Math 431 Homework 6
Due 10/30

1. Let P and Q be two points and l and m two lines. What can you say about these points and lines if you know that side$(P, l) \cap$ side$(Q, m) = \emptyset$? In the event that there is a point $L \in \{l\}$ such that $P * L * Q$ and $M \in \{m\}$ such that $P * M * Q$ show that $L * M * Q$ and $P * L * M$.

Solution: We have side$(P, l) \cap$ side$(Q, m) = \emptyset$. We first note that $P \notin \{l\}$ and $Q \notin \{m\}$. There are three possibilities:

1. l and m are distinct lines that share a point,
2. l and m are parallel, and
3. $l = m$

Let us consider each case in turn.

1. l and m are distinct lines that share a point. Then either
   (a) P lies on m
      i. Q lies on l: Let R be a point such that $P * R * Q$, which exists by axiom *B-2*. By Lemma 3.2.2 P and R are on the same side of l and R and Q are on the same side of m. Hence, $P \in$side$(P, l) \cap$ side(Q, m), which is a contradiction.
      ii. Q does not lie on l. First note that P and Q have to be on opposite sides of l, for if they were not then Q would belong to side$(P, l) \cap$ side(Q, m), which contradicts the assumption. Since P and Q are on opposite sides of l the segment PQ intersects l in a point, call it L, so that $P * L * Q$. By axiom *B-2* there is a point R such that $L * R * P$. Using Proposition 3.3, we can conclude that $Q * R * P$, that is Q and P are on the same side of m. Further, since $L * R * P$ we see that P and R are on the same side of l, hence $R \in$side$(P, l) \cap$ side(Q, m), contradicting our assumption.
   (b) P does not lie on m
      i. Q lies on l: proof follows the proof in case of $P \in \{m\}, Q \notin \{l\}$.

ii. Q does not lie on l. Take any point M on m so that P and M are on the same side of l (such point exist, because if it did not then m and l would be parallel, which contradicts our hypothesis), and take any point L on l so that Q and L are on the same side of m. By axiom *B-2* there is a point R such that $L * R * M$. By Lemma 3.2.2 and axiom *B-4* R and P are on the same side of l. Similarly, Q and R are on the same side of m, hence $R \in$side$(P, l) \cap$ side(Q, m), once again contradicting our assumption.

Hence, this case can not happen, given our hypothesis.

2. $l||m$. As in the previous case we have few possible configurations depending on where the given points lie with respect to the given lines.

   (a) P lies on m

   i. Q lies on l. Let R be a point such that $P * R * Q$. Using Lemma 3.2.2 we conclude that R and P are on the same side of l and R and Q are on the same side of m, that is $R \in$side$(P, l) \cap$ side(Q, m).

   ii. Q does not lie on l. As we noted above P and Q must lie on opposite sides of l, so by Lemma 3.2.5 there is a point $L \in \{l\}$ such that $P * L * Q$. Let R be such that $P * R * L$ (*B-1*). Then Lemma 3.2.2 and *B-4* give us that $R \in$ side(P, l). Similarly, $R \in$ side(Q, m). Contradiction.

   (b) P does not lie on m

   i. Q lies on l. As 2(ii).

   ii. Q does not lie on l. Using the arguments above we can show that P and Q must lie on opposite sides of both l and m. Let $L \in \{l\}$ such that $P * L * Q$ and $M \in \{m\}$ such that $P * M * Q$. By *B-3* one of the following happens:

   - $P * M * L$ – then P and M are on the same side of l (Lemma 3.2.5). Let S be such that $M * S * L$, so that M and S are on the same side of l. By *B-4*, P and S are on the same side of l. Also, $P * M * L$ and $P * L * Q$ give us $M * L * Q$ bu Proposition 3.3, so L and Q are on the same side of m (Lemma 3.2.2). From $M * S * L$ we conclude that L and S are on the same side of m, so by *B-4* we have that S and Q are on the same side of m. Hence, $S \in$side$(P, l) \cap$ side(Q, m).

- $P * L * M$ together with $P * M * Q$ gives, by Proposition 3.3 $L * M * Q$, hence our claim holds.
- $M * P * L$ together with $P * L * Q$ gives $M * P * Q$, so P and Q are on the same side of m, contradiciton.

Note: I could have done this whole argument without using Proposition 3.3, but that would have made it much longer than it already is, so I chose not to.

3. $l = m$. P and Q lie on opposite sides of l, for if they did not then side$(P, l) =$ side(Q, m), so side(P, l) is their intersection, and that set is nonempty.

This exhaust all the possible cases.

2. Prove Proposition 3.8: If D is in the interior of an $\sphericalangle CAB$ then:

1. so is every point on $\overrightarrow{AD}$ except A,
2. no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\sphericalangle BAC$
3. if $C * A * E$, then B is in the interior of $\sphericalangle DAE$

Proof of (1). Suppose that $E \in \overrightarrow{AD}$ and $E \neq A$. Since D is in the interior of $\sphericalangle CAB$, D and B are on the same side of $\overleftrightarrow{AC}$; by Lemma 3.7.5, E and D are on the same side of $\overleftrightarrow{AC}$; hence, by *B-4* E and B are on the same side of $\overleftrightarrow{AC}$. By the same reasoning, since D and C are on the same side of $\overleftrightarrow{AB}$, we deduce from Lemma 3.7.5 and *B-4* that E and C are on the same side of $\overleftrightarrow{AB}$. Thus E is in int$\sphericalangle CAB$. □

Proof of (2). Suppose that E is on the ray opposite to $\overrightarrow{AD}$. Then $E = A$ or $E * A * D$ by definition of the ray opposite to $\overrightarrow{AD}$ (as discussed in class). If $E = A$, then E lies on $\overleftrightarrow{AC}$, so E and B are not on the same side of $\overleftrightarrow{AC}$, so E is not in the interior of $\sphericalangle CAB$. Suppose that $E * A * D$. By Lemma 3.2.4, E and D are on opposite sides of $\overleftrightarrow{AB}$. Since $D \in$ int$\sphericalangle CAB$, D and C are on the same side of $\overleftrightarrow{AB}$. Thus, by Corollary to *B-4* E and C are on opposite sides of $\overleftrightarrow{AB}$. Thus, E is not in int$\sphericalangle CAB$. □

Proof of (3). There are two things to show: first is B and D are on the same side of $\overleftrightarrow{AE}$, and second is B and E are on the same side of line $\overleftrightarrow{AD}$.

Since $C * A * E$, by *B-1* and *I-1* we have $\overleftrightarrow{AC} = \overleftrightarrow{AE}$. Since D is in the interior of $\sphericalangle CAB$, B and D are on the same side of $\overleftrightarrow{AC}$, hence $\overleftrightarrow{AE}$.

To prove that B and E are on the same side of $\overleftrightarrow{AD}$, suppose on the contrary that E and B not on the same side of $\overleftrightarrow{AD}$. Before we can say that B and E are on opposite sides of $\overleftrightarrow{AD}$, we must first check that neither B nor E is on $\overleftrightarrow{AD}$. Since $D \in \text{int}\sphericalangle CAB$, D and C are on the same side of $\overleftrightarrow{AB}$, so D is not on $\overleftrightarrow{AB}$ (definition of same sides), so A, B, D are not collinear, so B is not on $\overleftrightarrow{AD}$. Also, since $D \in \text{int}\sphericalangle CAB$, D and B are on the same side of $\overleftrightarrow{AC} = \overleftrightarrow{AE}$, so D is not on $\overleftrightarrow{AE}$, so A, D, E are not collinear, so E is not on $\overleftrightarrow{AD}$. Now that we know that B and E do not lie on $\overleftrightarrow{AD}$ and are not on the same side of $\overleftrightarrow{AD}$, they must be on opposite sides of $\overleftrightarrow{AD}$.

By definition of opposite sides and segment, there is a point F lying on $\overleftrightarrow{AD}$ such that $E * F * B$. By Proposition 3.7, F is in the interior of $\sphericalangle BAE$. (Note that $\sphericalangle BAE$ is an angle because B does not lie on $\overleftrightarrow{AE}$.) In particular F and B are on the same side of $\overleftrightarrow{AE}$. Since B and D are on the same side of $\overleftrightarrow{AC} = \overleftrightarrow{AE}$, by *B-4* F and D are on the same side of $\overleftrightarrow{AE}$. Thus, either $D = F$, $A * D * F$ or $A * F * D$ by Lemma 3.2.3. In any case, D is on ray $\overrightarrow{AF}$. Thus, by Proposition 3.8(a) (applied to $\overrightarrow{AF}$ and $\sphericalangle BAE$) D is in the interior of $\sphericalangle BAE$.

By definition of interior, D and E are on the same side of $\overleftrightarrow{AB}$. We also know that D and C are on the same side of $\overleftrightarrow{AB}$ because D is in interior of $\sphericalangle CAB$. Therefore, by B-4(i), C and E are on same side of $\overleftrightarrow{AB}$. But, since $C * A * E$, by Lemma 3.2.4 C and E are on opposite sides of $\overleftrightarrow{AB}$. This is a contradiction. Therefore, B and E are on the same side of $\overleftrightarrow{AD}$.

□

3. If B and D are distinct points there exists a point C such that $B*C*D$.

1. There exists line $\overleftrightarrow{BD}$ through B and D – by axiom *I-1*, since B and D are distinct points.

2. There exists a point F not lying on $\overleftrightarrow{BD}$ – by Proposition 2.3.

3. There exists a line $\overleftrightarrow{BF}$ through B and F – by axiom *I-1*, since F does not lie on $\overleftrightarrow{BD}$ we must have $F \neq B$, and also $\overleftrightarrow{BD} \neq \overleftrightarrow{BF}$.

4. There exists a point G such that $B * F * G$ – by axiom *B-2*, since B and F are distinct points.

5. Points B, F, and G are collinear – by axiom *B-1* and step 4.

6. G and D are distinct points and D, B and G are not collinear – G lies on $\overleftrightarrow{BF}$, D lies on $\overleftrightarrow{BD}$, and the intersection of those two lines is B. Since the lines are distinct (step 3), by Proposition 2.1 B is the only point they have in common, hence G and D are distinct points. If D, B and G were collinear, they would have to lie on a unique line $\overleftrightarrow{BD}$ (axiom *I-1*), so $\overleftrightarrow{BD} = \overleftrightarrow{BF}$ which contradicts step 3.

7. There exists a point H such that $G * D * H$ – step 6 guarantees that we can apply axiom *B-2* to points G and D.

8. There exists a line $\overleftrightarrow{GH}$ – by axiom *I-1*.

9. H and F are distinct points – If they were the same then we would have $B*F*G$ and $G*D*F$, so by axiom *B-1* B, G and D are collinear points contradicting step 6.

10. There exists a line $\overleftrightarrow{FH}$ – previous step and axiom *I-1*.

11. D does not lie on $\overleftrightarrow{FH}$ – F does not lie on $\overleftrightarrow{GD}$ (step 9), so $\overleftrightarrow{GD} \neq \overleftrightarrow{FH}$. Since H lies on each of those lines, and since $H \neq D$ by step 7, by Proposition 2.1, D does not.

12. B does not lie on $\overleftrightarrow{FH}$ – If it did, then H would lie on the unique line $\overleftrightarrow{BF}$ determined by B and F (axiom *I-1*). Lines $\overleftrightarrow{BF}$ and $\overleftrightarrow{GD}$ now have two points in common: G and H. By Proposition 2.1 they would have to be equal, which contradicts the previous step.

13. G does not lie on $\overleftrightarrow{FH}$ – if it did we would have: G, F, H collinear, G, D, H collinear, hence G, D, B collinear (usinga axiom *I-1*) which contradicts step 6.

14. Points D, B and G determine $\triangle DBG$ – step 6 and definition of a triangle,

    and $\overleftrightarrow{FH}$ intersects side BG in a point between B and G – steps 4, 12 and 13.

15. H is the only point lying on both $\overleftrightarrow{FH}$ and $\overleftrightarrow{GH}$ – these two lines are distinct, eg. step 13, so by Proposition 2.1 they share exactly one point: H.

16. No point between G and D lies on $\overleftrightarrow{FH}$ – step 7 and axiom *B-3*.

17. Hence, $\overleftrightarrow{FH}$ intersects side BD in a point C between D and B – step 14, 16 and Pasch's theorem (note that it can't be point D since $G * D * H$).

18. Thus, there exists a point C between points B and D.