Math 431 Homework 4
Due 10/13

1. Suppose that $A * B * C$ and $A * C * D$.

(a) (**3 pts**) Prove that A, B, C, D are four distinct points.

Solution: By B-1, A, B, C are distinct and A, C, D are distinct. The only pair of points that do not appear in both sets is the pair B, D. If $B = D$ then substituting D for B in the hypothesis would yield $A * D * C$ and $A * C * D$, contradicting B-3. Therefore $B \neq D$.

(b) (**3 pts.**) Prove that A, B, C, D are collinear.

Solution: Axiom B-1 and the assumption $A * B * C$ together imply that the points A, B, C are collinear. Furthermore, the uniqueness part of I-1 guarantees that these points all line on line $\overleftrightarrow{AC}$. Similarly, Axiom B-1 and $A * C * D$ together imply that A, C, D are collinear. Again the uniqueness part of I-1 guarantees that these points all lie on $\overleftrightarrow{AC}$. So all four points lie on $\overleftrightarrow{AC}$.

2. Prove Proposition 3.1(ii): For any two distinct points A and B, $\overrightarrow{AB} \cup \overrightarrow{BA} = \{\overleftrightarrow{AB}\}$.

Proof. Step 1 (**5 pts.**): $\overrightarrow{AB} \cup \overrightarrow{BA} \subset \{\overleftrightarrow{AB}\}$.

Let $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$. The proof will be complete once we show that $P \in \{\overleftrightarrow{AB}\}$. If $P = A$ or $P = B$ then P is on line $\overleftrightarrow{AB}$ hence in set $\{\overleftrightarrow{AB}\}$. Now suppose that P, A, B are distinct. If $P \in \overrightarrow{AB}$, then by definition of ray, $P \in AB$ or $A * B * P$. Having ruled out the possibilities $P = A$ or $P = B$, if $P \in AB$ then $A * P * B$ by definition of segment. Therefore $A * P * B$ or $A * B * P$. In both cases A, P, B all lie on the same line according to B-1; this line is $\overleftrightarrow{AB}$ by the uniqueness part of I-1. Thus $P \in \{\overleftrightarrow{AB}\}$. By the same logic, if $P \in \overrightarrow{BA}$ then $B * P * A$ or $B * A * P$, and again P, A, B all lie on $\overleftrightarrow{AB}$, so $P \in \{\overleftrightarrow{AB}\}$.

Step 2 (**5 pts.**): $\{\overleftrightarrow{AB}\} \subset \overrightarrow{AB} \cup \overrightarrow{BA}$.

Let $P \in \{\overleftrightarrow{AB}\}$. The proof will be complete once we show that $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$. If $P = A$ or $P = B$, then $P \in AB$ by definition of segment.

$AB \subset \overrightarrow{AB}$ by definition of ray, and $\overrightarrow{AB} \subset \overrightarrow{AB} \cup \overrightarrow{BA}$ by definition of union, so $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$.

Now suppose that P, A, B are distinct. These points are collinear because we assumed that P lies on $\overleftrightarrow{AB}$. Thus B-3 gives us $P * A * B$ or $A * P * B$ or $A * B * P$.

- If $P * A * B$ then $P \in \overrightarrow{BA}$ by definition of ray.
- If $A * P * B$ then $P \in AB$ by definition of segment, so then $P \in \overrightarrow{AB}$ by definition of ray.
- If $A * B * P$ then $P \in \overrightarrow{AB}$ by definition of ray.

In all cases P is in $\overrightarrow{AB}$ or $\overrightarrow{BA}$, meaning that $P \in \overrightarrow{AB} \cup \overrightarrow{BA}$.

□

3. (**14 pts.**) Let $\mathcal{A}$ be an affine plane. Show that the projective completion of $\mathcal{A}$, $\mathcal{A}^*$ satisfies axioms I1, I2+, I3 and elliptic parallel postulate.
Axiom I2+: For every line l there are at least three distinct points incident with it.

Solution: See pages 59 – 60 in the book. Although few more details could be supplied.