the statement given in problem 3 is a theorem in affine geometry.)

Proof. Let l, m and n be three distinct lines. Suppose that $l \parallel m$ and $n \nparallel l$. Since $n \nparallel l$ there is a point P that lies on both n and l. Note that P is not on m since l and m have no points in common. By the Euclidean parallel postulate there is only one line through P that is parallel to m. Since n and l are two different lines passing through P and l is parallel to m, n must not be parallel to m. □